Ergodic properties of queuing networks with batch arrivals and batch service

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Ergodic Properties of Queueing Networks with Batch Arrivals and Batch Service

Propriétés ergodiques dans les réseaux de communications avec arrivées et services simultanés.

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Abstract

We analyze queueing networks with batch arrivals and batch service. Lyapounov functions are used for obtaining necessary and sufficient conditions for ergodicity of such networks.

Résumé

Pour une classe de réseaux avec arrivéses et services simultanés, on obtient les conditions nécessaires et suffisantes d'ergodicité.

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Ergodic Properties of Queueing Networks with Batch Arrivals and Batch Service

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Abstract

We analyze queueing networks with batch arrivals and batch service. Lyapounov functions are used for studying ergodicity. We also give an ergodicity criterion for such networks. Exponentially fast convergence to stationary equilibrium is proved.

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1 Introduction.

In this paper, we study ergodic properties of queueing networks with batch arrival and batch service. In these networks simultaneous arrivals of jobs at different nodes are allowed and service rates can depend on the queue size in the nodes.

One of the main methods of studying ergodic properties is to find explicit expressions for a stationary distribution, for example, in the product form. But for many queueing networks it is impossible to obtain effective formulae for equilibrium state probabilities. We consider exactly such case.

To study ergodicity we shall use Lyapounov functions. This method was applied to queueing networks in papers [1, 2, 3, 4]. The Lyapounov function method allows to get an ergodicity criterion. As in case of classical Jackson network ergodicity conditions are expressed in terms of loads.

From existence of Lyapounov function it will follow important corollaries.

1. Time t transition probabilities of the Markov chain which represents the network exponentially fast converge to the stationary probabilities as $t \to \infty$.

2. If we consider a network which is a small perturbation of the initial network then the stationary probabilities of the perturbed network depend analytically on perturbing parameters.

2 Main Definitions and Results.

We consider a network in which batch simultaneous arrivals at different nodes and batch servicing are allowed. Moreover, the service rate of a job can depend on the queue size. Let N be the number of nodes in the network. Assume that the service time of a batch of calls has an exponential distribution. Let

$$\mu_i^{(n)}(x_i),\ n \leq x_i$$

be the service rate of a batch consisting of n jobs in the i-th node under the condition that the queue size is x_i. A number of jobs which can be served simultaneously does not exseed $s_i < \infty$. The arrivals into the network are assumed to be Poisson. More exactly, for any subset

$$\Lambda = \{i_1, \ldots, i_k : 1 \leq i_1 < i_2 < \ldots < i_k \leq N\} \subseteq \{1, \ldots, N\}$$

let $\lambda_\Lambda\left(n_{i_1},\ldots,n_{i_k}\right)$ be the rate of simultaneous arrivals of batches consisting of $n_{i_1},\ldots,n_{i_k}$ jobs in nodes $i_1,\ldots,i_k$ respectively. Define the total rate of the external flow in the i-th node by formula

$$\bar{\lambda}_i = \sum_{\Lambda:i\in\Lambda}\sum_{n_{i_1},\ldots,n_{i_k}} n_i\lambda_\Lambda\left(n_{i_1},\ldots,n_i,\ldots,n_{i_k}\right), \tag{1}$$

where the external sum is performed over all $\Lambda = \{i_1,\ldots,i_k\}$ such that $1 \leq i_1 < i_2 < \ldots < i_k \leq N$, including i, the internal sum is performed over all collections $(n_{i_1},\ldots,n_{i_k})$.

Transitions of jobs between the nodes in the network are controled by a Markov chain with the state space $\{0,1,\ldots,N\}$, and the matrix of transition probabilities $P = \{p_{ij}\}$. State 0 is interpreted as the outside and this state is absorbing, i.e. $p_{00} = 1$. We shall consider *two types of job transfer between the nodes* in the network.

1. After completing service of a batch in the i-th node all jobs of the batch are independently routed among the nodes with probabilities p_{ij}.

2. After completing service of a batch all jobs of the batch enter the j-th node with the probability p_{ij}.

We assume that there is at least one i such that

$$p_{i0} = 1 - \sum_j p_{ij} > 0, \tag{2}$$

where p_{i0} is the probability that a job departs from the network after completing service in the i-th node. We also assume that the service discipline FIFO is used in all nodes.

The above network is represented by a continuous-time Markov chain ξ_t with the state space Z_+^N. Let e_i be the vector with the i-th coordinate equal to 1 and the other coordinates equal to 0. We denote by e_0 vector with zero coordinates.

In case of *the first type of job transfer between the nodes* the transition intensities λ_{xy} from the state $x = (\,x_1,\ldots,x_N\,)$ to the state $y = (\,y_1,\ldots,y_N\,)$ are defined as follows:

$$\lambda_{xy} = \lambda_\Lambda\left(\,n_{i_1},\ldots,n_{i_k}\,\right),$$

if

$$y - x = \sum_{i_m \in \Lambda} n_{i_m} e_{i_m}$$

and

$$\lambda_{xy} = \mu_i^{(n)}(x_i) \frac{n!}{\prod_{i=0}^{N} n_i!} \prod_{j=0}^{N} (p_{ij})^{n_j},$$

if

$$y - x = -ne_i + \sum_{j=0}^{N} n_j e_j,$$

where

$$\sum_{j=0}^{N} n_j = n.$$

In case of *the second type of job transfer between the nodes* the transition intensities λ_{xy} from the state x to the state y are

$$\lambda_{xy} = \lambda_\Lambda \left(n_{i_1}, \ldots, n_{i_k} \right),$$

if

$$y - x = \sum_{i_m \in \Lambda} n_{i_m} e_{i_m}$$

and

$$\lambda_{xy} = \mu_i^{(n)}(x_i) p_{ij}$$

if

$$y - x = n(-e_i + e_j).$$

Define a discrete time Markov chain ξ_n with the state space Z_+^N and the transition probabilities

$$p_{xy} = w\lambda_{xy}, \; x \neq y, \; p_{xx} = 1 - \sum_{y \neq x} p_{xy},$$

where the constant w satisfies the condition

$$w \leq \min_x \left(\sum_y \lambda_{xy} \right)^{-1}.$$

The initial chain ξ_t is ergodic if and only if the discrete time chain ξ_n is ergodic.

Note that the chain ξ_n satisfies the boundedness of jumps condition: there exists the constant $d > 0$ such that

$$p_{xy} = 0, \quad \text{if} \quad \| y - x \| > d,$$

where $\| x \| = \max_i | x_i |$.

Consider now Jackson's system

$$\nu_j = \bar{\lambda}_j + \sum_i \nu_i p_{ij}, \; j = 1, \ldots, N. \tag{3}$$

which can be rewritten in matrix form as

$$\nu = \nu P + \lambda,$$

where $\nu = (\nu_1, \ldots, \nu_N)$, $\lambda = (\bar{\lambda}_1, \ldots, \bar{\lambda}_N)$. If the network is in equilibrium then ν_i is the rate of the input flow at the i-th node from outside the network and from the other nodes. A solution of the system (3) can be obtained by means of iterations

$$\nu_j = \bar{\lambda}_j + \sum_{m=1}^{\infty} \sum_{i=1}^{N} \bar{\lambda}_i p_{ij}^{(m)}, \tag{4}$$

where $p_{ij}^{(m)}$ are the m-step transition probabilities. Convergence of the series (4) follows from the condition (2) as one can easily get the bound

$$p_{ij}^{(m)} \le C(1 - \epsilon)^m$$

for some $\epsilon > 0$, $C > 0$. It is convenient to write the solution (4) as

$$\nu = \sum_{n=0}^{\infty} \lambda P^{(m)},$$

where $P^{(0)} = I$ is the identity matrix, $P^{(m)} = \{p_{ij}^{(m)}\}$. If we denote

$$M = (I - P)^{-1} = \sum_{m=0}^{\infty} P^{(m)}, \tag{5}$$

then $\nu = M\lambda$.

Define the total service rate in the i-th node under the condition that the queue size in the node is x_i:

$$\bar{\mu}_i(x_i) = \sum_{n=1}^{s_i \wedge x_i} n\mu_i^{(n)}(x_i). \tag{6}$$

We shall assume to be fulfilled the following *homogeneity condition*: for all $i = 1, \ldots, N$ there exist limits

$$\lim_{x_i \to \infty} \bar{\mu}_i(x_i) = \bar{\mu}_i > 0. \tag{7}$$

Define *the load in the i-th node* by formula

$$\rho_i = \frac{\nu_i}{\bar{\mu}_i}. \tag{8}$$

We define now the vectors of the mean jumps of ξ_n. For any subset $\Lambda \subseteq \{1, \ldots, N\}$ introduce the face of R_+^N

$$B(\Lambda) = \{(r_1, \ldots, r_N) : \ r_i > 0, i \in \Lambda, \ x_i = 0, i \notin \Lambda\}.$$

The vector of the mean jump of ξ_n from the point $x \in B(\Lambda) \cap Z_+^N$ is

$$M_\Lambda(x) = f_0 + \sum_{i \in \Lambda} f_i(x_i), \tag{9}$$

where

$$\begin{aligned} f_0 &= w \textstyle\sum_{i=1}^N \bar{\lambda}_i e_i, \\ f_i(x_i) &= w\bar{\mu}_i(x_i) \left(-e_i + \textstyle\sum_{j=1}^N p_{ij} e_j\right) \end{aligned} \tag{10}$$

and $\bar{\lambda}_i$, $\bar{\mu}_i(x_i)$ are defined by formulae (1), (6) respectively.

Remark. Notice that mean jump vectors are the same for both types of a job transfer between the nodes.

Define also the limit vector of the mean jump of the chain ξ_n from the point $x \in B(\Lambda) \cap Z_+^N$

$$M_\Lambda = f_0 + \sum_{i \in \Lambda} f_i, \tag{11}$$

where

$$f_i = w\bar{\mu}_i \left(-e_i + \sum_{j=1}^N p_{ij} e_j\right). \tag{12}$$

We are now able to give our main theorems.

Theorem 2.1 *1. If $\rho_i < 1$ for all $i = 1, \ldots, N$ then the network is ergodic. Besides that, the following exponential estimates are fulfilled:*
a). The exponential estimate for stationary probabilities

$$\pi(x) \leq C \exp(-\gamma \| x \|_1) \tag{13}$$

for some constants $C, \gamma > 0$ not depending on x, where $\| x \|_1 = \sum_i x_i$.
b). Exponential fast convergence to the stationary distribution: there exists the constant $\sigma > 0$ such that for all $x, y \in Z_+^N$ and $t > 0$

$$| \pi(y) - p_{xy}^{(t)} | \leq C_1(x) \exp(-\sigma t) \tag{14}$$

for some constant $C_1(x)$ depending only on x, where $p_{xy}^{(t)}$ are the time t transition probabilities.
2. If for some i, $\rho_i > 1$ then the network is transient.

Consider a small perturbation of the initial network. We allow any jumps from x to y but the intensities of these additional transitions are assumed to be sufficiently small and the boundedness of jumps condition is assumed to be conserved. Let κ_{xy} be the intensities of additional transitions.

Theorem 2.2 *If the initial network is ergodic then there exists $\kappa_0 > 0$ such that the perturbed network is ergodic under $\kappa_{xy} \leq \kappa_0$ and its stationary probabilities analytically depend on κ_{xy}.*

Remark. For perturbed networks we also have the exponential estimate (13) and exponential convergence to stationary distribution.

3 Geometrical lemmas.

We need some lemmas to prove main results.

Lemma 3.1 *The following equality holds*

$$f_0 + \sum_{i=1}^{N} \rho_i f_i = 0. \tag{15}$$

Proof. The equality (15) follows from (3). According to (10), (12) we have

$$w\left(\sum_{i=1}^{N}\bar{\lambda}_i e_i + \sum_{i=1}^{N}\rho_i\bar{\mu}_i\left(-e_i + \sum_{j=1}^{N} p_{ij}e_j\right)\right) =$$

$$w\left(\sum_{i=1}^{N}\bar{\lambda}_i e_i - \sum_{i=1}^{N}\nu_i e_i + \sum_{i=1}^{N}\nu_i\sum_{j=1}^{N} p_{ij}e_j\right) =$$

$$w\left(\sum_{i=1}^{N}\bar{\lambda}_i e_i - \sum_{i=1}^{N}\nu_i e_i + \sum_{j=1}^{N}e_j\sum_{i=1}^{N} \nu_i p_{ij}\right) =$$

$$w\sum_{j=1}^{N}e_j\left(-\nu_j + \bar{\lambda}_j + \sum_{i=1}^{N}\nu_i p_{ij}\right) = 0.$$

The last equality follows from (3). □

Further, notice that the vectors f_i, $i = 1,\ldots,N$, are linearly independent. If it is not true then there exist coefficients $c_i \not\equiv 0$ such that

$$\sum_{i=1}^{N} c_i f_i = 0.$$

Hence, the coefficients c_i have to satisfy the following system:

$$c_j = \sum_{i=1}^{N} c_i p_{ij},\ j = 1,\ldots,N$$

and so the matrix $I - P$ has to be singular. It contradicts the condition (2).

Choose a new coordinate system with the basis $f_1,\ldots,f_N$ and the origin f_0. Let y_i be the i-th coordinate in this system. Then

$$y_i = \frac{1}{\mu_i}\left(-\sum_{j=1}^{N} m_{ji}x_j + \bar{\lambda}_i\right), \tag{16}$$

where m_{ij} are the elements of matrix M.

Define the following geometrical objects. Let

$$\Gamma = \{(y_1,\ldots,y_N) : y_i \geq 0 \ \text{ for all } i\} \tag{17}$$

be the positive orthant in the new coordinates,

$$\Gamma_\Lambda = \{(y_1, \dots, y_N) : y_i \geq 0 \text{ if } i \in \Lambda, \quad y_i = 0 \text{ if } i \notin \Lambda\} \tag{18}$$

be the coordinate faces and

$$S = \cup \Gamma_\Lambda \tag{19}$$

be the surface of Γ. Here the union is performed over all hyperfaces. For $\Lambda = \{1, \dots, N\} \setminus \{i\}$ introduce the half-space

$$\Gamma_\Lambda^+ = \{(y_1, \dots, y_N) : y_i \geq 0\}.$$

Also define the scaled geometrical objects $\alpha\Gamma$, $\alpha\Gamma_\Lambda$, $\alpha\Gamma_\Lambda^+$, αS corresponding to the coordinate system with the basis $f_1, \dots, f_N$ and the origin αf_0, where $\alpha \geq 1$.

Lemma 3.2 *The intersection*

$$R_+^N \cap \alpha\Gamma \tag{20}$$

is a bounded set.

Proof. Consider the hyperplane $y_i = 0$. By formula (16) the equation of this hyperplane is

$$\sum_{j=1}^N m_{ji} x_j = \bar{\lambda}_i, \tag{21}$$

and so in the initial coordinates

$$\Gamma = \{(x_1, \dots, x_N) : \sum_{j=1}^N m_{ji} x_j \leq \bar{\lambda}_i, \text{ for all } i\}.$$

The set (20) is bounded as all $m_{ij} > 0$. $\square$

Lemma 3.3 *Let $\Lambda' \subseteq \Lambda$ and Γ_Λ is a hyperface. Then for any $n_1, \dots, n_N$ there exists $\alpha > 1$ such that*

$$\alpha\Gamma_\Lambda \cap \{x_i \geq n_i, \, i \in \Lambda', \; x_i < n_i, \; i \notin \Lambda'\} = \emptyset$$

Proof. Let $\Lambda = \{2, \ldots, N\}$. Then

$$\alpha\Gamma_\Lambda = \{\alpha f_0 + \sum_{i=2}^{N} y_i f_i\}.$$

According to (12) we have

$$\alpha f_0 + \sum_{i=2}^{N} y_i f_i = \sum_{i=1}^{N} c_i e_i,$$

where

$$c_1 = \alpha\tilde{\lambda}_1 + \sum_{j=1}^{N} y_j p_{ji} > n_1$$

under sufficiently large α. It follows the result of the lemma. □

Lemma 3.4 *Let the loads $\rho_i < 1$ and $\Lambda_1 \not\subset \Lambda$. Then the vector $M_{\Lambda_1}(x)$, where $x \in \alpha\Gamma_\Lambda$ lies in $x \in \alpha\Gamma_\Lambda^+$ under sufficiently large α.*

Proof. Let $\Lambda_1 = \{1, \ldots, N\}$ and the point x is sufficiently far from the coordinate faces. Then the vector $M_{\{1,\ldots,N\}}(x)$ approximates the limit vector $M_{\{1,\ldots,N\}}$. By formula (9)

$$M_{\{1,\ldots,N\}}(x) = f_0 + \sum_{i=1}^{N} f_i(x_i) = \sum_{i=1}^{N}(-\rho_i + 1 + \epsilon_i) f_i.$$

Take $\epsilon_i < 1 - \rho_i$. Then the coordinates of the vector $M_{\{1,\ldots,N\}}(x)$ in the basis $f_1, \ldots, f_N$ are positive. Hence, it lies in Γ_Λ^+. Thus, we consider the case when

$$x \in \{x_i \geq n_i \text{ for all } i\}.$$

Let now

$$x \in \{x_i \geq n_i, i \in \Lambda', x_i < n_i,\ i \notin \Lambda'\}, \tag{22}$$

where $\Lambda' \not\subseteq \Lambda$. The vectors $f_i(x_i),\ i \in \Lambda'$ approximate the limit vectors f_i. As $f_i(x_i) = c_i(x_i) f_i$, where $c_i(x_i)$ is a constant depending on x_i, we have

$$M_{\{1,\ldots,N\}}(x) = f_0 + \sum_{i \in \Lambda'}^{N} f_i(x_i) + \sum_{i \notin \Lambda'}^{N} f_i(x_i) =$$

$$\sum_{i\in\Lambda'}^{N}(-\rho_i+1+\epsilon_i)f_i+\sum_{i\notin\Lambda'}^{N}c_i(x_i)f_i.$$

Let $i_0 \notin \Lambda$. Then $i_0 \in \Lambda'$ and coordinates $y_{i_0} = -\rho_{i_0} + 1 + \epsilon_{i_0} > 0$. Hence, the vector $M_{\{1,\dots,N\}}(x)$ lies in $\Gamma\Lambda^+$.

If $\Lambda \subseteq \Lambda'$ the face $\alpha\Gamma_\Lambda$ does not intersect the set (22). □

The case when $\Lambda_1 \subset \{1,\dots,N\}$ is considered similarly.

Define the following piecewise linear function on R_+^N:

$$f(x) = \alpha, \text{ if } x \in \alpha S. \tag{23}$$

As we show below this function is our Lyapounov function.

4 Analytic Lyapounov families.

Here we give some results from [5] which are necessary to prove the main results.

Consider a family of Markov chains $\{\xi_n(z)\}$, $z \in D$, where D is an interval of the real axis containing 0, with the same state space A. Let $P(z) = \{p_{ij}(z)\}$, $i,\ j \in A$ be the matrix of transition probabilities. The matrix $P(z)$ can be considered as a bounded linear operator in the Banach space $l_1(A)$. Assume that $P(z)$ is analytic in z as a function on D with values in the Banach algebra of bounded operators in $l_1(A)$. It means that $P(z)$ can be Taylor expanded as

$$P(z) = \sum_{n=0}^{\infty} P_n z^n,$$

where P_n are bounded linear operators with

$$\|P_n\| \le Ca^n$$

for some $C, a > 0$, i.e. the series is convergent for sufficiently small $|z|$.

A family of Markov chains $\{\xi_n(z)\}$ is called an analytic Lyapounov family if also the following conditions are fulfilled: there exist nonnegative functions $f_i(z)$, $i \in A$, $z \in D$, on A and positive integer valued functions $k_i(z)$ such that

(1)

$$\sup_{i\in A, z\in D} k_i(z) < \infty;$$

(2) for some $C, m > 0$

$$f_i(z) > Ci^m$$

for all $i \in A, z \in D$; (3) there exists $d > 0$ such that $p_{ij} = 0$ for all $z \in D$ whenever $|f_i(z) - f_j(z)| > d$;
(4) there exist $k > 0$ and $\delta > 0$ such that for any $i \in A$ and any

$$j \in V_i = \{j : \sup_{z \in D} p_{ji} > 0\}$$

$p_{ji}^{(n)}(0) > \delta$, where $p_{ji}^{(n)}(z)$ are the n-step transition probabilities of the chain $\xi_n(z)$;
(5) for all $z \in D$ and $i \in A \setminus B$, where B is a finite subset of A

$$\sum_{j \in A} p_{ij}^{(k_i(z))}(z) f_j(z) - f_i(z) \leq -\epsilon \tag{24}$$

for some $\epsilon > 0$.

By Foster criterion chain $\xi_n(z)$ is ergodic for every $z \in D$. Let $\pi_i(z)$ be the stationary probabilities of $\xi_n(z)$.

A Markov chain ξ_n is called an analytic Markov chain if the family $\xi_n(z) \equiv \xi_n$ is an analytic Lyapounov family.

Theorem 4.1 *([1]) Let $\xi_n(z)$ be analytic Lyapounov family. Then*
1) there exist constants $C_1, \sigma_1 > 0$ such that

$$\pi_i(z) \leq C_1 \exp(-\sigma_1 f_i(z))$$

for all $i \in A, z \in D$;
2) there exists a constant $\sigma_2 > 0$ such that for all $i \in A$, $z \in D$ and $n \in Z_+$

$$\sum_{j \in A} |p_{ij}^{(n)}(z) - \pi_j(z)| \leq C_2(j) \exp(-\sigma_2 n)$$

for some constant $C_2(i)$ depending only on i;
3) there exists $z_0 > 0$ such that the stationary probabilities $\pi_i(z)$ are analytic in z for $|z| < z_0$ for all $i \in A$.

5 Proof of Main Results.

We shall construct a Lyapounov function f_x such that for some $\epsilon > 0$ and $k \in Z_+$

$$\sum_y p_{xy}^{(k)} f_y - f_x < -\epsilon \tag{25}$$

for all $x \in Z_+^N \setminus B$ where B is a finite set. Then ergodicity will follow from Foster criterion. Moreover, following [1] we give two constructions of Laypounov functions.

First construction. Here we shall use the piecewise linear function (23) as our Lyapounov function. We have to verify the inequality (25) for this function. This fact follows from the following lemmas.

Lemma 5.1 *There exist $r_0 > 0$, and $k \in Z_+$ such that for all $x \in Z_+^N$ satisfying the condition*

$$\| x \| \doteq \max_i x_i > r_0$$

the inequality

$$\sum_y p_{xy}^{(k)} f_y - f_x < -\epsilon$$

holds for some $\epsilon > 0$.

Proof. The proof of this lemma is analogous to the proof of lemma 5.3 from [1]. For any Λ, $|\Lambda| = N - 1$ define function f^Λ:

$$f_x^\Lambda = \alpha,$$

for x belonging to the hyperplane αP_Λ generated by $\alpha\Gamma_\Lambda$. By lemma 3.4 the random sequence

$$f_{\xi_1}^\Lambda, f_{\xi_2}^\Lambda, \ldots, f_{\xi_n}^\Lambda$$

is a supermartingale satisfying the following condition

$$E(f_{\xi_i}^\Lambda | f_{\xi_1}^\Lambda, \ldots, f_{\xi_{i-1}}^\Lambda) < f_{\xi_{i-1}}^\Lambda - \sigma$$

for some $\sigma > 0$. By lemma 1.1 from [5] the exponential estimate

$$P(f_{\xi_k}^\Lambda - f_{\xi_1}^\Lambda < -\epsilon_i k) > 1 - c_i \exp(-\delta_i k)$$

holds for some $c_i, \delta_i, \epsilon_i > 0$. Hence, choosing sufficiently large r_0 we get that for all Λ simultaneosly

$$P(f^{\Lambda}_{\xi_k} - f^{\Lambda}_{\xi_1} < -\epsilon k) > 1 - c\exp(-\delta k)$$

holds for some $c, \delta, \epsilon > 0$. The result of the lemma follows easily from this fact. □

Lemma 5.2 *There exist $r_i > 0$, and $k \in Z_+$ such that for all $x \in Z_+^N$ satisfying the condition*

$$x_i > r_i$$

the inequality

$$\sum_y p^{(k)}_{xy} f_y - f_x < -\epsilon$$

holds for some $\epsilon > 0$.

Proof is analogous to the proof of lemma 5.1. □

Second construction. This construction gives rise to to one-step Lyapounov function, i.e. $k = 1$ in formula (25).

Lemma 5.3 *For any $\epsilon > 0$ there exists a smooth convex hypersurface $S(\epsilon)$ which is an approximation for piecewise linear surface S, i.e. for any $x \in S(\epsilon)$*

$$\rho(x, S) < \epsilon$$

and for any $y \in S$

$$\rho(y, S(\epsilon)) < \epsilon.$$

This lemma was proved in [1].

Define the following Lyapounov function

$$f_x = \alpha, \text{if } x \in \alpha S(\epsilon). \tag{26}$$

This function is smooth by construction. Using lemma 3.4 and the principle of almoust linearlity [6] we get that function (26) is one-step Lyapuonov function.

The exponential estimates of theorem 2.1 and theorem 2.2 follow from theorem 4.1.

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