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Conditions suffisantes du deuxième ordre et croissance quadratique pour des minima non isolés

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Abstract. For standard nonlinear programming problems, the weak second-order sufficient condition is equivalent to the quadratic growth condition as far as the set of minima consists of isolated points and some qualification hypothesis holds. This kind of condition is instrumental in the study of numerical algorithms and sensitivity analysis. The aim of the paper is to study the relations between various types of sufficient conditions and quadratic growth in case when the set of minima may have non isolated points.

Résumé. Pour le problème standard de programmation nonlinéaire, la condition faible suffisante du deuxième ordre est équivalente à la croissance quadratique, pour des minimas isolés et qualifiés; ce type de condition intervient de façon essentielle dans l'étude des algorithmes numériques et l'analyse de sensibilité. Le but de ce papier est l'étude des relations entre différents types de conditions suffisantes et la croissance quadratique quand l'ensemble des minima a des minima non isolés.

Keywords Optimality conditions, Lagrangian, composite functions, transversality, proximal normals, critical cone.

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1 Introduction

The importance of second order sufficient conditions is largely determined by their role in sensitivity analysis and numerical optimization. More attentive analysis of existing proofs [2], [3], [4], [5], [10], [6] show, however, that, at least as far as sensitivity analysis is concerned, what is needed is not a second order sufficient condition as such but rather an estimate of the kind [13]

$$f(x) \geq c + \beta\, dist^2(S, x), \tag{1.1}$$

(f is the cost function, c the value of the problem and S the solution set) which usually follows from the condition. The standard second order sufficient condition for an isolated minimum (e.g. [1], [8] [11]) is equivalent to (1.1) provided the Mangasarian-Fromovitz constrained qualification is valid [3]. But very little has been known so far about sufficient conditions and (1.1) like estimates in situations when the set of solution has a more complicated structure than just a finite collection of isolated points.

This article is an attempt to fill the gap. We establish several sufficient condition, based on second order information, critical cones and proximal normals to the solution set at different levels of generality and simplicity of formulations which imply a general "quadratic growth condition" similar to (1.1). The formulation of the most general of them - we call it the "general sufficient condition" in the paper - seems to be fairly awkward at the first glance. It requires information which is not "intrinsic" in the sense that it cannot be expressed on terms of derivatives of the Lagrangian function and relies upon the existence of a certain "projection" map to the solution set with some special properties. (Although the proofs provide information on possible structure of the map, we cannot offer much practical advice for its construction). What makes us introduce this condition as the basic sufficiency statement is that it is equivalent to the general growth condition under an additional "transversality" assumption which has a simple and natural formulation.

Transversality considerations are also instrumental in describing a (fairly general) structure of solution sets for which a sufficient condition very close to the standard second order sufficient condition can be formulated. They also help to highlight the "bottleneck" at which all the main difficulties caused by non-unicity of solutions are accumulated, namely the critical directions close to the contingent cone to the set of solutions. Much effort has been spent in the article to investigate the behaviour of the problem near such directions. Still some interesting questions remain unsolved.

A big portion of the paper is devoted to discussions on unconstrained optimization of a simple composite function (maximum of a finite collection of smooth functions) and only at the final section we reformulate all the main results for constrained optimization problems, using some simple reduction arguments. An advantage of such an approach (already tested for necessary conditions [8] and sensitivity analysis [10]) is that it allows to get rid of feasibility problems in the course of main arguments.

2 The main results. Statements and discussions

2.1 Notation and terminology

So we begin by considering the function

$$f(x) = \max_{1 \leq i \leq m} f_i(x).$$

The functions f_i **are assumed twice continuously differentiable** from $\mathbb{R}^q$ into $\mathbb{R}$ throughout the paper. We use the following notation and terminology :

$$I(x) = \{i; f_i(x) = f(x)\}$$

the set of **active** indices,

$$\mathcal{L}(\lambda, x) = \sum_{i=1}^{m} \lambda_i f(x)$$

the **Lagrangian** of f,

$$\Omega(x) = \{\lambda = (\lambda_i, \dots, \lambda_m) : \lambda_i \geq 0, \lambda_i = 0 \text{ if } i \notin I(x) \sum_{i=1}^{m} \lambda_i = 1, \sum \lambda_i \nabla f_i(x) = 0\}$$

(where as usual $\nabla f_i(x)$ is the gradient of f_i at x) the set of **Lagrange multipliers** for f at x and

$$\Omega_\delta(x) := \{\lambda = (\lambda_i, \dots, \lambda_m) \; ; \; \lambda_i \geq 0, \; \lambda_i = 0 \text{ if } i \notin I(x); \; \sum_{i=1}^{m} \lambda_i = 1, \; \|\sum \lambda_i \nabla f_i(x)\| \leq \delta\}$$

the set of **Lagrange** δ- multipliers.

We call a point x **stationary** if $\Omega(x) \neq \emptyset$ and δ-stationary if $\Omega_\delta(x) \neq \emptyset$. We set further

$$C(x) = \{h : \nabla f_i(x) h \leq 0, \; \forall i \in I(x)\},$$

the cone of critical vectors of f at x.

In what follows **we fix a compact set** S **of stationary points** of f such that $f(x) \equiv$ const $= c_0$ on S.

Definition 1 A mapping π from a neighborhood U of S onto S will be called a *regular projection* to S if $\pi(x) = x$ for $x \in S$ and there is an $\varepsilon > 0$ such that

$$\varepsilon \|x - \pi(x)\| \leq dist(S, x), \quad x \in U.$$

Given a set $C \subset R^q$ and $x \in C$; we denote by $T_C(x)$ the contingent cone to C at x :

$$T_C(x) = \limsup_{t \to +0} t^{-1}(C - x).$$

Definition 2 Let C, D be sets and $x \in C \cap D$. We say that C and D are *transversal* at x if

$$T_C(x) \cap T_D(x) = \{0\}.$$

Definition 3 We say that a closed set $C \subset \mathbb{R}^k$ is *nice* if for every $x \in C$ there is a neighborhood U of x and a diffeomorphism F of U into $\mathbb{R}^k$ such that $C \cap U$ can be represented as a union of a finite number of (relatively closed) sets C_i which are transversal to each other at x and such that the sets $F(C_i)$ are convex. We shall call the C_i *components* of C at x.

2.2 The basic properties

We say that f satisfies the **quadratic growth condition on** S if

(QGC) there are a $\beta > 0$ and a neighborhood U of S such that

$$f(x) \geq c_0 + \beta dist^2(S, x), \quad \forall x \in U. \tag{2.1}$$

We say that f satisfies the **general second order sufficient condition** on S if

(GSO) for any $\delta > 0$ there are a neighborhood U of S and, regular projection $\pi : U \to S$ and an $\alpha > 0$ such that for $x \in U \backslash S$, denoting $h := x - \pi(x)$:

$$\max_{\lambda \in \Omega_\delta(\pi(x))} [\mathcal{L}_x(\lambda, \pi(x))h + \frac{1}{2}\mathcal{L}_{xx}(\lambda, \pi(x))(h, h)] \geq \alpha \|h\|^2. \tag{2.2}$$

Finally, we say that f satisfies the **transversality condition** on $D \subset \mathbb{R}^q$ if

(TC) for any $i \in I(x)$ either $i \in I(y)$ for all $y \in D$ sufficiently close to x, or D and $\{y : f_i(y) = f_i(x) = c_0\}$ are transversal at x.

2.3 The theorems

Theorem 1 *The following implications hold :*

$$\text{(GSO)} \Rightarrow \text{(QGC)},$$

$$\text{(QGC) \& (TC)} \Rightarrow \text{(GSO)}.$$

Theorem 2 *Assume that*

(i) *S is a nice compact set of stationary points of f and f is constant on S,*

(ii) *f satisfies (TC) on every component of S,*

(iii) *for any $x \in S$ and any $h \in C(x) \backslash T_S(x)$*

$$\liminf_{u \xrightarrow{S} x} \max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, u)(h, h) > 0. \tag{2.3}$$

Then (GSO) holds.

Theorem 3 *If (QGC) holds, then*

$$\max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h, h) \geq \beta dist^2(T_S(x), h), \quad \forall h \in C(x), \quad \forall x \in S,$$

β being the same as on the (QGC). In particular

$$\max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h, h) > 0, \quad \forall x \in S, \quad \forall h \in C(x) \backslash T_S(x). \tag{2.4}$$

2.4 Comments and Corollaries

2.4.1 Strictly speaking, (GSO) is not a second order condition. It holds, for instance, for piecewise linear functions (or, equivalently, for linear program) in which case we actually have a stronger "linear growth condition" [10]. A "pure" second order condition we can distill from Theorem 1 is the following.

Corollary 1 *Assume that the following property holds :*

(GSO_1) there are $\alpha, \beta > 0$, a neighborhood U of S and a regular projection $\pi : U \to S$ such that for $h := x - \pi(x)$, we have

$$\max_{\lambda \in \Omega(\pi(x))} \mathcal{L}_{xx}(\lambda, \pi(x))(h,h) \geq \alpha \|h\|^2$$

whenever $x \in U$ satisfies $f(x) \leq c_0 + \beta dist^2(S,x)$.

Then (QGC) holds.

Proof We observe that the proof of Theorem 1 actually show that the implication (2.2) $\Rightarrow$ (2.1) always holds for any given x. Therefore if

$$f(x) \leq c_0 + \beta . dist^2(S,x)$$

(otherwise (2.1) is trivial), then, as every point of S is stationary and $\Omega(y) \subset \Omega_\delta(y)$,

$$\begin{aligned} &\max_{\lambda \in \Omega_\delta(x)} \left\{ \mathcal{L}_x(\lambda, \pi(x))h + \frac{1}{2}\mathcal{L}_{xx}(\lambda, \pi(x))(h,h) \right\} \\ &\geq \frac{1}{2} \max_{\lambda \in \Omega(\pi(x))} \mathcal{L}_{xx}(\lambda, \pi(x))(h,h) \geq \alpha \|h\|^2, \end{aligned}$$

which is (2.2). □

2.4.2 The main advantage of Theorems 2 and 3 over Theorem 1 is that they are intrinsic, i.e. stated in terms of the original data only, while Theorem 1 requires a foreign object such as a "regular projection". Further intrinsic sufficient criteria which are weaker but easier to verify than that of Theorem 2 can be found in §4.

Here we only observe that the standard second order sufficient condition is an easy corollary of Theorem 2, for the conditions (i) and (ii) of the Theorem are automatically satisfied if S is a finite set, and $T_S(x) = \{0\}$ for any $x \in S$. On the other hand if S is finite and (QGC) is satisfied, then any $x \in S$ is a local minimum of the function $f(x+h) - \alpha\|h\|^2$ for some $\alpha > 0$ and applying the second order necessary condition we finally arrive at the following local characterization of the (QGC) in this case.

Corollary 2 *Let S be a finite set of stationary points of f. Then f satisfies (QGC) on S if and only if*

$$\max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h,h) > 0, \quad \forall x \in S, \quad \forall h \in C(x), \quad h \neq 0.$$

2.4.3 The proof of Theorem 2 on the next section actually shows that the conclusion of the theorem remains valid if we replace (iii) by the following more precise version of the condition :

(iii') condition (2.4) holds and there is an $\varepsilon > 0$ such that (2.3) is valid for all $h \in C(x) \cap (T_S^\varepsilon(x) \backslash T_S(x))$, where

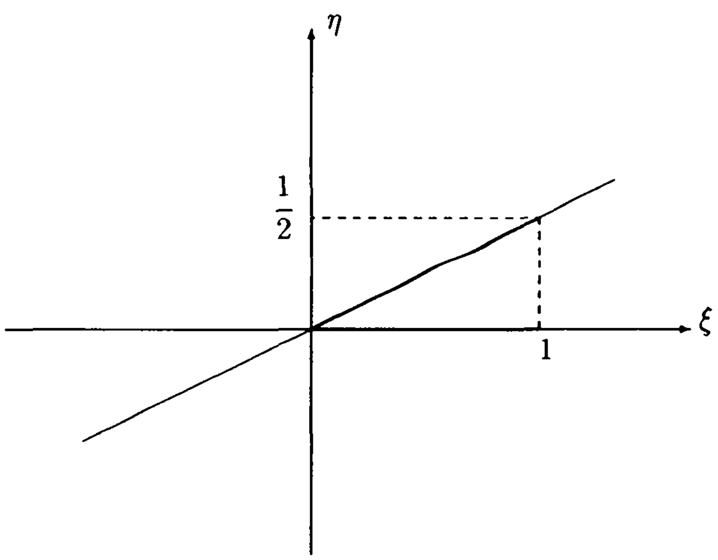
$$T_S^\varepsilon(x) = \{h : dist(T_S(x), h) \leq \varepsilon \|h\|\}$$

We observe further that (2.4) is actually necessary for (QGC) to hold as follows from Theorem 3. It is therefore natural to ask whether it is possible to get rid of (iii) or (iii') altogether and to replace it by (2.4) in Theorem 2. The following example shows that (iii) cannot be a necessary condition for (QGC) even in its modified (iii') form.

Let $X = \mathbb{R}^2$, $x = (\xi, \eta)$, and

$$f(x) = \max\{-\xi\eta + \eta^2, \xi\eta - 2\eta^2, -\xi, 2\eta - \xi, \xi - 1\};$$

$S = \{x = (\xi, \eta) : 0 \leq \xi \leq 1,\ \eta = 0 \text{ or } \eta = \xi/2\}$ (see the picture).



It can be easily verified that f satisfies (QGC) on S. Indeed :

if $\eta \leq 0$, $0 \leq \xi \leq 1$, then $f(x) \geq \eta^2 = dist^2(S, x)$;

if $\eta \leq 0$, $\xi \leq 0$, $\eta \leq \xi/2$, then

$$\begin{aligned} f(x) &\geq \max\{-\xi, (-\eta)(\xi - \eta)\} \leq \max\{\xi^2, |\eta|(|\eta| - |\xi|)\} \\ &\geq \frac{1}{2}[\xi^2 + (\eta^2 - \xi\eta)] \geq \frac{1}{4}(\xi^2 + \eta^2) \geq \frac{1}{4}\|x\|^2; \end{aligned}$$

if $0 \leq \eta \leq \xi/2$, $0 \leq \xi \leq 1$, then

$$f(x) \geq 2\eta\left(\frac{\xi}{2}-\eta\right) \geq 2\min\left\{\eta^2, \left(\frac{\xi}{2}-\eta\right)^2\right\} \geq 2dist^2(S,\alpha)$$

etc. ...

We notice furthermore that $T_S(x) = C(x)$ at any $x \in S$, $x \neq 0$ whereas

$$\begin{array}{rcl} T_S(0) & = & \{h = (\alpha,\beta) : \alpha \geq 0, \quad \beta = 0 \text{ or } \beta = \alpha/2\} \\ C(x) & = & \{h = (\alpha,\beta) : \alpha \geq 0, \quad \eta \leq \alpha/2\}. \end{array}$$

If $x = (\xi, 0) \in S$, $\xi > 0$, then $I(x) = \{1,2\}$ and $\mathcal{L}_x(x) = -\lambda_1\xi + \lambda_2\xi = 0$ which implies $\lambda_1 = \lambda_2 = 1/2$. Therefore for any $h = (\alpha, \beta)$

$$\max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h,h) = -\beta^2/2.$$

Now taking $h = (\alpha,\beta) \in C(0)\backslash T_S(0)$ which means that $\beta \neq 0$ (and $\beta < \alpha/2$) we see that

$$\liminf_{\substack{x\to 0\\ S}} \max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda,x)(h,h) \leq -\frac{1}{2}\beta^2 < 0.$$

Hence (iii) or (iii') are not satisfied, as was to be proved.

On the other hand condition (2.4) alone is not sufficient for (QGC), even if S is smooth. Indeed, consider the cost

$$f(x) = \max(x_1x_2^2, -x_1, -x_2, x_2 - x_1^2, 1 - x_1).$$

Then the minimum value 0 is attained on $S = [0,1] \times \{0\}$. It happens that the set of critical directions is equal to the contigent set of S at all x in S, so that (2.4) is trivially satisfied. However $x(t) := (t,t)$ with $t > 0$, $t \to 0$ satisfies $f(x(t)) = t^3$ and $\text{dist}(x(t), S) = t$, hence (QGC) does not hold. We note that Theorem 2 excludes this case as (TC) is not satisfied.

2.4.4 The other question suggested by Theorem 2 is : are there simply verifiable criteria for conditions (iii) or (iii'). We note that, thanks to Theorem 3, (iii) is satisfied if (2.4) hold and $\lim_{\substack{S\\ u\to x}} \Omega(u)$ exists and is equal to $\Omega(x)$ (we take the limit, lim sup, lim inf of sets in the sense of Painlevé-Kuratowski). However in general it only holds that $\liminf_{\substack{S\\ u\to x}} \Omega(u) \subset \Omega(x)$. For simplicity denote

$$\varphi_x(h) := \max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda,x)(h,h).$$

We shall show that, under the transversality condition, a discontinuity of $x \to \varphi_x(h)$ cannot be caused by functions which are active at x and not necessarily at $u \in S$ close to x, (at least if h satisfies the condition (iii') for ε small enough), but rather by new linear relations for gradient of functions active at x and around x which appear at x and are absent at certain point near x.

Proposition 1 *Assume that $x^n \xrightarrow{S} x$ in such a way that $\|x^n - x\|^{-1}(x^n - x) \to \bar{h}$, $I(x^n)$ are all equal to a certain I, and (TC) holds at x. Then :*

(i) $\lambda_i = 0$ for all $i \in I(x)\backslash I$ such that $\nabla f_i(x) \neq \emptyset$.

(ii) there is an $\varepsilon > 0$ such that

$$\varphi_x(h) = \max_{\substack{\lambda \in \Omega(x) \\ \lambda_i = 0, i \notin I}} \mathcal{L}_{xx}(\lambda, x)(h, h)$$

provided $h \in C(x)\backslash T_S(x)$, $\|h - \bar{h}\| < \varepsilon$.

Proof (i)This is simple. For every $i \in I(x)$ we have

$$\begin{aligned} \nabla f_i(x)h &= \lim t_n^{-1}(f_i(x^n) - f_i(x)), \\ &\leq \lim t_n^{-1}(f_i(x^n) - f_i(u^n)), \\ &= \lim t_n^{-1} \nabla f_i(u^n)(x^n - u^n) = 0, \end{aligned}$$

as $\|x^n - u^n\| = p(t^n)$.

On the other hand, if $j \notin J$ and $\nabla f_i(x) \neq 0$, the equality $\nabla f_j(x)h = 0$ is impossible due to (TC) for $h \in T_S(x)$. Thus $\nabla f_j(x)h < 0$. Finally if $\lambda = (\lambda_1, \ldots, \lambda^n) \in \Omega(x)$, then $\sum \lambda_i \nabla f_i(x)h = 0$. As $\lambda_i \geq 0$ and $\nabla f_i(x)h \leq 0$ we must conclude that $\lambda_j = 0$.

(ii) Take $i_0 \notin I$. If $\nabla f_{i_0}(x) \neq 0$ it follows from point (i) that $\lambda_{i_0} = 0$, $\forall \lambda \in \Omega(x)$, hence the multiplier that attains the max of $\mathcal{L}_{xx}(\lambda, x)(h, h)$ satisfies $\lambda_{i_0} = 0$.

It remains to analyse the case when $\nabla f_{i_0}(x) = 0$. Define

$$J := \{i_0 \in I(x) - I \; ; \; \nabla f_{i_0}(x) = 0\}.$$

We claim that

$$f''_{i_0}(\bar{x})(h, h) \leq 0. \tag{2.5}$$

Define $h^n := \|x^n - x\|^{-1}(x^n - x)$. As $f_{i_0}(x^n) \leq f(x^n) = f_{i_0}(x)$ it follows that

$$0 \geq f_{i_0}(x) + \|x^n - x\| \underbrace{\nabla f_{i_0}(x)h^n}_{0} + \frac{1}{2}\|x^n - \bar{x}\|^2 f_{i_0}(x)(h^n, h^n) + o(\|x^n - \bar{x}\|^2)$$

which proves (2.5).

Now observe that as $\nabla f_{i_0}(x) = 0$, we have

$$\Omega(x) = \bigcup_{\substack{\alpha_i \geq 0 \\ \sum_J \alpha_i \leq 1}} \{\sum_{i \in J} \alpha_i e_{i_0} + (1 - \sum_{i \in J} \alpha)\Omega^J(x)\}$$

with e_i the ith basis vector in $\mathbb{R}^m$ and

$$\Omega^J(x) := \{\lambda \in \Omega(x); \lambda_{i_0} = 0, \quad i_0 \in J\},$$

therefore

$$\varphi_x(h) = \max_{\substack{\alpha_i \geq 0 \\ \sum_J \alpha_i \leq 1}} \{\sum_J \alpha_i f''_{i_0}(x)(h,h) + (1 - \sum_J \alpha_i) \max_{\lambda \in \Omega^{i_0}(x)} \mathcal{L}_{xx}(\lambda, x)(h,h)\}. \tag{2.6}$$

Now by (TC) we know that h is a critical direction if ε is small enough ; from the classical necessary condition we know that $\varphi_x(h) \geq 0$, i.e.

$$\exists \lambda \in \Omega(x); \quad \mathcal{L}_{xx}(\lambda, x)(h,h) \geq 0$$

$$\lambda = \sum_J \alpha_i e_{i_0} + (1 - \sum_J \alpha_i)\hat{\lambda}, \quad \hat{\lambda} \in \Omega^{i_0}(x)$$

Hence by (2.5)

$$\max_{\lambda \in \Omega^J(x)} \mathcal{L}_{xx}(\lambda, x)(h,h) \geq 0$$

and with (2.5), (2.6)

$$\varphi_x(h) = \max_{\lambda \in \Omega^J(x)} \mathcal{L}_{xx}(\lambda, x)(h,h)$$

as was to be proved. □

Thus, "new"active indices at x have no effect on the value of $\varphi_x(h)$.

We observe further that $\Omega(x)$ is a polyhedral convex set, hence is the convex hull of (a finite set of) its extreme points. For any $\lambda \in \Omega(x)$ denote by *supp*(λ) the support of λ, i.e. the set of indices i for which $\lambda_i > 0$. It is an easy matter to verify that λ is an extreme point of $\Omega(x)$ if and only if the vectors $(\partial f_i(x)\partial x_1, \ldots, \partial f_i(x)\partial x_q, -1) \in R^{q+1}$, with $i \in \text{supp}(\lambda)$ are linearly independent (L.I.).

Now as $\lambda \to \mathcal{L}_{xx}(\lambda, x)(h,h)$ is linear it follows that the maximum over $\Omega(x)$ is attained for some extremal element, i.e.

$$\varphi_{x^k}(h) = \mathcal{L}_{xx}(\lambda^k, x)(h,h)$$

with λ^k being an extremal element of $\Omega(x^k)$. Taking if necessary a subsequence we may assume that supp $(\lambda^k) = I$ not depending on k. If $\{\nabla f_i(x)\}_{i \in I}$ is linearly independent, it follows that $\lambda^k \to \bar{\lambda}$, where $\bar{\lambda}$ is the extremal element of $\Omega(x)$ such that supp$(\bar{\lambda}) = I$. Otherwise, λ^k being bounded, we only know that any limit point $\bar{\lambda}$ of $\{\lambda^k\}$ satisfies supp$(\bar{\lambda}) \subset I$.

From the above discussion it follows that, defining

$J^1 := \{I \subset \{1, \ldots, m\}$; support of an extremal point λ^k of $\Omega(x^k)$;
$\{\nabla f_i(\bar{x})\}_{i \in I}$ is linearly independent$\}$,

$J^2 := \{I \subset \{1, \ldots, m\}$; support of an extremal point λ^k of $\Omega^k(x^k)$;
$\{\nabla f_i(\bar{x})\}$ is not linearly independent$\}$,

we get

$$\liminf \varphi_{x^k}(h) \geq \max \left\{ \max_{\substack{I \in J^1 \\ \lambda \in \Omega(x) \\ supp(\lambda) = I}} \mathcal{L}_{xx}(\lambda, x)(h,h), \quad \min_{\substack{I \in J^2 \\ \lambda \in \Omega(x) \\ supp(\lambda) \subset I}} \mathcal{L}_{xx}(\lambda, x)(h,h) \right\}.$$

We have the following corollary

Corollary 3 *If J^2 is empty and (TC) holds, then $\Omega(x) = \lim_{u \xrightarrow{S} x} \Omega(u)$ and (iii) holds.*

Proof Assume that $J := \{i; f_i(x^k) = c_0\}$ constant and define

$$\hat{\Omega}(x) := \{\lambda \in \Omega(x); \lambda_i = 0 \text{ if } i \notin J\}.$$

By Proposition 1 we have

$$0 < \varphi_x(h) = \max_{\lambda \in \hat{\Omega}(x)} \mathcal{L}_{xx}(\lambda, x)(h,h).$$

Now, J^2 being empty it follows that $\hat{\Omega}(x) = \lim \Omega(x^k)$.□

3 Proofs of the Theorems

3.1 Proof of Theorem 1

3.1.1. (GSO) $\Rightarrow$ (QGC) . Fix a $\delta > 0$ and choose a neighborhood U of S and a regular projection $\pi : U \to S$ such that (2.2) holds. Take a $0 < \beta < \alpha$ and a $\sigma > 0$ small enough that

$$|\mathcal{L}(\lambda, x+h) - \mathcal{L}(\lambda, x) - \mathcal{L}_x(\lambda, x)h - \frac{1}{2}\mathcal{L}_{xx}(\lambda, x)(h,h)| < (\alpha - \beta)\|h\|^2$$

provided $x \in S$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$ and $\|h\| < \sigma$. With no loss of generality we may assume that $\|x - \pi(x)\| < \sigma$ for $x \in U$. Then for any such x we have, setting $h = x - \pi(x)$ as in the statement,

$$\begin{aligned} f(x) - c_0 &= f(\pi(x) + h) - f(\pi(x)), \\ &\geq \max_{\lambda \in \Omega_\delta(\pi(x))} \{\mathcal{L}(\lambda, \pi(x) + h) - \mathcal{L}(\lambda, \pi(x))\}, \\ &\geq \max_{\lambda \in \Omega_\delta(\pi(x))} \{\mathcal{L}_x(\lambda, \pi(x))h + \frac{1}{2}\mathcal{L}_{xx}(\lambda, \pi(x))(h,h) - (\alpha - \beta)\|h\|^2\}, \\ &\geq \beta\|h\|^2 \geq \beta . dist(S, x)^2. \end{aligned}$$

3.1.2. Analysis of (TC)

Lemma 1 *Assume (TC) and set for $u \in S$*

$$B(u) := \{v : I(v) \backslash I(u) \neq \emptyset\}.$$

Then there is a $\gamma > 0$ such that $x^n \in S$, $x^n \to x$ implies that

$$dist(B(x^n), x^n) \geq \gamma \|x - x^n\|.$$

Proof Assuming the contrary, we find a sequence of $x^n \in S$ converging to an $x \in S$ and such that

$$dist(B(x^n), x^n) = o(\|x - x^n\|).$$

Set $h^n = \lim \|x^n - x\|^{-1}(x^n - x)$, (which we may assume to exists). It is clear that $h \in T_S(x)$. On the other hand, if $i \in I(x)$ is such that $i \notin I(x^n)$ and $i \in I(x^n + v^n)$ for some v^n with $v^n = o(\|x - x^n\|)$, then h belongs to the contingent cone to $\{y; f_i(y) = f_i(x)\}$ which together with $h \in T_S(x)$ must imply by virtue of (TC) that $\|h\| = 0$. But $\|h\| = 1$ by definition. □

3.1.3. (QGC) & (TC) $\Rightarrow$ (GSO) . Assume the contrary. Then there are a $\delta > 0$ and a sequence of x^n converging to a certain $x \in S$ (as S is compact) such that for any $u \in S$ with $\|u - x^n\| \leq n\ dist(S, x^n)$ we have

$$\max_{\lambda \in \Omega_\delta(u)} (\mathcal{L}_x(\lambda, u)h + \frac{1}{2}\mathcal{L}_{xx}(\lambda, u)(h, h)) \leq \frac{1}{n}\|h\|^2 \tag{3.1}$$

where $h = x^n - u$.

We consider two possible situations.

(A) $dist(S, x^n) = o(\|x - x^n\|)$. Let u^n be a nearest to an element of S ; that is $\|x^n - u^n\| = dist(S, x^n)$; set $h_m = x^n - u^n$. By Lemma 1, $I(x^n) \subset I(u^n)$ for large n, hence by (QGC)

$$\begin{aligned}
\beta\|h^n\|^2 &\leq f(u^n + h^n) - f(u^n) \\
&= \max_{i \in I(u^n)} \{f_i(u^n + h^n) - f_i(u^n)\} \\
&= \max_{\lambda \in \Omega_\infty(u^n)} \{\sum \lambda_i (f_i(x^n + h^n) - f_i(u^n))\} \\
&= \max_{\lambda \in \Omega_\infty(u^n)} \{\mathcal{L}_x(\lambda, u^n)h^n + \frac{1}{2}\mathcal{L}_{xx}(\lambda, u^n)(h^n, h^n)\} + o(\|h^n\|^2).
\end{aligned} \tag{3.2}$$

Set

$$\xi = \max\{\|\mathcal{L}_x(\lambda, x)\|; x \in S, \quad \lambda \in \Omega_\infty(x)\}.$$

Then (note that $\Omega_\infty(x)$ is a standard simplex)

$$\delta\xi^{-1}\Omega_\infty(x) \subset \Omega_\delta(x),$$

so that by (3.1), (3.2)

$$\begin{aligned}\beta\|h^n\|^2 &\le \frac{\xi}{\delta}\max_{\lambda\in\Omega_\delta(x^n)}(\mathcal{L}_x(\lambda,u^n)h^n+\frac{1}{2}\mathcal{L}_{xx}(\lambda,u^n)(h^n,h^n))+o(\|h^n\|^2)\\ &\le \frac{\xi}{n\delta}\|h^n\|^2+o(\|h^n\|^2)=o(\|h^n\|^2)\end{aligned}\tag{3.3}$$

which may only happen if $\beta = 0$ contrary to (QGC).

(B) There is a $\theta > 0$ such that

$$dist(S, x^n) \ge \theta(\|x^n - x\|).$$

As $I(u)$ is an upper semicontinuous map, we have $I(x^n) \subset I(x)$ for large n. Therefore (3.2) is valid if we replace u^n by x and take $h^n = x^n - x$. On the other hand, as $\|x - x^n\| \le n dist(S, x^n)$ if n is large enough, (3.1) holds with $u = x$. Therefore (3.3) also holds with u^n replaced by x and we arrive at the same contradiction as in the first case.

This completes the proof of Theorem 1.

3.2 Proof of Theorem 2

Suppose the theorem is wrong and (GSO) is not valid. Then, as in the proof of Theorem 1, we find a $\delta > 0$ and a sequence of x^n converging to an $x \in S$ such that for any $u \in S$ with $\|u - x^n\| \le n dist(S, x^n)$, (3.1) holds.

Let $u^n \in S$ be a nearest to x^n, $h^n = t_n^{-1}(x^n, x)$, $t^n = \|x^n - x\|$, and let h^n converge to an h, $\|h\| = 1$. We consider the same two possibilities as in 3.1.3 (but at the opposite order).

(A) $\|x - x^n\| = O(dist(S, x^n))$. Then $h \notin T_S(x)$ and $\|x - x^n\| \le n dist(S, x^n)$ for large x, so (3.1) must hold with h replaced by $x^n - x$ and u replaced by x. Therefore

$$\max_{\lambda\in\Omega_\delta(x)}[\mathcal{L}_x(\lambda,x)h^n+\frac{t^n}{2}\mathcal{L}_{xx}(\lambda,x)(h^n,h^n)]\le\frac{t^n}{n}\tag{3.4}$$

and, consequently, for any $\delta > 0$:

$$\max_{\lambda\in\Omega_\delta(x)}\mathcal{L}_x(\lambda,x)h\le 0.$$

This may happen only if $h \in C(x)$. Thus $h \in C(x)\backslash T_S(x)$ and inequality (2.3) is valid for h and x, in particular

$$\max_{\lambda\in\Omega(x)}\mathcal{L}(\lambda,x)(h,h)>0.$$

On the other hand it follows from (3.4) that

$$\max_{\lambda\in\Omega(x)}\frac{t^n}{2}\mathcal{L}_{xx}(\lambda,x)(h^n,h^n)\le\frac{t^n}{n},$$

and therefore

$$\max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda,x)(h,h) \le 0.$$

Hence we arrived at a contradiction.

(B) $dist(S,x^n) = \|u^n - x^n\| = o(\|x - x^n\|)$. Then $h \in T_S(x)$. In what follows we agree (taking if necessary a subsequence) that $I(u^n)$ is the same for all n, and denote this set by J. Obviously $J \subset I(x)$.

The proof is based on the following lemma.

Lemma 2 *Assume that $\|u^n - x^n\| = o(\|x - x^n\|)$. Then there is a sequence of $w^n \in S$ such that*

$$dist(S,x^n) = O(\|x^n - w^n\|),$$

and $e^n := \|x^n - w^n\|^{-1}(x^n - w^n)$ have, among their limit points as $n \to \infty$, a vector $e \notin T_S(x)$ and such that $\nabla f_i(x)e \le 0$ for all $i \in I(x)\backslash J$. Moreover, given $\varepsilon > 0$, the sequence of w^n can be chosen in such a way that $\|h - e\| < \varepsilon$.

Assume for the moment that the lemma has been already proved. Find a sequence of w^n as in Lemma 2 and let $e \notin T_S(x)$ be a corresponding limit point.

As $\|x^n - w^n\|$ is of the same order as $dist(S,x^n)$, we have $\|x^n - w^n\| \le n.dist(S,x^n)$ so that (3.1) holds with $x = w^n$ and $h = x^n - w^n$, that is to say

$$\max_{\lambda\in\Omega_\delta(w^n)} [\mathcal{L}_x(\lambda,w^n)e^n + \frac{\tau^n}{2}\mathcal{L}_{xx}(\lambda,w^n)(e^n,e^n)] \le \frac{\tau^n}{n}, \tag{3.5}$$

where $\tau^n = \|x^n - w^n\|$.

We observe further that $\|w^n - u^n\|$ is both $o(\|w^n - x\|)$ and $(o\|u^n - x\|)$, so by Lemma 1 $I(w^n) = I(u^n) = J$ for large n.

It follows from (3.5) that

$$\nabla f_i(x)e = \lim \nabla f_i(x^n)e^n = 0 \quad \forall i \in J,$$

and, by Lemma 2, $\nabla f_i(x)e \le 0$ if $i \in I(x)\backslash J$.Consequently, $e \in C(x)\backslash T_S(x)$, and (iii) implies that

$$0 < \liminf_{n\to\infty} \max_{\lambda\in\Omega(w^n)} \mathcal{L}_{xx}(\lambda,w^n)(e^n,e^n),$$

in contradiction with (3.5).

Thus it remains to prove Lemma 2.

Proof of Lemma 2 By (i) there is a finite collection of closed convex sets $C_1, \dots, C_k$ (say, containing zero) and a diffeomorphism Q of a neighbourhood V of zero onto a neighbourhood U of x such that

$$S \cap U = Q(C \cap V)\ ; \text{ where } C = \cup C_j.$$

Let y^n and v^n be defined by

$$Q(y^n) = x^n, \quad Q(v^n) = u^n.$$

Then $v^n \to 0$, $y^n \to 0$, $v^n \in C$,

$$\|x^n - u^n\| = O(\|y^n - v_u\|), \quad \|x^n - x\| = O(\|v^n\|) \tag{3.6}$$

and the sets $S_j = Q(C_j)$ are transversal at x.

We may assume that all v^n belong to the same C_j say to C_1, hence $u^n \in S_1$ and, consequently, $h \in T_{S_1}(x)$, $h \notin T_{S_j}(x)$, $j = 2, \dots, k$. It follows from Lemma 2 that $\nabla f_i(x)h < 0$ if $i \in I(x)\backslash J$ and $\nabla f_i(x) \neq 0$. Therefore we can find a $\gamma > 0$ such that $\|e - h\| < \gamma$ implies that $e \notin T_{S_j}(x)$, $j = 2, \dots, k$ and $\nabla f_i(x)e \leq 0$ if $i \in I(x)\backslash J$, $\nabla f_i(x) \neq 0$. We can always assume that ε is smaller than the given γ. Take an $M > 1 + 2\gamma^{-1}$ and let

$$\begin{aligned} \alpha^n &= M\frac{\|x^n - u^n\|}{\|x^n - x\|} \\ z^n &= (1 - \alpha^n)v^n \\ w^n &= Q(z^n). \end{aligned} \tag{3.7}$$

Then $\alpha^n \to 0$, $z^n \in C_1$ and $w^n \in S_1$. We further define e^n as in the statement by means of the w^n. As always, we assume that $e^n \to e$. We have to show that $e \notin T_S(x)$ and that $\|h - e\| \leq \varepsilon$. We have

$$w^n = Q((1 - \alpha^n)v^n) = Q(0) + Q'(0)(1 - \alpha^n)v^n + o(\|v^n\|)$$

and, on the other hand,

$$w^n = Q(v^n - \alpha^n v^n) = Q(v^n) + Q'(v^n)(-\alpha^n v^n) + o(\alpha^n\|v^n\|).$$

Multiplying the first equality by α^n, the second by $(1 - \alpha^n)$ and adding, we have

$$\begin{aligned} w^n &= \alpha^n x + (1 - \alpha^n)u^n + [Q'(0) - Q'(v^n)]\alpha^n(1 - \alpha^n)v^n + o(\alpha^n\|v^n\|), \\ &= \alpha^n x + (1 - \alpha^n)u^n + o(\alpha^n\|v^n\|), \\ &= \alpha^n x + (1 - \alpha^n)u^n + o(\|u^n - x^n\|), \end{aligned}$$

or

$$w^n - x^n = \alpha^n(x - x^n) + (1 - \alpha^n)(u^n - x^n) + o(\|u^n - x^n\|). \tag{3.8}$$

It follows from (3.7), (3.8) that

$$\left|\frac{\|w^n - x^n\|}{\|u^n - x^n\|} - M\right| \leq 1 + r^n, \tag{3.9}$$

where $r^n \to 0$. In particular, $\|w^n - x^n\| = O(dist(S_1, x^n))$ from which, using the fact that S_1 is diffeomorphic to a convex set, we conclude that $e \notin T_{S_1}(x)$.

Thanks to the choice of γ, all we have to show is that $\|h - e\| < \varepsilon$. We have from (3.8) setting $g^n = \dfrac{x^n - u^n}{\|x^n - u^n\|}$:

$$e^n = \alpha^n \frac{\|x^n - x\|}{\|x^n - w^n\|} h^n + \frac{\|x^n - u^n\|}{\|x^n - w^n\|} g^n + r^n$$

where $\|r^n\| \to 0$ or (by (3.7))

$$e^n = \frac{\|x^n - u^n\|}{\|x^n - w^n\|}(Mh^n + g^n) + r^n$$

which together with (3.9) gives

$$\|e^n - h^n\| \leq \frac{2}{M-1} + r^n,$$

that is (see the choice of M) :

$$\|e - h\| \leq \frac{2}{M-1} < \gamma.$$

Q.E.D.

3.3 Proof of Theorem 3

We have (see e.g. [9], Corollary 5)

$$\liminf_{\substack{\sigma \to 0 \\ h' \to h}} \frac{f(x + \sigma h') - f(x)}{\sigma^2} = \max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h, h) \tag{3.10}$$

for any $x \in S$ and any $h \in C(x)$. Assume now that the (QGC) holds. Then

$$dist(S, x + \sigma h) \geq \sigma\ dist(T_S(x), h) + o(\sigma)$$

according to the definition of $T_S(x)$, hence

$$f(x + \sigma h) \geq f(x) + \beta\sigma^2 dist^2(T_S(x), h)^2 + o(\sigma^2)$$

which, together with (3.10) immediately implies the theorem.

4 Further intrinsic sufficient conditions

The proof of Theorem 1 suggests that the orthogonal projection onto S has a special importance for (GSO). We shall obtain some simple intrinsic sufficient conditions using this idea. Recall that a vector h is called a proximal normal to S at $x \in S$ if

$$t\|h\| = dist(S, x + th)$$

for sufficiently small $t > 0$. (Enough to require that there is at least one $t > 0$ with such property). We shall denote by $PN(S, x)$ the collection of proximal normals to S at x. It is always a closed convex cone.

We also denote by $C_\varepsilon(x)$ the ε-critical cone for f at x :

$$C_\varepsilon(x) = \{h : \nabla f_i(x)h \leq \varepsilon\|h\|, \quad i \in I(x)\}.$$

Lemma 3 *Let $\varepsilon > 0$, $x^n \in S$ and $h^n \to 0$ be such that for a certain $\delta > 0$*

$$\max_{\lambda \in \Omega_\delta(x^n)} [\mathcal{L}_x(\lambda, x^n)h^n + \frac{1}{2}\mathcal{L}_{xx}(\lambda, x^n)(h^n, h^n)] \leq O(\|h^n\|^2).$$

Then $h^n \in C_\varepsilon(x^n)$ for all sufficiently large n.

Proof We already observed in 3.1.3 that $\Omega_\infty(x) \subset \delta\xi^{-1}\Omega_\delta(x)$ for some $\xi > 0$. It follows from the assumption that

$$\max_{\lambda \in \Omega_\infty(x^n)} [\mathcal{L}_x(\lambda, x^n)h^n + \frac{1}{2}\mathcal{L}_{xx}(\lambda, x^n)(h^n, h^n)] \leq O(\|h^n\|^2),$$

hence

$$\max_{\lambda \in \Omega_\infty(x)} \mathcal{L}_x(\lambda, x^n)h^n \leq O(\|h^n\|^2).$$

On the other hand, for any $u \in S$ any h and any $i \in I(x)$, $\nabla f_i(x)h \leq \max_{\lambda \in \Omega_\infty(x)} \mathcal{L}_x(\lambda, x)h$. The conclusion follows. □

Proposition 2 *Suppose that there are $\varepsilon > 0$, $\alpha > 0$ such that*

$$\max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h, h) \geq \alpha\|h\|^2$$

for any $x \in S$ and any $h \in C_\varepsilon(x) \cap PN(S, x)$. Then GSO holds.

Proof Let π be an orthogonal projection onto S, i.e. $\|x - \pi(x)\| = dist(S, x)$. We shall show that (GSO) holds with such a π. Assuming the contrary we shall conclude that for any $\delta > 0$ there are sequences of $x^n \in S$ and $h^n \to 0$ such that $\|h^n\| = dist(S, x^n + h^n)$ and (3.1) holds with $u = x^n$, $h = h^n$. By Lemma 3, $h^n \in C_\varepsilon(x^n)$ if n is large enough and, by definition $h^n \in PN(S, x^n)$. So we get a contradiction as soon as $\alpha > n^{-1}dist(S, x^n)$. □

Calculation of ε-critical vectors may present certain difficulties compared with calculation of "regular" critical vectors. The next proposition gives a sufficient criterium in terms of the latter. For any $x \in S$ and h we set

$$f'(x;h) = \max_{i \in I(x)} \nabla f_i(x)h,$$

which is the directional derivative of f at x. Then $h \notin C(x)$ if and only if $f'(x;h) > 0$. For such h we set

$$\begin{aligned} I_1(x;h) &= \{i \in I(x), \nabla f_i(x)h = f'(x;h)\}; \\ M(x;h) &= \{\lambda : \lambda_i \geq 0;\ \lambda_i = 0 \ if\ i \notin I_1(x;h); \textstyle\sum \lambda_i = 1,\ \mathcal{L}_x(\lambda, x)h \geq 0\}; \\ \mu(x;h) &= \min\{\|\mathcal{L}_x(\lambda, x)\| : \lambda \in M(x;h)\}. \end{aligned}$$

We also set

$$PN_\delta(S,x) = \{h : dist(PN(S,x), h) \leq \delta\|h\|\}.$$

Proposition 3 *Assume that there is an $\bar{\mu} > 0$ such that*

$$\mu(x;h) \geq \bar{\mu}, \quad \forall x \in S, \quad \forall h \notin C(x).$$

Suppose also that there are $\alpha > 0$, $\delta > 0$ such that

$$\max_{\lambda \in \Omega(x)} \mathcal{L}_{xx}(\lambda, x)(h,h) \geq \alpha\|h\|^2$$

for all $x \in S$, $h \in C(x) \cap PN_\delta(S,x)$. Then (GSO) holds.

Proof We will apply Proposition 2 in order to get the result. So, let h be in $C_\varepsilon(x) \cap PN(S,x)$. It follows from A. Ioffe [7] that

$$dist(C(x), h) \leq \bar{\mu}^{-1} f'(x;h)$$

(due to homogeneity of $f'(x;.)$). Therefore

$$h \in C_\varepsilon(x) \Rightarrow dist(C(x);h) \leq \frac{\varepsilon}{\bar{\mu}}\|h\|. \tag{4.1}$$

Choose $\delta_1 \in (0, 1/2)$ such that

$$|\mathcal{L}_{xx}(\lambda, x)(h,h) - \mathcal{L}_{xx}(\lambda, x)(h', h')| \leq \alpha/2 \tag{4.2}$$

if $x \in S$, $\lambda_i \geq 0$, $\sum \lambda_i = 1$, $\|h\| = 1$, $\|h - h'\| \leq \delta_1$. Let $\varepsilon > 0$ be so small that

$$\frac{\varepsilon}{\bar{\mu}} < \min\left\{\frac{\delta_1}{1+\delta_1}, \frac{\delta}{1+\delta}\right\}. \tag{4.3}$$

By (4.1) and (4.3) for any $h \in C_\varepsilon(x) \cap PN(S,x)$ there is a $e \in C(x)$ such that

$$\|h - e\| \leq \frac{\varepsilon}{\bar{\mu}}\|h\| \leq \delta\|e\|. \tag{4.4}$$

This means that $e \in C(x) \cap PN_\delta(S,x)$. Then by hypothesis

$$\alpha\|e\|^2 \le \max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda,x)(e,e)$$

and, as $\|h-e\| \le \delta_1\|e\|$ by (4.3), (4.2) implies that

$$\begin{aligned}\max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda,x)(h,h) &\ge \max_{\lambda\in\Omega(x)} \mathcal{L}_{xx}(\lambda,x)(e,e) - \frac{\alpha}{2}\|h\|^2 \\ &\ge \alpha(|e|^2 - 1/2)\|h\|^2).\end{aligned}$$

Taking ε small enough and using (4.3) we can minorize the right hand side by, say, $\frac{\alpha}{4}\|h\|^2$. We now just have to apply Proposition 2.□

It can be be observed that Propositions 1 and 2, though much simpler to formulate, are weaker results then Theorems 1 and 2. To see this, we can consider the function

$$f(x) = \max\{\xi\eta, -\xi, -\eta, \xi+\eta-1\}$$

(where $x = (\xi,\eta) \in R^2$), and

$$S = \{x : \xi\eta = 0\ ;\ 0 \le \xi,\eta, \quad \xi+\eta \le 1\}$$

It can be easily verified that the conditions of Theorems 1, 2 and even Corollary 1 are satisfied on this case but not the conditions of Proposition 1 and 2.

5 Problems with constraints

5.1 General case

This section is essentially devoted to the reformulation of the main results for constrained non-linear programs:

$$(P) \qquad \begin{array}{l}\text{minimize } f_0(x) \\ \text{subject to } f_i(x) \le 0,\ i=1,\dots,k\ ;\ f_i(x)=0,\ i=k+1,\dots,m\end{array}$$

The very fact that theorems on a maximum function as considered above can be applied to (P) follows from the simple observation (cf. [10]) :

Proposition 4 *Let S be a closed set of feasible elements of (P) such that $f_0(x) \equiv const = c$ on S. Set*

$$f(x) = \max\{f_0(x)-c,\ f_1(x),\dots,f_k(x),\ |f_{k+1}(x)|,\dots,|f_m(x)|\}. \tag{5.1}$$

Then the following two properties are equivalent :

(a) *there is a neighborhood U of S such that $f_0(x) > c$ for any $x \in U\backslash S$ which is feasible for (P);*

(b) *$f(x) > 0$ for any $x \in U\backslash S$.*

Proof The implication (b) $\Rightarrow$ (a) is obvious. Conversely, if (a) holds, then $f(x) \geq f_0(x) > 0$ for any feasible $x \in U \backslash S$. On the other hand if x is not feasible then either $f(x) > 0$ for some $i = f_i, \dots, k$, or $|f_0(x)| > 0$ for some $i = k+1, \dots, m$; in either case $f(x) > 0$. $\square$

Thanks to this proposition we can easily reformulate the basic properties, i.e. the quadratic growth condition and the second order sufficient condition, as well as all the theorems for (P), using the specific form of the function f given by (5.1).

The reformulation procedure actually consists on (a) replacing $|f_i(x)|$ by $\max\{f_i(x), -f_i(x)\}$ on (5.1) followed by application of all the formulae to the so obtained function and the subsequent return to the original notation and (b) the observation that $f(x)$ and $f_i(x)$ for $i = k+1, \dots, m$ are constant on S.

The results of the reformulation can be summerized as follows. Consider the set $\Lambda(x)$ of Lagrange multipliers of (P) at x:

$$\Lambda(x) = \{\lambda = (\lambda_0, \dots, \lambda_m) : \lambda_i \geq 0,\ i = 0, \dots, k;\ \lambda_i f_i(x) = 0,\ i = 1, \dots, k;\ \sum \lambda_i \nabla f_i(x) = 0\},$$

the set of δ-multipliers:

$$\Lambda_\delta(x) = \{\lambda = (\lambda_0, \dots, \lambda_m) : \lambda_i \geq 0,\ i = 0, \dots, k,\ \lambda_i f_i(x) = 0,\ i = 1, \dots, k;\ \|\sum \lambda_i \nabla f_i(x)\| \leq \delta\},$$

the subset of *normalized* multipliers and δ-multipliers:

$$\begin{aligned} \Lambda^N(x) &= \{\lambda \in \Lambda(x)\ ;\ \sum |\lambda_i| \leq 1\} \\ \Lambda_\delta^N(x) &= \{\lambda \in \Lambda_\delta(x)\ ;\ \sum |\lambda_i| \leq 1\}. \end{aligned}$$

and the critical cone for (P) at x:

$$K(x) = \{h : \nabla f_i(x) h \leq 0,\ i = 0, \dots, k,\ \nabla f_i(x) h = 0,\ i = k+1, \dots, m\}.$$

Now let us say that

(QGC_P) Problem (P) satisfies the quadratic growth condition on S if $f(x)$ defined by (5.1) satisfies (QGC) on S;

(GSO_P) Problem (P) satisfies the general second order sufficient condition on S if there are a neighborhood U of S and, regular projection $\pi : U \to S$ and an $\alpha > 0$, such that (2.2) is valid with $\Omega_\delta(\pi(x))$ replaced by $\Lambda_\delta^N(\pi(x))$.

(TC_P) For any $x \in S$ and any $i \in I_P(x) := \{i = 1, \dots, k : f_i(x) = 0\}$ either $i \in I(x)$ for all $y \in S$ sufficiently close to x, or S and $\{y : f_i(y) = 0\}$ are transversal at x.

Then the theorems are reformulated as follows.

Theorem 1 (P) *The following implications hold:*

$$(GSO_P) \Rightarrow (GQFC_P),$$

$$(QGC_P)\ \&\ (TC_P) \Rightarrow (GSO_P).$$

Theorem 2 (P) *Assume that*

(i) *S is a nice compact set of stationary points of (P) and f is constant on S*

(ii) *(P) satisfies (TC_P) on every component of S*

(iii) *For any $x \in S$ and any $h \in K(x)\backslash T_C(x)$*

$$\liminf_{u \xrightarrow{S} x} \max_{\lambda \in \Lambda(u)} \mathcal{L}_{xx}(\lambda, x)(h, h) > 0$$

Then (GSO_P) holds.

Theorem 3 (P) *If (QGC_p) holds, then*

$$\max_{\lambda \in \Lambda^N(x)} \mathcal{L}_{xx}(\lambda, x)(h, h) \geq \beta dist^2(T_S(x), h), \quad \forall h \in K(x) \quad \forall x \in S,$$

β being the same as on the (QGC_p), in particular

$$\max_{\lambda \in \Lambda^N(x)} \mathcal{L}_{xx}(\lambda, x)(h, h), \quad \forall x \in S, \quad \forall h \in K(x)\backslash T_S(x).$$

The corresponding replacement can be also made in all other results.

5.2 Constraint qualification

Further specification of definition and results can be obtained under the assumption that the Mangasarian-Fromovitz constraint qualification holds at any $x \in S$. As S is compact, it follows that there is a constant $\eta > 0$ such that the distance from the origin to the affine manifold spanned by the gradients of the equality constraint functions is greater than η and there is an h in X with the unit norm such that :

$$\nabla f_i(x)h = 0, \quad i = k+1, \ldots, m \; ; \; \nabla f_i(x)h \leq -\eta, \quad i \in I_p(x),$$

and

$$\inf\{\lambda_0 : \lambda \in \Lambda^N(x) \; ; \; \sum |\lambda_i| = 1\} \geq \eta$$

which means that the standardly normalized sets of Lagrange multipliers:

$$\Lambda^1(x) = \{\lambda \in \Lambda(x), \quad \lambda_0 = 1\}$$

are uniformly bounded on S. This immediately implies

Proposition 5 *If the (MF) constraind qualification condition is satisfied for any $x \in S$ than on (GSO_P) we can replace $\Lambda^N(x)$ by $\Lambda^1(x)$.*

The change which occurs with the growth condition is more substantial.

Proposition 6 *If the (MF) constant qualification condition is satisfied for all $x \in S$ then (QGC_P) is equivalent to the following*

(QGC_{MF}) there are a $\beta > 0$ and a neighborhood U of S such that

$$f_0(x) \geq c + \beta dist^2(S, x)$$

for all feasible $x \in U$.

Proof It is clear that (QGC$_P$) $\Rightarrow$ (QGC$_{MF}$). Conversely, assume that (QGC$_{MF}$) holds. Let

$$A = \{x : f_0(x) \leq 0, \quad i = 1, \dots, k, \quad f_i(x) = 0, \quad i = k+1, \dots, m\}$$

be the set of feasible elements. It follows from the Robinson regularity theorem [12] that for any $x \in S$, there are a $\gamma(x) > 0$ and an $L(x) > 0$ such that

$$dist(A, n) \leq L(x). \max\{f_1(x), \dots f_0(x), \ |f_{k+1}(x)|, \dots, |f_m(x)|\} \tag{5.2}$$

if $\|u - x\| \leq \gamma(x)$. As S is compact, we can chose $\gamma > 0$ and $L > 0$ such that (5.2) is valid with $\gamma(x)$ and $L(x)$ replaced respectively by γ and L. Assuming that (QGC$_P$) does not hold we shall find a sequence of $\{u^n\}$ such that $dist(S, x^n) \to 0$ and

$$f(x^n) \leq \frac{1}{n} dist^2(S, u^n)$$

given by (5.1). Then (5.2) implies that

$$dist(A, u^n) \leq \frac{L}{n} dist^2(S, u^n),$$

hence there is $x^n \in A$ with $\|x^n - u^n\| \leq \frac{L}{n} dist^2(S, u^n)$. Such an x^n cannot belong to S and, in fact, $dist(S, x^n) \sim dist(S, u^n)$.

On the other hand, as all functions are Lipschitz continuous near S we have

$$f(x^n) = o(dist^2(S, u^n)) = o(dist^2(S, x^n)).$$

Since $x^n \in A \backslash S$, we have

$$\beta. dist^2(S, x^n) \leq f_0(x^n) - c \leq o(dist^2(S, x^n))$$

a contradiction. □

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