Condensation in large closed Jackson networks

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Condensation dans les grands réseaux de Jackson fermés.

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Résumé

On considère des réseaux de Jackson finis, fermés, avec N noeuds $1, ..., N$ et M clients. Nous obtenons les asymptotiques de la fonction de partition et des fonctions de corrélation, lorsque $M \to \infty$, $N \to \infty$, $\frac{M}{N} \to \lambda > 0$. Nous obtenons les conditions pour lesquelles les tailles moyennes des files sont finies. Nous trouvons aussi les conditions pour qu'il existe au moins un noeud dont la file ne soit pas bornée (phénomènes de condensation, embouteillages de circulation, et.c.).

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Condensation In Large Closed Jackson Networks

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Abstract

We consider finite closed Jackson networks with N nodes $1, ..., N$ and M customers. We get the asymptotics of the partition function and of correlation functions when $M \to \infty$, $N \to \infty$, $\frac{M}{N} \to \lambda > 0$. We get conditions when mean queue lengths are finite and when there exists a node where the mean queue length tends to infinity under the above limit (condensation phenomena, traffic jams).

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1 Introduction.

We consider finite closed Jackson networks J_N with N nodes $(1, ..., N)$, $M = M_N$ customers, exponential service rates $\mu_{i,N}$ at the nodes i and the transition probability (routing) matrix P_N defining a finite Markov chain. The latter is assumed to be irreducible (consisting of one class of essential states, with no inessential states) aperiodic. Let $\rho_N = (\rho_{1,N}, ..., \rho_{N,N})$ be the positive vector (unique up to a positive factor) satisfying the following equality:

$$\rho_N P_N = \rho_N \tag{1}$$

Let $\{r_{i,N}\}_{i=1}^N$ be defined by $r_{i,N} = C\frac{\rho_{i,N}}{\mu_{i,N}}$, where C is chosen so that

$$\max_i r_{i,N} = 1. \tag{2}$$

Note that the right hand sides of equations (3) and (4) below do not depend on the choice of C.

The partition function for J_N is defined as

$$Z_N = Z_{M,N} = \sum_{n_1+n_2+\ldots+n_N=M} \prod_{i=1}^N r_{i,N}^{n_i}, \; n_i \geq 0, \; i = 1, \ldots, N$$

Denote $\xi_{i,N}$ the random number of customers at node i. Then, as it is well known (e.g. [1]),

$$P_{M,N}\{\xi_{1,N} = n_1, \xi_{2,N} = n_2, \ldots, \xi_{N,N} = n_N\} = \frac{\prod_{i=1}^N r_{i,N}^{n_i}}{Z_{M,N}},$$
$$0 \leq n_i \leq M, i = 1, \ldots, N \tag{3}$$

and the mean number of customers at the node i

$$m_{i,N} = m_{i,M,N} := E\xi_{i,N} = \frac{r_{i,N}\frac{\partial Z_{M,N}}{\partial r_{i,N}}}{Z_{M,N}} \tag{4}$$

The goal of the paper is threefold :

Firstly, we study the asymptotics of the partition function when $M \to \infty$, $N \to \infty$, $\frac{M}{N} \to \lambda > 0$. The useful remark is that although a large Jackson network strongly resembles an ideal gas (canonical ensemble) in statistical mechanics, the crucial difference between the two is that Jackson networks can be extremely inhomogeneous in the properties of the particles which build up the system. This together with other pecularities can give rise to features in the mere definition of the thermodynamic limit. To study the limit behaviour we have to assume something about the families $\{J_N\}_{N=1}^{\infty}$ of finite closed Jackson networks. The weakest assumption could be made in terms of the distribution of $r_{i,N}$ in the segment $[0,1]$ of the real axis $\mathbf{R}$.

For any Borel set A we define the measure $I_N(A) = \frac{1}{N}\sharp(i : r_{i,N} \in A)$.

Throughout this paper we will suppose that the following assumption holds true:

Assumption W.

As $N \to \infty$ the measures I_N converge weakly to a (probability) measure I on $[0,1]$.

To study the limiting distribution of the queue at a node or of the correlation functions we will also suppose that $r_{i,N}$ for the nodes in consideration have limits as $N \to \infty$: $r_i := \lim\limits_{N\to\infty} r_{i,N}$.

The assumptions above concern the properties of solutions of the systems of linear equations (1) and are related to the network topology (see section 6 for the discussion).

Under these assumptions we get the condition for the mean lengths of queues at all nodes to be uniformly bounded (**Theorem 1**). In this case we get the limiting distribution for the sequence of closed Jackson network in equilibrium. It is of course the product of geometrical distributions.

We study the asymptotics in question by means of the integral representations of the partition function of the network, and of its derivatives. We apply to those integrals the techniques of the "saddle-point" method ([2]). The necessary knowledge of the theory of functions of a complex variable can be found, for example, in ([3]).

Secondly, we show (**Theorem 2**) that the conditions of **Theorem 1** are not only sufficient but also necessary. Otherwise for the node with the maximal load the mean queue length tends to infinity. This is a traffic jam phenomenon. It resembles the Bose condensation phenomenon in quantum statistical physics: particles condense in the minimal energy state. This phenomenon is governed by a density parameter, denoted by λ, we find the critical value λ_{cr} of this parameter. This gives us the means to change $\mu_{i,N}$ to avoid traffic jams.

The rate of divergence of queue lengths can be quite arbitrary, since it depends on the way the measures I_N and the values of $r_{i,N}$ converge. In **Theorem 3** we provide an example where the mean queue length diverges linearly with N.

Our third goal is to give some useful techniques for fast numerical computation for closed product form networks with large number of nodes (see [4] about this problem). The main difficulty here is to compute $Z_{M,N}$, while we are providing the asymptotic formula for it.

Various asymptotics for certain type of closed networks, posessing a product form solution, and consisting of fixed finite number of queuing nodes, a fixed number of processor-sharing CPUs and a finite number of classes of jobs, and for large populations, were studied in the series of papers ([5],[6],[7]). In thas case the load distribution in the system was in fact fixed throughout the limit transition.

We hope that our results can be extended to more general product form networks (BCMP etc.) Our results give an easy way to avoid traffic jams by appropriately choosing service rates μ_i.

2 Main results.

Let us put $\epsilon_N = \frac{M}{N\lambda} - 1$.

Denote for $z \in \mathbf{C}\backslash[1, +\infty)$

$$h(z) := \int_0^1 \frac{r}{1 - zr} dI(r) \tag{5}$$

$$S_N(z) := -\lambda(1 + \epsilon_N)\ln z - \frac{1}{N}\sum_{i=1}^{N} \ln(1 - zr_{i,N}) \tag{6}$$

$$S(z) := -\lambda \ln z - \int_0^1 \ln(1 - zr) dI(r) \tag{7}$$

It is obvious that $h(z)$ is strictly monotone on $[0, 1)$. We will denote $\lambda_{cr} := \lim\limits_{z \to 1-} h(z)$.

Remark 1. Note that if $dI(r)$ is absolutely continuous with a smooth density $f(r)$, then $\lambda_{cr} < \infty$ if, e.g. $f(1) = 0$.

Remark 2. The case when $\lambda_{cr} = 0$ is, of course, possible. Then for any $\lambda > 0$ we observe the overflow behaviour in the system, but we do not study it in this paper. In the sequel, we will assume that

$$\lambda_{cr} > 0$$

The main results of the paper are the following theorems.

Theorem 1. Let $\lambda < \lambda_{cr}$. Then

$$(i) \qquad Z_N \sim \frac{1}{\sqrt{2\pi N S''(z_0)}} \frac{1}{z_0} exp(N S_N(z_{0,N})), \; N \to \infty$$

and the asymptotics of the free energy

$$F_N = \frac{1}{N} \ln Z_N \sim S(z_0).$$

(ii) $m_{i,N}$ are uniformly bounded in N and i,
i.e. there exists some constant Q, that $m_{i,N} < Q, \; N \geq 1, \; 1 \leq i \leq N$.

(iii) if there exist $r_i = \lim\limits_{N \to \infty} r_{i,N}, \; 1 \leq i \leq K$ for some $K, \; K > 0$, then

$$\forall n_1, n_2, \ldots, n_K \geq 0$$

$$P_N\{\xi_1 = n_1, \xi_2 = n_2, \ldots, \xi_K = n_K\} \to \prod_{i=0}^{K} (1 - z_0 r_i)(z_0 r_i)^{n_i}, \; N \to \infty$$

(this is, of course, true for any finite set of nodes, but we choose the "first" ones for the sake of simplicity of notations)

(iv) if for some i $\exists r_i = \lim_{N\to\infty} r_{i,N}$, then $m_{i,N} \to \dfrac{z_0 r_i}{1 - z_0 r_i}$, $N \to \infty$

where

$P_N\{A\}$ is the probability of an event A in the N-th network J_N, z_0 is the root of the equation

$$\frac{\partial S(z)}{\partial z} = 0 \Leftrightarrow h(z) = \frac{\lambda}{z},$$

such that $0 < z_0 < 1$ and $z_{0.N}$ is the least root of the equation

$$\frac{\partial S_N(z)}{\partial z} = 0$$

Theorem 2. Let $\lambda \geq \lambda_{cr}$, and let $i(N)$ be such that $r_{i(N),N} = 1$. Then $m_{i(N),N} \to \infty$, $N \to \infty$.

We also provide a particular example of overflow behaviour.

Theorem 3. Let for all N $r_{1,N} = 1$ and $r_{i,N} \leq B < 1, i \neq 1$, $\lambda > \lambda_{cr}$, and in some neighbourhood to the left of B $\frac{dI(r)}{dr}$ exists, $\frac{dI(r)}{dr} > L$ for some $L, L > 0$.

Then

(0) $$\lambda_{cr} < \infty$$

If $\lambda > \lambda_{cr}$, then:

(i) $$Z_N \sim \prod_{i=2}^{N} \frac{1}{1 - r_{i,N}}, \text{ and } m_{1,N} \sim -NS'(1),\ N \to \infty$$

where in fact $S(z) = -\lambda \ln z - \int\limits_0^B \ln(1 - zr)dI(r)$, and hence its domain of definition comprises a neighbourhood of $z = 1$.

(ii) $m_{i,N}$ are uniformly bounded in N and $i, 2 \leq i \leq N$;

(iii) if for some i, $i \geq 2$ $\exists r_i = \lim_{N\to\infty} r_{i,N}$, then $m_{i,N} \to \frac{r_i}{1-r_i}, N \to \infty$

If $\lambda < \lambda_{cr}$, then asymptotics of **Theorem 1** hold true.

3 Auxiliary results.

Note that $r_i \neq 0, i = 1, \ldots, N$.

Lemma 1. For any $N, N > 1$, any $j, 1 \leq j \leq N$ $P_{M+1}\{\xi_j \neq 0\} > P_M\{\xi_j \neq 0\}$.

Proof. We will prove that $P_{M+1}\{\xi_j = 0\} < P_M\{\xi_j = 0\}$. Without loss of generality we will assume $j = 1$.

Denote

$$Z_K^{L,R} = \sum_{\sum_{l=L}^{R} n_l = K} \prod_{k=L}^{R} r_k^{n_k}, \quad L \leq R,\ K \geq 0, n_i \geq 0, L \leq i \leq R.$$

Then

$$P_M\{\xi_1 = 0\} = \frac{Z_M^{2,N}}{Z_M^{1,N}}$$

Let $N = 2$. In this case we have simple explicit formula:

$$P_M\{\xi_1 = 0\} = \frac{1}{\sum_{n_1=0}^{M} (\frac{r_1}{r_2})^{n_1}},$$

and this probability decreases with M.

Let $N > 2$ and assume by induction in N that for all $M, M = 0, 1, 2, \ldots$

$$\frac{Z_{M+1}^{2,N-1}}{Z_{M+1}^{1,N-1}} < \frac{Z_M^{2,N-1}}{Z_M^{1,N-1}} \tag{8}$$

But we have

$$Z_{M+1}^{2,N} = r_N^{M+1} Z_0^{2,N-1} + r_N^M Z_1^{2,N-1} + \ldots + Z_{M+1}^{2,N-1} = r_N Z_M^{2,N} + Z_{M+1}^{2,N-1}$$
$$Z_{M+1}^{1,N} = r_N^{M+1} Z_0^{1,N-1} + r_N^M Z_1^{1,N-1} + \ldots + Z_{M+1}^{1,N-1} = r_N Z_M^{1,N} + Z_{M+1}^{1,N-1}$$

and then the condition

$$\frac{Z_{M+1}^{2,N}}{Z_{M+1}^{1,N}} < \frac{Z_M^{2,N}}{Z_M^{1,N}}, \ M \geq 0$$

is equivalent to the condition

$$p_{M+1,N-1} := \frac{Z_{M+1}^{2,N-1}}{Z_{M+1}^{1,N-1}} <$$
$$\frac{Z_M^{2,N}}{Z_M^{1,N}} = \frac{r_N^M Z_0^{2,N-1} + r_N^{M-1} Z_1^{2,N-1} + \ldots + Z_M^{2,N-1}}{r_N^M Z_0^{1,N-1} + r_N^{M-1} Z_1^{1,N-1} + \ldots + Z_M^{1,N-1}}, \ M \geq 0 \tag{9}$$

But due to (8) $Z_L^{2,N-1} > Z_L^{1,N-1} p_{M+1,N-1}$, $0 \leq L \leq M$, i.e. the fraction of the corresponding terms in the nominator and the denominator in (9) is greater than $p_{M+1,N-1}$, hence the whole fraction is also greater than $p_{M+1,N-1}$, i.e. the inequality in (9) holds. ∎

Lemma 2. For any $M, M \geq 0$, any $N, N \geq 1, j, 1 \leq j \leq N$ $m_{j,M+1,N} > m_{j,M,N}$

Proof. For $M = 0$ or $N = 1$ the inequality is trivial. Let $M > 0$ and $N > 1$.

Note that

$$P_{M+1}\{\xi_j = l | \xi_j \neq 0\} = P_M\{\xi_j = l - 1\}, \ \ 1 \leq l \leq M + 1 \tag{10}$$

Indeed, for $l, 1 \leq l \leq M + 1$

$$P_{M+1}\{\xi_j = l | \xi_j \neq 0\} = \frac{P_{M+1}\{\xi_j = l \cap \xi_j \neq 0\}}{P_{M+1}\{\xi_j \neq 0\}} =$$

$$\frac{P_{M+1}\{\xi_j = l\}}{P_{M+1}\{\xi_j \neq 0\}} = \frac{\sum\limits_{n_1+\ldots+n_N=M+1, \ n_j=l, \ n_k \geq 0, \ 1 \leq k \leq N} \prod\limits_{i=1}^{N} r_i^{n_i}}{b} =$$

$$\frac{r_j \sum\limits_{n_1+\ldots+n_N=M,\ n_j=l-1,\ n_k\geq 0,\ 1\leq k\leq N} \prod\limits_{i=1}^{N} r_i^{n_i}}{b} = \frac{P_M\{\xi_j = l-1\}}{b^*},$$

where b and b^* are some constants, but $\sum\limits_{l=1}^{M+1} P_{M+1}\{\xi_j = l|\xi_j \neq 0\} = 1$, hence $b^* = 1$ and (10) is true.

Now we can write:

$$m_{j,M+1,N} = \sum_{l=1}^{M+1} lP_{M+1}\{\xi_j = l\} =$$

$$P_{M+1}\{\xi_j \neq 0\} \sum_{l=1}^{M+1} lP_{M+1}\{\xi_j = l|\xi_j \neq 0\} =$$

$$P_{M+1}\{\xi_j \neq 0\} \sum_{l=0}^{M} (l+1)P_M\{\xi_j = l\} = P_{M+1}\{\xi_j \neq 0\}(m_{j,M,N}+1) \quad (11)$$

As N and j are fixed, let's denote $m_M := m_{j,M,N}$ and $P_M := P_M\{\xi_j \neq 0\}$.

From (11) we can then see that the condition $m_{M+1} > m_M$ is equivalent to

$$m_M < \frac{P_{M+1}}{1-P_{M+1}} \quad (12)$$

Let's check (12) for $M = 1$. Indeed, $m_1 = P_1$, and $\frac{P_2}{1-P_2} > P_2 > P_1 = m_1$, where the second inequality is due to **Lemma 1**.

Let (12) holds for $M, M \geq 1$. Then

$$m_{M+1} = P_{M+1}(m_M + 1) < P_{M+1}(\frac{P_{M+1}}{1-P_{M+1}} + 1) =$$

$$\frac{P_{M+1}}{1-P_{M+1}} < \frac{P_{M+2}}{1-P_{M+2}}$$

where the latest inequality is again due to **Lemma 1**. Therefore, (12) holds for $M' = M + 1$, too. ■

For every fixed N we consider the grand partition function

$$\Xi_N(z) := \sum_{M=0}^{\infty} z^M Z_{M,N} =$$

$$\sum_{M=0}^{\infty} z^M \sum_{n_1+n_2+\ldots+n_N=M} \prod_{i=1}^{N} r_{i,N}^{n_i} = \sum_{M=0}^{\infty} \sum_{n_1+n_2+\ldots+n_N=M} \prod_{i=1}^{N} z(r_{i,N})^{n_i} =$$

$$\sum_{n_1,n_2,\ldots,n_N} \prod_{i=1}^{N} z(r_{i,N})^{n_i} = \prod_{i=1}^{N} \frac{1}{1 - zr_{i,N}}, \text{ when } |z| < 1 \tag{13}$$

In accordance with the Cauchy formula on residues $Z_{M,N} = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{\Xi_N(z)}{z^{M+1}} dz$, where γ is a circle of radius less than 1 around the point $z = 0$, i.e.

$$Z_{M.N} = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{1}{z^{M+1}} \prod_{i=1}^{N} \frac{1}{1 - zr_{i,N}} dz$$

Note that the integrand is a meromorphic function with poles $z_{i,N} = \frac{1}{r_{i,N}} \geq 1,\ i = 1, \ldots, N$.

Hence.

$$Z_N = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{1}{z} \exp(NS_N(z))dz \tag{14}$$

$$m_{i,N} = \frac{r_{i,N} \frac{\partial Z_N}{\partial r_{i,N}}}{Z_N} = \frac{\frac{1}{2\pi i} \int\limits_{\gamma} \frac{zr_{i,N}}{1-zr_{i,N}} \frac{1}{z^{N\lambda(1+\epsilon_N)+1}} \prod_{i=1}^{N} \frac{1}{1-zr_{i,N}} dz}{Z_N} =$$

$$\frac{\frac{1}{2\pi i} \int\limits_{\gamma} \frac{r_{i,N}}{1-zr_{i,N}} \exp(NS_N(z))dz}{Z_N} \tag{15}$$

For the joint distributions we have:

$$P_N\{\xi_1 = n_1, \xi_2 = n_2, \ldots, \xi_K = n_K\} = \frac{r_{1,N}^{n_1} r_{2,N}^{n_2} \ldots r_{K,N}^{n_K} Z^K_{M-\sum\limits_{i=1}^{K} n_i,N}}{Z_N}$$

where

$$Z_{L,N}^K = \sum_{n_{K+1}+n_{K+2}+\ldots+n_N=L} \prod_{i=K+1}^{N} r_{i,N}^{n_i}$$

We can again consider the characteristic function of the sequence:

$$\{Z_{L,N}^K\}_{L=0}^{\infty} : \quad \Xi_N^K(z) = \sum_{L=0}^{\infty} z^L Z_{L,N}^K = \prod_{i=K+1}^{N} \frac{1}{1 - zr_{i,N}}, |z| < 1$$

Again, we can derive, that

$$Z_{L,N}^{K} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z^{L+1}} \prod_{i=K+1}^{N} \frac{1}{1 - zr_{i,N}} dz$$

Therefore,

$$P_N\{\xi_1 = n_1, \xi_2 = n_2, \ldots, \xi_K = n_K\} =$$

$$\frac{r_{1,N}^{n_1} r_{2,N}^{n_2} \cdots r_{K,N}^{n_K}}{2\pi i Z_N} \int_{\gamma} \frac{1}{z^{M-\sum_{i=1}^{K} n_i+1}} \prod_{i=K+1}^{N} \frac{1}{1 - zr_{i,N}} dz =$$

$$\frac{1}{Z_N} \frac{1}{2\pi i} \int_{\gamma} \frac{\prod_{i=1}^{K} (1 - zr_{i,N})(zr_{i,N})^{n_i}}{z^{M+1}} \prod_{i=K+1}^{N} \frac{1}{1 - zr_{i,N}} dz =$$

$$\frac{1}{Z_N} \frac{1}{2\pi i} \int_{\gamma} \frac{\prod_{i=1}^{K} (1 - zr_{i,N})(zr_{i,N})^{n_i}}{z} \exp(N S_N(z)) dz \qquad (16)$$

Lemma 3. All roots of the equation

$$\frac{\partial S_N(z)}{\partial z} = 0 \qquad (17)$$

are real and positive. The least of the roots $z_{0,N} < 1$.
Proof.
Rewriting (17) we obtain:

$$0 = \frac{\partial S_N(z)}{\partial z} = -\frac{\lambda(1+\epsilon_N)}{z} + \frac{1}{N} \sum_{i=1}^{N} \frac{r_{i,N}}{1 - zr_{i,N}} \qquad (18)$$

Let $z = r\exp(i\phi), r > 0$. Then for the imaginary part of (18) we get the necessary condition:

$$\frac{\lambda(1+\epsilon_N)}{r} \sin\phi + \frac{1}{N} \sum_{i=1}^{N} \frac{r_{i,N}^2 \sin\phi}{1 - 2r_{i,N} r\cos\phi + r_{i,N}^2 r^2} = 0$$

which cannot hold true if $\sin\phi \neq 0$. Hence, the roots of (18) are real. But both terms of the sum in (18) become positive if $z < 0$, which proves that the roots are positive.

Let's now look how $\frac{\partial S_N(z)}{\partial z}$ behaves when z lies on the positive part of the real axis.

At the critical points (the simple poles of $\frac{\partial S_N(z)}{\partial z}$) $z = 0, \frac{1}{r_{1,N}}, \dots, \frac{1}{r_{N,N}}$ there are vertical asymptotes for $\frac{\partial S_N(z)}{\partial z}, z \in \mathbf{R}_+$. Its number is greater by 1 than the number of different $\{r_{i,N}\}, i = 1, \dots, N$. In the intervals between any neighbouring asymptotes the expression in (18) is strictly monotone as all its addends are strictly monotone, and it has limits of different signs when z approaches the bounds of an interval. Hence, the derivative has only one zero in any of these intervals. It is easy to see that it has no zeroes to the right of these intervals.

Let us denote $z_{0,N}$ the least of the roots of (17). Then obviously $0 < z_{0,N} < 1$ due to (2). ■

Lemma 4. For any fixed $\lambda, \lambda > 0$ there exists $\lim_{N\to\infty} z_{0,N} =: z_0 = z_0(\lambda) > 0$. It has the following properties:

(i) If $\lambda < \lambda_{cr}$, then $z_0(\lambda)$ is the root of the equation

$$h(z) = \frac{\lambda}{z}, \tag{19}$$

$z_0(\lambda)$ is strictly monotone and $0 < z_0(\lambda) < 1$.

(ii) $\lim_{\lambda\to\lambda_{cr}-} z_0(\lambda) = 1$.

(iii) If $\lambda \geq \lambda_{cr}$, then $z_0 = 1$.

Proof. a) $\lambda < \lambda_{cr}$. Note that $zh(z)$ is strictly monotone on $[0,1)$. Since $\lambda < \lim_{z\to 1-} h(z) = \lim_{z\to 1-} zh(z)$, the equation (19) obviously has a unique solution $z_0 = z_0(\lambda) \in (0,1)$, strictly monotone in λ. Hence, there exists $\lim_{\lambda\to\lambda_0-} z_0(\lambda)$. It is easy to see that it equals 1.

Let us show that $\lim_{N\to\infty} z_{0,N} = z_0$.

For any N let $g_N(z) := \frac{\partial S_N(z)}{\partial z}$, it is a monotonically increasing (see (18)) continuous function on $(0,1)$; $g(z) := h(z) - \frac{\lambda}{z}$ is also a monotonically increasing function on the same set of z, and $\forall z, z \in (0,1)$ $g_N(z) \to g(z), N \to \infty$.

Take any small interval around z_0: $(z_0 - \epsilon, z_0 + \epsilon)$. Then $g(z)$ will be separated from zero at the end points of that interval, and will have different signs. But for N large $g_N(z)$ will have the same signs at the respective points. Hence, the interval will contain $z_{0,N}$.

b) Let $\lambda \geq \lambda_{cr}$

It is obvious that $\limsup_{N\to\infty} z_{0,N} \le 1$ (since $z_{0,N} \le \frac{1}{r_{i,N}}, 1 \le i \le N$.)

For arbitrary $\epsilon > 0$ sufficiently small $g(1-\epsilon) < \lim_{z\to 1-} g(z) \le 0$. For N sufficiently large, $g_N(1-\epsilon)$ will also be negative for $z, 0 < z \le 1-\epsilon$, i.e. $z_{0,N} \in (1-\epsilon, 1)$. ∎

Lemma 5. $ReS_N(r\exp(i\phi))$ monotonically decreases when $\phi \in (0,\pi)$ and monotonically increases when $\phi \in (\pi, 2\pi)$, $r > 0$.

Proof. Indeed,

$$exp(NReS_N(z)) = \frac{1}{|z|^{N\lambda(1+\epsilon_N)}} \prod_{i=1}^{N} \frac{1}{|1 - zr_{i,N}|} \tag{20}$$

The first factor in the r.h. side of (20) has constant value on the curve $r\exp(i\phi)$. Let $d_N(\phi) = |1 - rr_{i,N}\exp(i\phi)|^2$. Then $d_N'(\phi) = 2rr_{i,N}\sin\phi$, so the denominators in (20) increase when $0 < \phi < \pi$ and decrease when $\pi < \phi < 2\pi$. ∎

4 Proof. The Regular Case.

We will prove first a slightly more general theorem, from which **Theorem 1** will follow.

In the sequel, we shall denote by $U_d(v) := \{z \in \mathbf{C} : |z - v| < d\}$.

Let $\gamma = \{z \in \mathbf{C} : |z| = z_0(\lambda)\}$.

Theorem 4. Assume

I) $\lambda < \lambda_{cr}$;

II) $f(\theta, z), \theta \in \Theta$ is a family of functions, holomorphic in the ring $\{z \in \mathbf{C} : z_0(\lambda) - \sigma_0 < |z| < z_0(\lambda) + \sigma_0\}$ for some $\sigma_0 > 0$, and uniformly bounded in that ring, and such that for a given ϵ sufficiently small, $\epsilon > 0$ there exists such $\sigma_u, \sigma_u > 0$, that $\left|\frac{f(\theta,z)}{f_u} - 1\right| < \epsilon$, $z \in U_{2\sigma_u}(z_0)$, $\theta \in \Theta$ for some constant $f_u \in \mathbf{R}$, $f_u \ne 0$.

Then there exists such N_ϵ that for any $N, N > N_\epsilon$, $\theta \in \Theta$

$$\frac{1}{2\pi i}\int_{\gamma} f(\theta, z) exp(NS_N(z))dz =$$

$$\frac{f_u}{\sqrt{2\pi N S''(z_0)}} exp(NS_N(z_{0,N}))(1+\zeta_N), \zeta_N \in \mathbf{R}, |\zeta_N| < 25\epsilon$$

Proof. Essentially, we will apply to the integral along γ the techniques of the "saddle point" method. But the way $S_N(z)$ depends on N produces certain specific features in the application of this method, and we were not able to find a suitable result in the literature.

Proposition 1. For any $\sigma, \sigma > 0$, there exists such N' : for any N, $N > N'$ $U_\sigma(z_{0,N}) \subset U_{2\sigma}(z_0)$.

Proof obviously follows from the convergence of $z_{0,N}$ to z_0 (see **Lemma 4**).

Lemma 6. For any given $\epsilon, \epsilon > 0$ sufficiently small and any given $\sigma_u, \sigma_u > 0$ there exist such N' and σ_r, $\sigma_u \geq \sigma_r > 0$ that the following seven properties hold true for $N > N'$, and for $z \in U_{\sigma_r}(z_{0,N})$, where applicable:

a) $S_N(z)$ can be expanded into a power series in $U_{\sigma_r}(z_{0,N})$:

$$S_N(z) = S_N(z_{0,N}) + \frac{(z - z_{0,N})^2}{2} S''_N(z_{0,N})(1 + R_{2,N}(z))$$

where

$$R_{2,N}(z) = \frac{2}{S''_N(z_{0,N})}[(z - z_{0,N})\frac{S'''_N(z_{0,N})}{3!} + (z - z_{0,N})^2 \frac{S^{(iv)}_N(z_{0,N})}{4!} + \ldots];$$

b) $S''_N(z_{0,N}) > F$ for some constant $F, F > 0$;

c) $\left|\frac{\sqrt{S''(z_0)}}{\sqrt{S''_N(z)}} - 1\right| < \epsilon$;

d) $|R_{2,N}(z)| < \epsilon$;

e) $\left|(z - z_{0,N})R'_{2,N}(z)\right| < \epsilon$;

f) $\left|\arctan \frac{ImR_{2,N}(z)}{1+ReR_{2,N}(z)}\right| < \frac{\pi}{4}$.

g) $z_{0,N} + \sigma_r < z_0 + \sigma_0, z_{0,N} - \sigma_r > r_0$;

Proof. We can prove separately each of the properties a)-g), with appropriate $N'_a, \ldots, N'_g$ and $\sigma_a, \ldots, \sigma_g$. Then by choosing $\sigma_r = \min(\sigma_a, \ldots, \sigma_g)$ and $N' = \max(N'_a, \ldots, N'_g)$ we will ensure that they hold true simultaneously.

To ensure a) in $U_{\sigma_a}(z_{0,N})$ it is sufficient to require that $0 < \sigma_a < \frac{1-z_0}{2}$, for $\lambda < \lambda_{cr}$ implies $z_0 < 1$. Then any $S_N(z)$ can be expanded into the power series when $z \in U_{\sigma_a}(z_{0,N})$:

$$S_N(z) = S_N(z_{0,N}) + \frac{(z-z_{0,N})^2}{2!} S_N''(z_{0,N}) + \frac{(z-z_{0,N})^3}{3!} S_N'''(z_{0,N}) + \ldots,$$

all $S_N^{(n)}(z_{0,N}) \in \mathbf{R}$, $S_N^{(n)}(z) = (-1)^n (n-1)! \frac{\lambda(1+\epsilon_N)}{z^n} + (n-1)! \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(\frac{1}{r_{i,N}} - z)^n}$

Remark. It is easy to see, that there exists such N_a', that $z_{0,N} \in U_{\frac{\sigma_a}{2}}(z_0)$, for $N > N_a'$, and the above expansions become valid in $U_{\frac{\sigma_a}{2}}(z_0)$.

Note that

$$\forall n \; \exists \lim_{N\to\infty} S_N^{(n)}(z) =:$$

$$S^{(n)}(z) = (-1)^n (n-1)! \frac{\lambda}{z^n} + (n-1)! \int_0^1 \frac{1}{(\frac{1}{r_{i,N}} - z)^n} dI(r) \qquad (21)$$

uniformly on every compact outside $[1, +\infty)$, and hence for $z \in \overline{U}_{\frac{\sigma_a}{2}}(z_0)$.

Hence,

$$\forall n \exists \lim_{N\to\infty} S_N^{(n)}(z_{0,N}) = S^{(n)}(z_0)$$

and

$$S''(z_0) > 0 \qquad (21')$$

This obviously implies b).

Using uniform convergence of $S_N''(z)$ in $U_{\frac{\sigma_a}{2}}(z_0)$, $N \to \infty$ and of $(21')$, we can choose σ_c to be small enough, so that $\left| \frac{\sqrt{S''(z_0)}}{\sqrt{S_N''(z)}} - 1 \right| < \frac{\epsilon}{2}$, $z \in U_{2\sigma_c}(z_0)$. From **Proposition 1** it follows that there exists N_c' such that c) is also true if $N > N_c'$.

From (21) it follows that $\exists N_d'$, $\exists M$, $M > 0 : |S_N(z)| < M, z = z_{0,N} + \mu \exp(i\phi), \phi \in [0, 2\pi)$, for some small μ, $N > N_d'$. Then $\left| \frac{S_N^{(n)}(z_{0,N})}{n!} \right| \leq \frac{M}{\mu^n}$. Let $|z - z_{0,N}| \leq \sigma < \mu$. Then

$$\left| \frac{S_N'''(z_{0,N})}{3!} (z - z_{0,N})^3 + \ldots \right| \leq$$

$$M\left(\left| \frac{(z-z_{0,N})^3}{\mu^3} \right| + \left| \frac{(z-z_{0,N})^4}{\mu^4} \right| + \ldots \right) \leq M \frac{\sigma^3}{\mu^3} \frac{1}{1 - \frac{\sigma}{\mu}}$$

and

$$\frac{M\frac{\sigma^3}{\mu^3}\frac{1}{1-\frac{\sigma}{\mu}}}{\frac{S''_N(z_{0,N})}{2!}\sigma^2} \longrightarrow 0, \sigma \to 0$$

This entails that in the expansion

$$S_N(z) = S_N(z_{0,N}) + \frac{(z-z_{0,N})^2}{2} S''_N(z_{0,N})(1 + R_{2,N}(z))$$

we have

$$|R_{2,N}(z)| \to 0 \text{ if } |z - z_{0,N}| \to 0 \tag{22}$$

uniformly for all $N > N'_d$, which yields d).

To ensure e) we can just evaluate:

$$\left|(z - z_{0,N})R'_{2,N}(z)\right| =$$

$$\frac{2}{\left|S''_N(z_{0,N})\right|}\left|(z - z_{0,N})\frac{S'''_N(z_{0,N})}{3!} + 2(z - z_{0,N})^2\frac{S^{iv}_N(z_{0,N})}{4!} + \ldots\right| \leq$$

$$\frac{2M}{\left|S''_N(z_{0,N})\right|\lambda^2}\left(\frac{\sigma}{\lambda} + 2\left(\frac{\sigma}{\lambda}\right)^2 + 3\left(\frac{\sigma}{\lambda}\right)^3 + \ldots\right), |z - z_{0,N}| < \sigma \leq \mu, N > N'_d$$

and by choosing appropriate $N'_e \geq N'_d$ and appropriate small σ_e we can provide e).

Property f) follows from d), with some σ_f, N'_f.

It is obvious that the property g) can be easily satisfied too, with some σ_g, N'_g.

So **Lemma 5** is proved.

Now we proceed to the proof of the theorem. From now on, we shall refer to the properties formulated in **Lemma 5** simply by a)-g), the conditions of **Lemma 5** evidently following from the conditions of the theorem.

First, we deform the integration contour γ into the new integration contour γ_N, depending on N, in the following way. There are two level curves of the function $ImS_N(z)$, which pass through $z_{0,N}$: $ImS_N(z) = ImS_N(z_{0,N}) =$

0. We choose the one orthogonal to the real axis of the z-plane. As the only finite singularity points of $S_N(z)$ are $0, \frac{1}{r_{i,N}}, 1 \le i \le N$, this level curve goes out of $U_\sigma(z_{0,N})$, crossing its boundary at two points, symmetrical with respect to the real axis. We denote the intersection of this curve with $U_\sigma(z_{0,N})$ by $\gamma_{\sigma,N}$, and by $z_{1,N}, z_{2,N}$ we denote the ends of $\gamma_{\sigma,N}$. Then we connect $z_{1,N}$ and $z_{2,N}$ by an arc with the center at $z = 0$, which is located outside of $U_\sigma(z_{0,N})$, thus obtaining the integration curve γ_N. Since γ_N lies inside of the domain of analiticity of the integrands by g) and by the conditions of the theorem, it would give the same integral values as those counted along γ.

Now we will evaluate the integral along the part of γ_N outside of $U_\sigma(z_{0,N})$. But $z_{j,N} - z_{0,N} = \sigma \exp(i\phi_j)$, $j = 1,2$ and $Im((z_j - z_{0,N})^2(1 + R_{2,N}(z_j))) = 0$, $j = 1,2$, which yields $\tan(2\phi_j) = -\frac{ImR_{2,N}(z_j)}{1+ReR_{2,N}(z_j)}$. From b) and (22) and f) we can derive that $\exists C$, $C > 0$ such that $\forall N, N > N'$ $ReS_N(z_{j,N}) < ReS_N(z_{0,N}) - C$, $j = 1,2$.

If we take into account the uniform boundedness of $\{f(\theta,z)\}_{\theta\in\Theta}$, we can obtain:

$$\left| \frac{1}{2\pi i} \int\limits_{\gamma_N \setminus \gamma_{\sigma,N}} f(\theta,z) exp(NS_N(z)) dz \right| \le G \frac{exp(NS_N(z_{0,N}))}{exp(CN)} \tag{23}$$

for some $C > 0, G > 0$ and any $N > N'$, $\theta \in \Theta$.

Now we proceed to evaluation of the main part (along $\gamma_{\sigma,N}$) of the integral.

Let us introduce

$$\psi_N(z) = (z - z_{0,N})\sqrt{S''_N(z_{0,N})}\sqrt{1 + R_{2,N}(z)}, z \in U_{\sigma_r}(z_{0,N}),$$

where the positive branch of the square root is used. It is obvious that $\psi'_N(z_{0,N}) = \sqrt{S''_N(z_{0,N})}$ and that $-i\psi_N(z) \in \mathbf{R}, z \in \gamma_{\sigma,N}$.

From d) it follows that $\psi_N(z)$ in $U_\sigma(z_{0,N})$ has only one zero at $z_{0,N}$, and that $\frac{1}{2\pi i} \int\limits_{\gamma_s} \frac{\psi'_N(z)}{\psi_N(z)} dz = 1$, where $\gamma_s = \{z : z = z_{0,N} + \frac{\sigma_r}{2} exp(i\phi), 0 \le \phi < 2\pi\}$

Let $\delta = \inf\limits_{|z - z_{0,N}| = \frac{\sigma_r}{2}, N > N'} |\psi_N(z)|$. Note that $\delta > 0$ due to c). And let $\sigma = \min(\frac{\sigma_r}{2}, \inf\limits_{N>N'} \sup\limits_{\sigma'>0}\{\sigma' : \sigma' exp(i\phi) \in \psi_N^{-1}(U_\delta(0)), 0 \le \phi < 2\pi\})$ Note also that $\sigma > 0$.

Then for any $z' \in U_\sigma(z_{0,N})$ $\frac{1}{2\pi i}\int\limits_{\gamma_s} \frac{\psi'_N(z)}{\psi_N(z)-\psi_N(z')}dz = 1$, which, due to the Argument Principle, implies that there are no other points $z'' \in U_\sigma(z_{0,N})$: $\psi_N(z'') = \psi_N(z')$.

Hence, $\psi_N(z)$ performs analytical isomorphism of $U_\sigma(z_{0,N})$, $N > N'$ into some neighbourhood of zero.

Put $z = \psi_N^{-1}(w), w \in \psi_N(U_\sigma(z_{0,N}))$

Then

$$\frac{1}{2\pi i}\int\limits_{\gamma_{\sigma,N}} f(\theta, z)exp(NS_N(z))dz =$$

$$\frac{1}{2\pi i}exp(NS_N(z_{0,N}))\int\limits_{\psi_N(\gamma_{\sigma,N})} f(\theta, \psi_N^{-1}(w))exp(\frac{Nw^2}{2})\frac{dw}{\psi'_N(\psi_N^{-1}(w))} =$$

$$\frac{1}{2\pi}exp(NS_N(z_{0,N}))\int\limits_{-i\psi_N(\gamma_{\sigma,N})} exp(-\frac{Nu^2}{2})\frac{f(\theta, z_N(u))du}{\psi'_N(z_N(u))}$$

where $w = iu$ and $z_N(u) := \psi_N^{-1}(iu)$. Note again that $u \in \mathbf{R}$.

But $\frac{1}{\psi'_N(z_N(u))} = \frac{\sqrt{1+R_{2,N}(z_N(u))}}{\sqrt{S''_N(z_{0,N})}[1+R_{2,N}(z_N(u))+\frac{(z_N(u)-z_{0,N})}{2}R'_{2,N}(z_N(u))]}$

From c),d),e) together with the condition II) of the theorem it follows, that $\frac{f(\theta,z_N(u))}{\psi'_N(z_N(u))} = \frac{f_u}{\sqrt{S''(z_0)}}(1 + \zeta_{1,N}(u)), \zeta_{1,N}(u) \in \mathbf{C}$, $|\zeta_{1,N}(u)| < 10\epsilon$, $iu \in \psi_N(U_\sigma(z_{0,N}))$, $\theta \in \Theta$ (as ϵ is small).

So, for $N > N'$, $\theta \in \Theta$

$$\int\limits_{-i\psi_N(\gamma_{\sigma,N})} exp(-\frac{Nu^2}{2})\frac{f(\theta, z_N(u))}{\psi'_N(z_N(u))}du =$$

$$\int\limits_{-i\psi_N(\gamma_{\sigma,N})} exp(-\frac{Nu^2}{2})du\frac{f_u}{\sqrt{S''(z_0)}}(1 + \zeta_{2,N}), \zeta_{2,N} \in \mathbf{C},\ |\zeta_{2,N}| < 15\epsilon$$

But $-i\psi_N(z_{j,N}) = (z_{j,N} - z_{0,N})\sqrt{S''_N(z_{0,N})}\sqrt{1 + R_{2,N}(z_{j,N})}$, $j = 1, 2$, and due to c), d) and f), for small ϵ and for $N > N'$

$$\frac{\sigma}{2}\sqrt{S''(z_0)} < \left|\frac{\psi_N(z_{j,N})}{i}\right| < 2\sigma\sqrt{S''(z_0)},\ j = 1, 2$$

and therefore

$$\int_{-\sigma\frac{D}{2}}^{\sigma\frac{D}{2}} exp(-\frac{Nu^2}{2})du < \int_{\frac{\psi_N(\gamma_{\sigma,N})}{i}} exp(-\frac{Nu^2}{2})du < \int_{-2\sigma D}^{2\sigma D} exp(-\frac{Nu^2}{2})du$$

where $D = \sqrt{S''(z_0)}$.

But both of the bounding integrals above $\sim \sqrt{\frac{2\pi}{N}}, N \to \infty$, hence there exists $N'', N'' \geq N'$, such that for any $N, N > N''$

$$\int_{-i\psi_N(\gamma_{\sigma,N})} exp(-\frac{Nu^2}{2})du = \sqrt{\frac{2\pi}{N}}(1 + \zeta_{3,N}), \zeta_{3,N} \in \mathbf{R}, \ |\zeta_{3,N}| < \epsilon, \ N > N''$$

and therefore

$$\int_{-i\psi_N(\gamma_{\sigma,N})} exp(-\frac{Nu^2}{2})\frac{f(\theta, z_N(u))}{\psi_N'(z_N(u))}du =$$

$$\sqrt{\frac{2\pi}{N}}\frac{f_u}{\sqrt{S''(z_0)}}(1 + \zeta_{4,N}), \zeta_{4,N} \in \mathbf{C}, \ |\zeta_{4,N}| < 20\epsilon, \ N > N'', \ \theta \in \Theta$$

From this estimate together with (23) it follows that the integrals along $\gamma_N \backslash \gamma_{\sigma,N}$ are exponentially small w.r.t. those along $\gamma_{\sigma,N}$, i.e. **Theorem 4** is proved. ∎

Now we will prove the statements of **Theorem 1**.

By letting $f(\theta, z) = f(z) = \frac{1}{z}, f_u = \frac{1}{z_0}$ we can see that for any small $\epsilon, \epsilon > 0$ we can find such small $\sigma_u, \sigma_u > 0$, that the condition II) of **Theorem 4** is satisfied.

Therefore, for any ϵ sufficiently small, $\epsilon > 0 \ \exists N_\epsilon : \forall N, N > N_\epsilon$

$$Z_N = \frac{1}{\sqrt{2\pi N S''(z_0)}}\frac{1}{z_0}\exp(NS_N(z_{0,N}))(1 + \zeta_N), |\zeta_N| < 25\epsilon, N > N_\epsilon$$

which means that the statement (i) of **Theorem 1** holds true.

To prove (ii) we fix some small $\epsilon, \epsilon > 0$ and assume $f(\theta, z) = \frac{A}{z} + \frac{\theta}{1 - z\theta}$, $\theta \in \Theta = [0,1]$, $A > 0$, $f_u = \frac{A}{z_0}$. Then $\frac{f(\theta,z)}{f_u} - 1 = \frac{z_0}{z} + \frac{z_0\theta}{1-z\theta}\frac{1}{A}$. By letting $\sigma_u = \frac{\epsilon}{8}$ and $A = \frac{16 z_0}{(1-z_0)\epsilon}$ we can ensure the condition II) of **Theorem 4.**

Then we obtain that $\exists N_\epsilon$: $\forall N$, $N > N_\epsilon$, $\theta \in \Theta$

$$\frac{1}{2\pi i}\int_\gamma \left(\frac{A}{z} + \frac{\theta}{1 - z\theta}\right) \exp(NS_N(z))dz =$$

$$\frac{1}{\sqrt{2\pi N S''(z_0)}}\frac{A}{z_0} exp(NS_N(z_{0,N}))(1 + \zeta_N), \zeta_N \in \mathbf{R}, |\zeta_N| < 25\epsilon$$

I.e.

$$A + \frac{\frac{1}{2\pi i}\int_\gamma \frac{\theta}{1-z\theta}\exp(NS_N(z))dz}{Z_N} =$$

$$A\frac{\frac{1}{\sqrt{2\pi N S''(z_0)}}\frac{1}{z_0}\exp(NS_N(z_{0,N}))}{Z_N}(1 + \zeta_N), \ N > N_\epsilon, \ \theta \in \Theta$$

Applying the statement (i) to the r.h. side of this equality, we can obtain that $\exists N_\epsilon^{'} : \forall N, N > N_\epsilon^{'}$

$$A + \frac{\frac{1}{2\pi i}\int_\gamma \frac{\theta}{1-z\theta}\exp(NS_N(z))dz}{Z_N} = A(1 + \zeta_N^{'}), \ \left|\zeta_N^{'}\right| < 30\epsilon$$

But

$$m_{i,N} = \frac{\frac{1}{2\pi i}\int_\gamma f_{i,N}(z)\exp(NS_N(z))dz}{Z_N}$$

where $f_{i,N}(z) = \frac{r_{i,N}}{1 - z r_{i,N}}$.

Since $r_{i,N} \in [0,1]$, $A + f_{i,N}(z)$ pertains to our family of $\{f(\theta, z)\}_{\theta\in\Theta}$ with $\theta = r_{i,N}$, and the above evaluation implies the following evaluation:

$$|m_{i,N}| < 30A\epsilon, \ N > N_\epsilon^{'}, \ 1 \le i \le N$$

Hence, the statement (ii) is also proved.

To prove (iv) we can assume for some fixed i

$$f(\theta, z) = f(N, z) = \frac{1}{1 - zr_{i,N}}, \ f_u = \frac{1}{1 - z_0 r_i}$$

Then for any ϵ, $\epsilon > 0$ sufficiently small we can obviously find such N_ϵ^0 and such σ_u, $\sigma_u > 0$, that for $\Theta = \{N_\epsilon^0, N_\epsilon^0 + 1, \ldots\}$ the condition II) of **Theorem 4** is satisfied.

Then we obtain:

$$\frac{\partial Z_N}{\partial r_{i,N}} = \frac{1}{1 - z_0 r_i} \frac{1}{\sqrt{2\pi N S''(z_0)}} \exp(N S_N(z_{0,N}))(1 + \zeta_N),$$

$$|\zeta_N| < 25\epsilon, \ N > \max(N_\epsilon^0, N_\epsilon)$$

Applying (i), we can see, that $\frac{\frac{\partial Z_N}{\partial r_{i,N}}}{Z_N} \to \frac{z_0}{1 - z_0 r_i}$, $N \to \infty$. And as $r_{i,N} \to r_i$, $N \to \infty$, we obtain in a straightforward manner the wanted result.

The statement (iii) can obviously be proved in exactly the same manner.

■

5 Proof. The Condensation Case.

Proof of **Theorem 2**.

Note that by **Lemma 4** the value $\frac{z_0(\lambda)}{1 - z_0(\lambda)}$ is strictly and infinitely increasing when $\lambda \nearrow \lambda_{cr}$. Hence, for any $m, m > 0$ we can choose such $\lambda' = \lambda'(m) < \lambda_{cr}$, that $\frac{z_0(\lambda')}{1 - z_0(\lambda')} = m + 1$.

Note, that instead of tracking the varying index $i(N)$ of the node, where $r_{i(N),N} = 1$, we can assume, without loss of generality, that $i(N) = 1$, i.e. $r_{1,N} = 1$. Indeed, the value of the expression for $m_{i,N}$ does not depend on i, but only on the value of $r_{i,N}$.

With $M'(N) = [\lambda' N]$ ($[\ldots]$ denotes the integer part of $(\ldots)$) the assumptions of **Theorem 1** hold true, and therefore exists $N' : \forall N > N'$ $m_{1,M'(N),N} > \frac{z_0(\lambda')}{1 - z_0(\lambda')} - 1 = m$.

But $\frac{M}{N} \to \lambda \geq \lambda_{cr} > \lambda'$, hence $\exists N'' : \ \forall N, N > N'', M(N) = N\lambda(1 + \epsilon_N) \geq M'(N)$. Therefore, by **Lemma 2**, $\forall N, N > \max(N', N'')$ $m_{1,N} = m_{1,M(N),N} \geq m_{1,M',N} > m$.

As m is arbitrary, **Theorem 2** is proved. ■

Proof of **Theorem 3**.

It is easy to see that under the assumptions of the theorem the measure $I(r)$ will be concentrated on the segment $[0, B]$, and $h(z)$ will have finite limit at the point 1 because $B < 1$. Thus, (0) holds true.

The case of $\lambda < \lambda_{cr}$ was treated in **Theorem 1.**

Proceeding to the case of $\lambda > \lambda_{cr}$, we can note that under the assumptions of the theorem it represents just one of the possibilities for the general case of $z_0 = 1$, which appears to be more difficult to study, than the regular one, due to unknown rate of convergence of the network parameters. But in this particular situation we can try to evaluate the integral for Z_N by taking into account the residue at $z = 1$, the new integration curve crossing the real axis to the right of that point. Indeed, the equation (19) is not satisfied with $z \leq 1$ if $\lambda > \lambda_{cr}$, and 1 is separated from other singularity points.

Let us rewrite:

$$Z_N = \frac{1}{2\pi i}\int_\gamma \frac{1}{z} exp(NS_N(z))dz = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\frac{1}{1-z} exp(NS^*_N(z))dz \qquad (24)$$

where

$$S^*_N(z) = -\lambda(1+\epsilon_N)\ln z - \frac{1}{N}\sum_{i=2}^{N}\ln(1 - zr_{i,N}) \qquad (25)$$

Let $z_{1,N}$ be the leftmost point, where

$$\frac{\partial S^*_N(z)}{\partial z} = 0.$$

Note that $\forall n\ S^{*(n)}_N(z) \longrightarrow \frac{\partial^n}{\partial z^n}(-\lambda \ln z - \int\limits_0^B \ln(1 - zr)dI(r)),\ N \to \infty$ uniformly on every compact set outside $[\frac{1}{B}, +\infty)$.

It is clear that exists $\lim\limits_{N\to\infty} z_{1,N} =: z_1 \geq 1$. But z_1 is a root of the equation:

$$-\frac{\lambda}{z} + \int\limits_0^B \frac{r}{1-zr}dI(r) = 0$$

Suppose that $z_1 = z_0 = 1$. Then $0 = \lim_{z \to z_0 -} S^{*'}(z) \leq \lim_{z \to z_0 -} S'(z) < 0$ - contradiction. Hence,

$$z_1 > z_0, S^*(z_1) < S^*(z_0), S^{*'}(z_0) < 0 \tag{26}$$

Note also that $z_1 < \frac{1}{B}$ since the measure $I(r)$ density is positive near B. Now we divide the integral for Z_N into a couple of them:

$$Z_N = -\frac{1}{2\pi i} \int_{\gamma'} \frac{1}{z} \frac{1}{1 - zr_1} exp(NS_N^*(z))dz + \frac{1}{2\pi i} \int_{\gamma_N''} \frac{1}{z} \frac{1}{1 - zr_1} exp(NS_N^*(z))dz \tag{27}$$

where γ' is a small circle around $z_0 = 1$, and γ_N'' embraces 0 and z_0 and goes through $z_{1,N}$.

The second integral can be evaluated in the same way as in **Theorem 1**, and we have:

$$\frac{1}{2\pi i} \int_{\gamma_N''} \frac{1}{z} \frac{1}{1 - zr_1} exp(NS_N^*(z))dz \sim \frac{const}{\sqrt{N}} exp(NS_N^*(z_{1,N})), \ N \to \infty \tag{28}$$

The first term, due to the residue formula, yields:

$$-\frac{1}{2\pi i} \int_{\gamma'} \frac{1}{z} \frac{1}{1 - zr_1} exp(NS_N^*(z))dz = exp(NS_N^*(\frac{1}{r_1})) \tag{29}$$

From (26) it follows that the r.h. in (28) is exponentially small with respect to (29), and the asymptotics for Z_N is therefore proved.

Let $i = 1$. We will differentiate the representation (27), taking into account (29):

$$\frac{\partial Z_N}{\partial r_1} = \frac{\partial}{\partial r_1} exp(NS_N^*(\frac{1}{r_1})) + \frac{1}{2\pi i} \int_{\gamma_N''} \frac{1}{z} \frac{zr_1}{1 - zr_1} \frac{1}{1 - zr_1} exp(NS_N^*(z))dz \tag{30}$$

The second integral can be again evaluated as in **Theorem 1**, and it is equivalent to $\frac{const}{\sqrt{N}} exp(NS_N^*(z_{1,N}))$.

The first term of (30) yields:

$$\frac{\partial}{\partial r_1} exp(NS_N^*(\frac{1}{r_1})) \sim \frac{N}{r_1^2}(-S'(1))exp(NS_N^*(1)),\ N \to \infty$$

The second term of (30) is again exponentially small w.r.t. the first one, and therefore

$$m_{1,N} = \frac{r_1 \frac{\partial Z_N}{\partial r_1}}{Z_N} \sim -NS'(1),\ N \to \infty$$

Hence, the statement (i) of the theorem is proved.

Recall that

$$Z_N = exp(NS_N^*(1)) + \frac{1}{2\pi i}\int\limits_{\gamma_N''} \frac{1}{z}\frac{1}{1-z} exp(NS_N^*(z))dz \tag{31}$$

If $i \neq 1$

$$\frac{\partial Z_N}{\partial r_{i,N}} = exp(NS_N^*(1))\frac{1}{1-r_{i,N}} + \frac{1}{2\pi i}\int\limits_{\gamma_N''} \frac{1}{z}\frac{1}{1-z}\frac{z}{1-zr_{i,N}} exp(NS_N^*(z))dz \tag{32}$$

By the reasoning, analogous to that of the proof of **Theorem 1**, it can be shown that the ratio of the second term in (32) and the second term in (31) is uniformly bounded when $N \to \infty$, $0 \le r_{i,N} \le B$. Since the second term in (31) is exponentially small w.r.t. the first one, we can see, firstly, that $m_{i,N} = \frac{r_{i,N}\frac{\partial Z_N}{\partial r_{i,N}}}{Z_N}$ are uniformly bounded in $i, i > 1$ and N, and, secondly, that $m_{i,N} \to \frac{r_i}{1-r_i}$, $N \to \infty$, if $r_{i,N} \to r_i$, $N \to \infty$, $i > 1$.

Therefore, **Theorem 3** is completely proved. ■

6 Remarks and examples.

The natural question arises: which measures I_N are possible? First, give trivial remarks:

1. For any J_N, P_N we can choose $\mu_i(N)$ so that multiplication on $\mu_i(N)$ gives any I_N we like; so any limiting measure can apear on $[0, 1]$.

2. Even if $\mu_i(N)$ are fixed, e.g. $\mu_i(N) \equiv 1$, then for any positive vector ρ_N there exists a stochastic routing matrix P_N such that $\rho_N P_N = \rho_N$ holds. It is sufficient to define

$$p_{ij,N} = \frac{\rho_{j,N}}{\sum\limits_{k=1}^{N} \rho_{k,N}}, \; i,j = 1,\dots,N$$

3. We can get the same results as in 2. even for much more restricted class of topologies and interactions. An example for the circle topology is:

Lemma 7. Let N be odd and the vector ρ_N be given such that

$$\rho_{1,N} \leq \rho_{2,N} \leq \dots \leq \rho_{N,N} \leq 2\rho_{1,N},$$

then there exists a stochastic routing matrix P_N satisfying $\rho_N P_N = \rho_N$ and also:

(i) $p_{ii,N} = 0$ for all i;

(ii) $p_{ij,N} = 0$ if $|i-j| > 1 (mod\ N)$.

We skip the proof.

4. If there exists a symmetry group G_N acting transitively and one-to-one on J_N so that

$$p_{ij} = p_{g(i)g(j)}$$

for all $1 \leq i,j \leq N$ and $g \in G_N$ then all ρ_i^N are equal and I_N is the point measure at 1. Examples: lattice on the torus with translation invariant probabilities, completely symmetric networks.

At the end we indicate one problem. It could be natural to define an "infinite" closed Jackson network in the following way. Let us take a countable Markov chain L with the state space S and transition probabilities p_{ij}. For any finite $\Lambda \subset S$ define the finite Markov chain S_Λ by truncating the stochastic matrix in some way, e.g. put

$$p_{ij}^{\Lambda} = \begin{cases} p_{ij}, & i,j \in \Lambda, i \neq j \\ p_{ii} + \sum\limits_{j \in S \setminus \Lambda} p_{ij}, & i = j \end{cases}$$

And suppose that for any $i \in S$ the service rate μ_i is given. Then we define a closed Jackson network J_Λ with routing probabilities p_{ij}^{Λ}, rates μ_i and $M_\Lambda = [\lambda |\Lambda|]$.

One of the questions is what limiting measures I can appear for various sequences

$$\Lambda_1 \subset \Lambda_2 \subset \ldots \subset \Lambda_N \subset \ldots$$

such that $|\Lambda_N| = N$, $\bigcup \Lambda_N = S$.

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