# Maxima in convex regions

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**N° 1832**
Janvier 1993

PROGRAMME 2

Calcul Symbolique,
Programmation
et Génie lociciel

Rapport
de recherche

1993

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**Abstract:** Consider a bounded convex region in $d$-dimensional space, and $n$ points drawn from the uniform distribution over this region. In this note we examine the expected number of points which are maximal, as $n$ gets large.

# Maxima dans les régions convexes

**Résumé :** On considère une région bornée et convexe de l'espace à $d$ dimensions, et $n$ points choisis de manière uniforme dans cette région. Dans cette note, nous étudions le nombre moyen de points qui sont maximaux lorsque $n$ tend vers l'infini.

*This work was supported by a Chateaubriand fellowship from the French Ministère des Affaires Étrangères, by the Basic Research Action of the E.C. under contracts No. 3075 and No. 7141 (Projects ALCOM I and ALCOM II), and by NSF Grant CCR-8918152.

# Maxima in Convex Regions (Extended Abstract)

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**Abstract**

Suppose that $C$ is a bounded convex region. Let $p_1, \ldots, p_n$ be points drawn from the uniform distribution over $C$ and let $M_n^C$ be the number of the points which are maximal. In this note we examine the asymptotics of $\mathbf{E}\left(M_n^C\right)$ as $n \to \infty$. We show, for example, that if $C$ is planar then, either $\mathbf{E}\left(M_n^C\right) = \Theta\left(\sqrt{n}\right)$ or $\mathbf{E}\left(M_n^C\right) = O(\log n)$, and give a simple geometric criterion that tells which of the two behaviors applies. This note also addresses the asymptotics of $\mathbf{E}\left(M_n^C\right)$ when $C$ is a higher-dimensional convex region, and discusses the asymptotic behavior of the higher moments of $M_n^C$ as well. Some immediate algorithmic implications that follow from knowing $\mathbf{E}\left(M_n^C\right)$ are also examined, e.g., the existence of heuristics for finding maxima that have fast expected running times when their input points are drawn from any of many different possible distributions.

## 1 Introduction

In this note we analyze $M_n^C$, the number of maximal points in a set of $n$ Independently Identically Distributed (I.I.D.) points drawn from the uniform distribution over some bounded convex region $C$.

The corresponding question for convex hull points has been well studied. Renyi and Sulanke [8] [9] proved that if $n$ points are chosen I.I.D. from $C$ then, if $C$ is a convex polygon, the expected number of convex hull points is $\Theta(\log n)$ while, if $C$ is convex and has a doubly continuously-differentiable boundary the answer is $\Theta\left(n^{1/3}\right)$. Dwyer [5] provides a survey of more recent results. These purely geometric facts have been useful to computational geometers because they lead directly to bounds on the expected running time for some convex hull finding algorithms. The classic example is in the analysis of Gift-Wrapping [7].

The expected number of maximal points has not been examined nearly as closely. It has been known for many years, see e.g., [2], that if $C$ is the unit square then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$. Recently Dwyer [5] proved that $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$ when $C$ is a circle and also proved a general upper bound $\mathbf{E}\left(M_n^C\right) = O\left(n^{1-1/d}\right)$ that is valid for any bounded convex $d$-dimensional region.

In this paper we study the asymptotics of $\mathbf{E}\left(M_n^C\right)$ in detail, concentrating on the planar case. The results of this study will enable us to study the average case behavior of some maxima-finding algorithms in much the same way that Renyi and Sulanke's results about the expected number of convex hull points permitted the analysis of some convex hull finding algorithms. Note that the condition that $C$ be convex is important. If it is abandoned then it can be shown that, for all $f(n) \le n/\log^2 n$ that are slowly varying at infinity, there is some $C$ such that $\mathbf{E}\left(M_n^C\right) = \Theta(f(n))$ (see [6]) for details). If $C$ is constrained to be convex the situation is very different. We prove in this paper that, for convex planar $C$, either $\mathbf{E}\left(M_n^C\right) = \Theta\left(\sqrt{n}\right)$ or $\mathbf{E}\left(M_n^C\right) = O(\log n)$; nothing between these two functions is possible. Our proof is constructive in that it provides necessary and sufficient geometric conditions on $C$ for when each of the two behaviors apply. We also give sufficient conditions for $\mathbf{E}\left(M_n^C\right) = \Theta(1)$ and $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$. In the non-planar case we give sufficient conditions for $\mathbf{E}\left(M_n^C\right) = O\left(\log^{d-1} n\right)$ and $\mathbf{E}\left(M_n^C\right) = \Theta(n^{1-1/d})$. Finally, a theorem due to Devroye will allow us to translate all of our statements about expectations into statements about higher moments as well.

Knowledge of the asymptotic behavior of $\mathbf{E}\left(M_n^C\right)$ will let us prove facts about the average running time of some maxima finding algorithms. For example, it will let us prove that the divide-and-conquer algorithm presented in [2] has an expected $O(n)$ running time when the input points are chosen from a large number of distributions and not only from the hypercube distribution assumed in that paper.

The remainder of this section introduces the notation that we will use. Section 2 presents our results along with some immediate algorithmic implications. Section 3 lists some useful tools. Section 4 contains the proof of our main theorem. In section 5 we conclude by presenting some open problems.

*Note:* In this extended abstract we only provide a proof of the main Theorem. Considerations of space require us to

*This work was supported by a Chateaubriand fellowship from the French Ministère des Affaires Étrangères, by the European Community, Esprit II Basic Research Action Program Number 3075 (ALCOM) and by NSF Grant CCR-8918152.

present the secondary theorems and lemmas without formal proofs.

**1.1 Notation And Helpful Facts.** If $p = (p.x, p.y)$ and $q = (q.x, q.y)$ are planar points we say that $p$ *dominates* $q$ if $p.x \ge q.x$ and $p.y \ge q.y$. Similarly, if $p = (p.1, p.2, \ldots, p.d)$ and $q = (q.1, q.2, \ldots, q.d)$ are $d$-dimensional points we say that $p$ dominates $q$ if $p.i \ge q.i$ for all $1 \le i \le d$. If $S = \{p_1, \ldots, p_n\}$ is a set of points we say that $P$ is *maximal* in $S$ if there is no $q \in S$, $q \ne p$, such that $q$ dominates $p$.

We set

$$\mathrm{MAX}(S) = \{p : p \text{ is maximal in } S\}.$$

See Figures 1 (a), (b) and (c). Suppose $C$ is a measurable set. Let $S = \{p_1, \ldots, p_n\}$ be a set of $n$ points drawn Independently Identically Distributed (I.I.D.) from the uniform distribution over $C$. Set $M_n^C = |\mathrm{MAX}(S)|$, to be the number of maximal points in $S$. We will study $\mathbf{E}\left(M_n^C\right)$, the expected number of maximal points.

A region $C$ is *convex* if for all points $p, q \in C$ the line segment connecting $p$ and $q$ is also in $C$. A function $f$ is convex in an interval $[\alpha, \beta]$ if $f((1-\lambda)x_1 + \lambda x_2) \le (1-\lambda)f(x_1) + \lambda f(x_2)$ for all $\alpha \le x_1 < x_2 \le \beta$ and $0 \le \lambda \le 1$. Note that if $f$ is convex then its left and right derivatives $f'_-(x)$ and $f'_+(x)$ are defined at all $\alpha \le x \le \beta$ except for the left derivative at $x = \alpha$ and the right derivative at $x = \beta$ which are both undefined. Furthermore

$$\begin{aligned} \frac{f(x_2) - f(x_1)}{x_2 - x_1} &\le f'_-(x_2) \\ &\le f'_+(x_2) \le \frac{f(x_3) - f(x_2)}{x_3 - x_2} \end{aligned} \tag{1.1}$$

for all $\alpha \le x_1 < x_2 < x_3 \le \beta$. This fact will be very useful in the proof of Theorem 1. Finally, a function $f$ is *concave* if $-f$ is convex.

## 2 Results and Applications

This section presents our results and some immediate algorithmic applications. The following notation is used throughout: $C$ is a bounded measurable region; $p_1, \ldots, p_n$ are points chosen Independently Identically Distributed (I.I.D.) from the uniform distribution over $C$; $M_n^C = |\mathrm{MAX}(\{p_1, \ldots, p_n\})|$, the number of the points which are maximal; $\mathbf{E}\left(M_n^C\right)$ is the expected number of maximal points.

We will always assume that $C$ is a *closed* region. We do this to ensure that $C$ contains its boundary, $\partial C$ : this assumption makes our proofs slightly simpler. Notice though that the assumption is not restrictive. If $C$ is *any* bounded convex region then $\mathbf{E}\left(M_n^C\right) = E\left(M_n^{\overline{C}}\right)$ because a point chosen from the uniform distribution over $\overline{C}$ is in $\partial C$ with probability zero. It thus suffices to analyze $\mathbf{E}\left(M_n^C\right)$ for closed $C$.

**2.1 Results.**

THEOREM 1. (THE GAP THEOREM) *Let $C$ be a planar convex region. We say that a point $p \in C$ is the* upper-right-hand-corner *of $C$ if $p$ dominates every point $q \in C$. The expected number of maxima among $n$ points chosen I.I.D. uniformly from $C$ is qualitatively dependent upon whether $C$ has an upper-right-hand-corner:*

- *If $C$ does not have an upper-right-hand-corner then* $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$.
- *If $C$ does have an upper-right-hand-corner then* $\mathbf{E}\left(M_n^C\right) = O(\log n)$.

The proof is given in Section 4. Note that the theorem implies that, for convex planar $C$, $\mathbf{E}\left(M_n^C\right)$ can not behave like a function asymptotically between $\log n$ and $\sqrt{n}$, i.e., there is a gap between the two possible behaviors.

EXAMPLE 1. Figures 1 (b), (c), (d), (f), and (h) all have upper-right-hand-corners and thus have $\mathbf{E}\left(M_n^C\right) = O(\log n)$ : figures 1 (a), (e), and (g) don't and so have $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$.

Theorem 1 tells us that when $C$ does not have an upper-right-hand-corner then $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$. When $C$ does have such a corner then all that we know is that $\mathbf{E}\left(M_n^C\right) = O(\log n)$. To derive tighter bounds it is necessary to have better information about the tangents to the boundary of $C$ at the corner. We digress to introduce notation describing these tangents.

Let $C$ be a convex region. If $C$ has an upper-right-hand-corner then the boundary curve of $C$ as it leaves $p$ can be divided into two parts: one curve that goes down and the other that goes to the left.

We define two functions $d(\alpha)$ and $l(\alpha)$ : $d(\alpha) = \beta$ where $\beta$ is such that $(p.x - \beta, p.y - \alpha)$ is on the down curve and $l(\alpha) = \beta$ where $\beta$ is such that $(p.x - \alpha, p.y - \beta)$ is on the left curve. See Figure 2.

While these functions are not defined for all real $\alpha$, the convexity of $C$ ensures that there is always some $\epsilon > 0$ such that both of the functions are well defined, convex and nondecreasing in $[0, \epsilon]$ with $d(0) = l(0) = 0$. Because the functions are convex, their left and right derivatives exist everywhere in the interval except for the undefined left derivative at 0 and the undefined right one at $\epsilon$.

The *down tangent* to $C$ at $p$ is the tangent to the down curve at $p$. The slope of this tangent line is totally determined by the value of the right derivative $d'_+(0)$. If

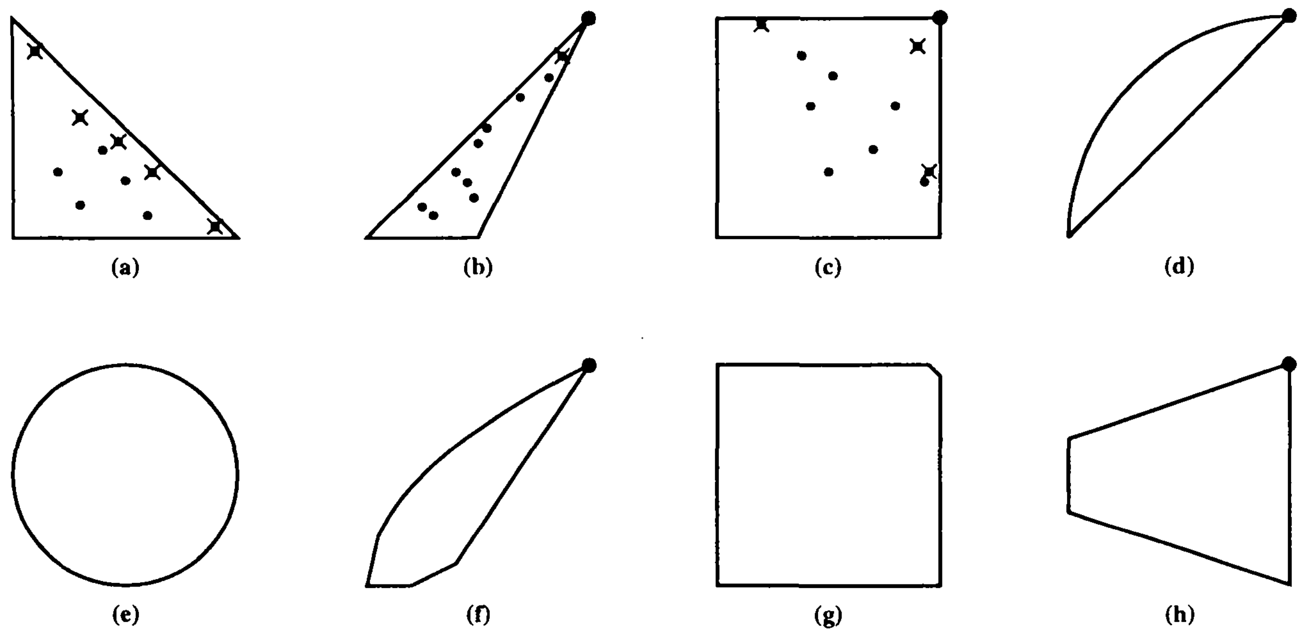


Figure 1: Figures (a), (b), and (c) each contain 10 points the maxima of which are marked by x-s: (a) contains 5 maximal points, (b) 1 maximal point and (c) 3 maximal points. Figures (b), (c), (d), (f) and (h) all have upper-right-hand-corners (marked with a large point); the other figures don't have a corner. Figures (c) and (d) have horizontal left tangents; figures (c) and (h) have vertical down ones. Figure (d) has a doubly differentiable left tangent curve coming out of its upper-right-hand-corner.

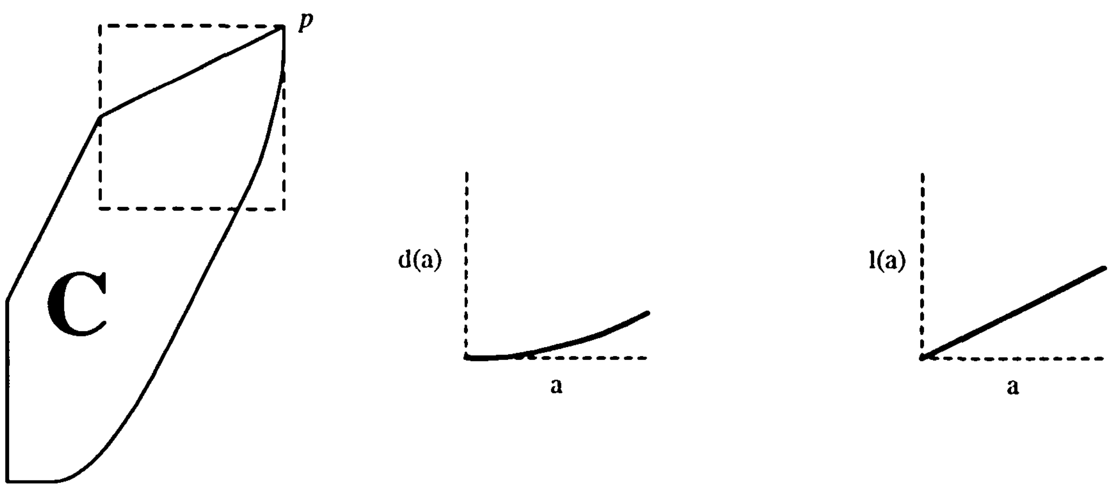


Figure 2: $C$ has an upper-right-hand-corner $p$ with a vertical down tangent at $p$ and a non-horizontal left one. The middle figure portrays $d(\alpha)$, the displacement of the down boundary curve from the vertical line through $p$ and the rightmost figure portrays $l(\alpha)$, the displacement between the left boundary curve and the horizontal line through $p$.

$d'_+(0) = 0$ then the tangent line is vertical. Similarly, if $l'_+(0) = 0$ then the *left tangent* line, the tangent to the left curve, is horizontal. This is illustrated in Figure 2.

The next two theorems discuss the behavior of $\mathbf{E}\left(M_n^C\right)$ when $C$ has an upper-right-hand-corner .

THEOREM 2. *Let $C$ be a convex planar region with upper-right-hand-corner $p$. If the down tangent at $p$ is not vertical and the left tangent at $p$ is not horizontal then $\mathbf{E}\left(M_n^C\right) = \Theta(1)$, i.e., it is bounded. Otherwise $\mathbf{E}\left(M_n^C\right) = \omega(1)$, i.e. it is unbounded.*

EXAMPLE 2. Figures 1 (b) and (f) have $\mathbf{E}\left(M_n^C\right) = \Theta(1)$; figures 1 (c), (d), and (h) have $\mathbf{E}\left(M_n^C\right) = \Omega(1)$.

We now present a tight lower bound for many of the cases in which the left tangent is horizontal and/or the down tangent is vertical.

THEOREM 3. *Let $C$ be a convex planar region with upper-right-hand-corner $p$. Suppose further that one of the following (Lipschitz-like) conditions is fulfilled:*

1. *The down tangent is not vertical and there are positive constants $\delta$ and $c$ such that $l(\alpha) \le c\alpha^{1+\delta}$.*
2. *The left tangent is not horizontal and there are positive constants $\delta$ and $c$ such that $d(\alpha) \le c\alpha^{1+\delta}$.*
3. *There are positive constants $\delta$ and $c$ such that $d(\alpha) \le c\alpha^{1+\delta}$ and $l(\alpha) \le c\alpha^{1+\delta}$.*

*Then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$.*

Note that the condition $l(\alpha) \le c\alpha^{1+\delta}$ forces the left tangent to be horizontal and the condition $d(\alpha) \le c\alpha^{1+\delta}$ forces the down tangent to be vertical. Actually, these conditions can be thought of as requiring not only the left (down) tangent to be horizontal (vertical) but the requiring the entire curve leaving $p$ itself to be "almost" horizontal (vertical) near $p$. These conditions might seem artificial but, in practice, are satisfied quite often as the following examples will illustrate.

EXAMPLE 3. Suppose $C$ has an upper-right-hand-corner $p$ and its down tangent at $p$ is vertical. If $d$ is continuously doubly differentiable in some interval $[0, \epsilon]$ then Taylor's theorem with remainder tells us that there is some $c$ such that $d(\alpha) \le c\alpha^2$ so Theorem 3 tells us that $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$. The same is true if its left tangent is horizontal and $l$ is any continuously doubly differentiable function. Thus Figure 1(d), whose left curve is the upper left quarter of a circle, has $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$.

EXAMPLE 4. As a special case of Example 3 we have that if $C$ is a convex *polygon* with an upper-right-hand-corner and either a vertical down tangent and/or a horizontal left tangent at the corner then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$, e.g. Figures 1 (c) and (h) have $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$. Combining Theorems 1, 2 and 3 we therefore find that if $C$ is a convex polygon then $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$, $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$, or $\mathbf{E}\left(M_n^C\right) = \Theta(1)$; nothing else is possible.

Suppose that $C$ has an upper-right-hand-corner with a vertical and/or horizontal tangent but this tangent does not satisfy the Lipschitz-type condition of Theorem 3. Is it possible to show that $\mathbf{E}\left(M_n^C\right)$ must be either $\Theta(1)$ or $\Theta(\log n)$ as Example 4 tells us must be the case when $C$ is restricted to be a polygon? The answer, unfortunately, is no, as the following example illustrates.

EXAMPLE 5. Let $C$ be the region whose boundary is the line segment connecting $(0,0)$ with $(-e^{-1}, -e^{-1})$ and the curve $\left\{\left(-\frac{y}{\ln|y|}, y\right) : 0 < y < -e^{-1}\right\}$. It can be shown that, for this $C$, $\mathbf{E}\left(M_n^C\right) = O(\ln\ln n)$ and from Theorem 2 we already know $\mathbf{E}\left(M_n^C\right) = \omega(1)$.

All the results mentioned so far are only applicable to convex *planar* regions. The situation is somewhat more complicated if the regions are higher-dimensional. We must now introduce some new notation. Let $C$ be a fixed $d$-dimensional region and $p = (p.1, p.2, \ldots, p.d) \in C$. We define $maximal\_rank(p) = k$ where $k$ is the size of the largest set of indices $I \subseteq \{1, 2, \ldots, d\}$, $|I| = k$ such that for all points $q \in S$ we have $p.i \ge q.i\ \forall i \in I$. Every maximal point in $C$ will have $maximal\_rank > 0$; the definition can be thought of as quantifying "how maximal" a point really is. We then define

$$maximal\_rank(C) = \max_{p \in C} maximal\_rank(p).$$

For example, if $C$ is two-dimensional and has an upper-right-hand-corner then $maximal\_rank(C) = 2$, otherwise, $maximal\_rank(C) = 1$.

If $C$ is a $d$-dimensional hypercube then $maximal\_rank(C) = d$; if $C$ is a $d$-dimensional hypersphere then $maximal\_rank(C) = 1$. We can now write down the $d$-dimensional analogue to the Gap Theorem:

THEOREM 4. *Let $C$ be a $d$-dimensional convex region.*

- *If $maximal\_rank(C) = 1$ then $\mathbf{E}\left(M_n^C\right) = \Theta\left(n^{1-1/d}\right)$.*
- *If $maximal\_rank(C) = d$ then $\mathbf{E}\left(M_n^C\right) = O\left(\log^{d-1} n\right)$.*

Note that this theorem subsumes the two-dimensional Gap Theorem. Note too that in the $d$-dimensional case, unlike in the two-dimensional one, it is possible to have $C$ with $\mathbf{E}\left(M_n^C\right)$ between $n^{1-1/d}$ and $\log^{d-1}$. It seems to be an open question to fully characterize the possible behavior of $\mathbf{E}\left(M_n^C\right)$ when $C$ is $d$-dimensional.

We close this section by quickly discussing the behavior of the higher moments $\mathbf{E}\left((M_n^C)^p\right)$, $p > 1$. Even though we have restricted ourselves to analyzing only the *expectation* of $M_n^C$ there is a remarkable result due to Devroye [4] which lets us transform all of the statements presented in this section into statements about the higher moments. Devroye's theorem tells us that if $\mathbf{E}\left(M_n^C\right) = O(g(n))$ where $g$ is a nondecreasing function then $\mathbf{E}\left((M_n^C)^p\right) = O\left(g^p(n)\right)$ (the same is true if $O()$ is replaced by $\Theta()$).

Whenever, in one of this sections results, we stated that $\mathbf{E}\left(M_n^C\right)$ was $O(g(n))$ or $\Theta(g(n))$ the function $g$ was nondecreasing and Devroye's theorem is applicable. For example, using Devroye's theorem we can rewrite Theorem 1 to say that if $C$ does not have an upper-right-hand-corner then $\mathbf{E}\left((M_n^C)^p\right) = \Theta\left(n^{p/2}\right)$ while if it does have a corner $\mathbf{E}\left((M_n^C)^p\right) = O\left(\log^p n\right)$. The other results about maxima in this section can all be generalized in exactly the same way.

**2.2 Algorithmic Applications.** Maxima have applications in statistics, economics, graphics, and computational geometry [7] to name just a few fields. For this reason there is a large literature describing algorithms to calculate them. Many of these algorithms have been designed to have good average case behaviors when the input points are chosen from certain distributions [1] [2] [3]. Detailed information describing the asymptotic behavior of $\mathbf{E}\left(M_n^C\right)$, such as presented in the first part of this section, can lead to better analyses of these algorithms. We present two such immediate applications.

Bentley, Kung, et. al [2] present an algorithm that they prove finds, in $O(n)$ expected time, the maxima of $n$ input points chosen I.I.D. from a $d$-dimensional hyper-rectangle rectangle with sides parallel to the Cartesian axes. Their proof is dependent only upon the fact that, when $C$ is the hyper-rectangle, $\mathbf{E}\left(M_n^C\right) = O\left(\log^{d-1} n\right)$. Theorem 4 therefore tells us that their algorithm continues to run in $O(n)$ expected time not only when the points are chosen from the hyper-rectangles, but when they are chosen from any convex $C$ with $maximal\_rank(C) = d$. In two dimensions this means that if $C$ has an upper-right-hand-corner then their algorithm finds the maxima in $O(n)$ time.

The algorithm discussed above is not particularly simple to program and, for this reason, Bentley, Clarkson and Levine introduce [1] a new easily coded maxima-finding algorithm, the Move-To-Front heuristic, which empirical evidence suggests runs extremely quickly. The expected running time of this algorithm can be shown to be bounded above by

$$O\left(\sum_{m<n} E\left(M_m^C\right)\right) \tag{2.2}$$

when the input points are chosen I.I.D. from the uniform distribution over $C$. Using the fact that, if $C$ is a rectangle with sides parallel to the Cartesian axes then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$, they used the above equation to show that the expected running time of their algorithm is bounded above by $O(n \log n)$ when $C$ is such a rectangle. Armed with the results of the previous subsection we can say much more. For example, if $C$ is *any* convex planar region that has an upper-right-hand-corner then plugging the result of Theorem 1 into (2.2) tells us that Bentley, Clarkson and Levine's algorithm will run in $O(n \log n)$ expected time on $n$ points chosen I.I.D. uniformly from $C$. If, furthermore, $C$'s down tangent at the corner is not vertical and its left tangent at the corner is not horizontal then Theorem 2 tells us that the algorithm will run in $O(n)$ time.

## 3 Tools of the Trade

In this section we describe some lemmas that will be useful in the analysis of $\mathbf{E}\left(M_n^C\right)$. The proofs of the lemmas are relatively straightforward but, because of space considerations, are omitted in this extended abstract.

There are a class of affine transformations under which $\mathbf{E}\left(M_n^C\right)$ remains invariant.

LEMMA 1. (SCALING) *Let $T$ be one of the following three types of transformations:*

1. *Translation: $T(x,y) = (x+a, y+b)$ where $a, b$ are any real numbers.*
2. *Reflection over the line $x = y$ : $T(x,y) = (y,x)$.*
3. *Positive Scaling: $T(x,y) = (ax, by)$ where $a, b > 0$.*

*Then, for any measurable region $C$, $\mathbf{E}\left(M_n^C\right) = E\left(M_n^{TC}\right)$, where $TC = \{T(x,y) : (x,y) \in C\}$.*

We will find it very useful to be able to upper-bound $\mathbf{E}\left(M_n^C\right)$ by upper-bounding the expected number of maxima in $C$'s component parts.

LEMMA 2. *Let $C = C_1 \cup C_2$ where $C_1$ $C_2$ are not necessarily disjoint. Let $S = \{q_1, \ldots, q_n\} \subset C$. Then (a) $MAX(S) \cap C_1 \subseteq MAX(S \cap C_1)$. Further suppose that $E\left(M_n^{C_1}\right) = O(f_1(n))$ and $E\left(M_n^{C_2}\right) = O(f_2(n))$ where $f_1$ and $f_2$ are nondecreasing functions. Then (b) $\mathbf{E}\left(M_n^C\right) = O\left(f_1(n) + f_2(n)\right)$.*

There is a corresponding lemma that, in special instances, lets us lower-bound $\mathbf{E}\left(M_n^C\right)$.

LEMMA 3. *Let $C, C'$ be measurable regions with $C' \subseteq C$ and the property that only points in $C'$ can* *dominate points in $C''$, i.e. if $p \in C$ dominates $q \in C'$ then $p$ must be in $C'$. Then if $E\left(M_n^{C'}\right) = \Omega(\log n)$ we must have $\mathbf{E}\left(M_n^C\right) = \Omega(\log n)$. Similarly if $E\left(M_n^{C'}\right) = \Omega(\sqrt{n})$ then $\mathbf{E}\left(M_n^C\right) = \Omega(\sqrt{n})$.*

We will need the following special case in the proof of Theorem 1. It can be proven by direct calculation.

EXAMPLE 6. Let $a, b > 0$ and let $C$ be a right triangle with vertices $(0,0)$, $(0,b)$, and $(-a,0)$. Then $\mathbf{E}\left(M_n^C\right) = \Theta(\log n)$. We emphasize that the constants implicit in the $\Theta$ notation are independent of $a$ and $b$. This follows from the positive-scaling part of Lemma 1.

We conclude this section with a lemma which will be crucial in the proof of Theorem 1. It lets us show that if $C = T \cup V$ such that $T$ dominates $V$ in a very particular way then $V$ will not contain many maxima.

LEMMA 4. *Let $a$, $b$ be positive constants and let $T$ be the triangle with vertices $(0,0)$, $(0,a)$, and $(-b,0)$. Let $V$ be any bounded measurable region totally contained in the quarter plane $\{(x,y) : x \le 0,\ y \le 0\}$. Set $C = T \cup V$ and let $S = \{p_1, \ldots, p_n\}$ be points chosen I.I.D. uniformly from $C$. The expected number of these points which are in $V$ and are maximal will be $O(1)$ :*

$$\mathbf{E}(|MAX(S) \cap V|) = O(1).$$

## 4 Proof of the Gap Theorem

*Proof of Theorem 1:* In the proof we assume that $S = \{p_1, \ldots, p_n\}$ is a set of points chosen I.I.D. uniformly from the $C$ under consideration, i.e. $\mathbf{E}\left(M_n^C\right) = \mathbf{E}\left(|\mathrm{MAX}(S)|\right)$.

*(a) The case that $C$ does not have an upper-right-hand-corner:*

We must prove that $\mathbf{E}\left(M_n^C\right) = \Theta(\sqrt{n})$ when $C$ has an upper-right-hand-corner. As mentioned in the first section Dwyer [5] has proven a general upper bound; for all convex $C$, $\mathbf{E}\left(M_n^C\right) = O(\sqrt{n})$. It will therefore suffice to prove that $\mathbf{E}\left(M_n^C\right) = \Omega(\sqrt{n})$ for $C$ of the type described. We prove this in two stages.

(i) We start by assuming that $C$ is a region that has the following form

$$C = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le f(x)\} \tag{4.3}$$

where $f(x) : [0,1] \to [0,1]$ is a concave function with $f(0) = 1$, $f(1) = 0$, and $f'_+(0) < 0$. See Figure 3 (a).

Note that the fact that $f$ is concave and has negative right derivative at 0 immediately implies that $f$ is a monotonically decreasing function in $[0,1]$. It actually implies more than that. Let $t = -f'_+(0) > 0$. Using equation (1.1) we see that

$$f(x) - f(x') \ge t(x' - x),\ \forall x, x',\ 0 \le x < x' \le 1. \tag{4.4}$$

Let $m = \lfloor \sqrt{n} \rfloor$ and set $x_i = \frac{i}{m}$, $0 \le i \le m$. Define $A_i$, $0 \le i < m$ to be the region

$$A_i = C \cap \{(x,y) : x_i \le x < x_{i+1},\ y > f(x_{i+1})\}.$$

We have defined the $A_i$ so that, with constant probability, at least one of the points in $S$ will be in $A_i$. To see this, note that $A_i$, contains the (open) triangle $T_i$ with vertices $(x_i, f(x_i))$, $(x_i, f(x_{i+1}))$, and $(x_{i+1}, f(x_{i+1}))$. Applying inequality (4.4) yields

$$Area(A_i) \ge Area(T_i) \ge \frac{1}{2}(x_{i+1} - x_i)\,(f(x_i) - f(x_i + 1)) \ge \frac{t}{2}$$

Letting $c = Area(C)$ we find that

$$\Pr\left(|A_i \cap S| > 0\right) = 1 - \left[1 - \frac{Area(A_i)}{Area(C)}\right]^n \ge 1 - e^{-\frac{t}{2c}}. \tag{4.5}$$

A point $q \in C$ can not dominate a point in $A_i$ unless $q \in A_i$ as well. One consequence of this is that $\mathrm{MAX}(S) \cap A_i = \mathrm{MAX}(S \cap A_i)$; another is that if $|S \cap A_i| > 0$ then $|MAX(S \cap A_i)| > 0$. Combining these two facts with inequality (4.5) we find that

$$\begin{aligned} \mathbf{E}\left(|\mathrm{MAX}(S)|\right) &\ge \sum_{0 \le i < m} \mathbf{E}\left(|\mathrm{MAX}(S \cap A_i)|\right) \\ &\ge m\left(1 - e^{-\frac{t}{2c}}\right) = \Omega(\sqrt{n}). \end{aligned}$$

(ii) We now prove the first part of the theorem in its full generality. Let $C$ be a bounded convex region that does not have an upper-right-hand-corner. We will show that $C$ contains some region $C_1$ equivalent, after translation and positive scaling, to the type of region analyzed in part (i) and therefore $E\left(M_n^{C_1}\right) = \Omega\left(\sqrt{n}\right)$. Furthermore no point in $C \setminus C_1$ will be able to dominate a point in $C_1$ so Lemma 3 will let us conclude that $\mathbf{E}\left(M_n^C\right) = \Omega\left(\sqrt{n}\right)$.

We define $x_{max} = \max\{x : (x,y) \in C\}$ and $y_{max} = \max\{y : (x,y) \in C\}$. We also define $\overline{x} = \max\{x : (x, y_{max}) \in C\}$ and $\overline{y} = \max\{x : (x_{max}, y) \in C\}$. The points $RH = (\overline{x}, y_{max})$ and $HR = (x_{max}, \overline{y})$ are, respectively, the rightmost point with maximal $y$-coordinate in $C$ and the highest point with maximal $x$-coordinate. Figures (3b) and (3c). Note that $RH \ne HR$ since otherwise $RH$ is the upper-right-hand-corner of $C$ and the assumption is that $C$ does not have an upper-right-hand-corner.

Let $l$ be the line segment connecting $RH$ and $HR$. Let the equation of this line (segment) be $y = l(x)$. By its definition, $l$ has a negative slope. It therefore makes sense to talk about a point being above or below $l$. Also, by the convexity of $C$ and the fact that $RH, HR \in C$ we know that $l \subset C$ as well.

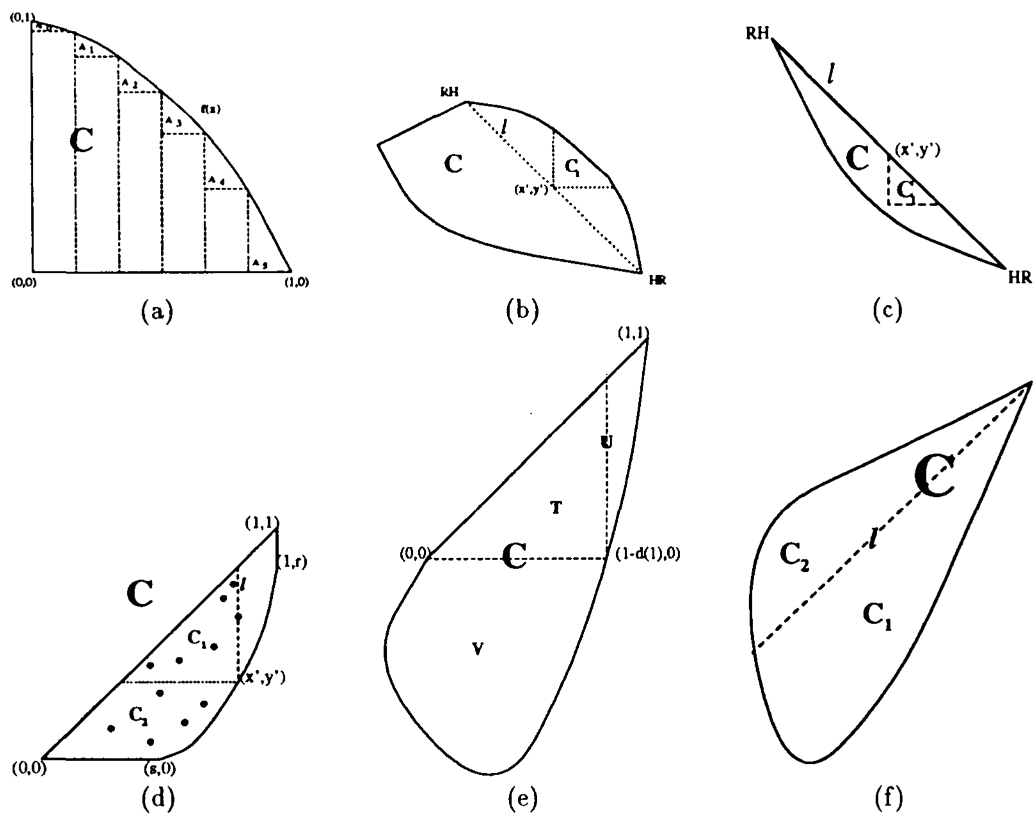


Figure 3: Steps in the proof of Theorem 1.

Let $f$ be the function $f(x) = \max\{y : (x,y) \in C\}$. For $x$, $\overline{x} \le x \le x_{max}$ this function is well defined, concave and monotonically decreasing. Since $l \subset C$ we know that $l(x) \le f(x)$ for all $x$ in this domain.

Let $(x', y')$ be the midpoint of $l : x' = (\overline{x} + x_{max})/2$, $y' = (\overline{y} + y_{max})/2 = l(x')$. Suppose that $f(x') > l(x') = y'$. Let $C_1 = C \cap \{(x,y) : x \ge x', y > y'\}$. See Figure 3 (b). This region is nonempty since it contains $(x', f(x'))$. Furthermore, after translation and positive scaling, $C_1$ is in exactly the form addressed by part (i) above so $E\left(M_n^{C_1}\right) = \Omega\left(\sqrt{n}\right)$. No point in $C \setminus C_1$ can dominate a point in $C_1$ so, from Lemma 3, we find that $\mathbf{E}\left(M_n^C\right) = \Omega\left(\sqrt{n}\right)$.

Assume then that $f(x') = y'$. Figure 3 (c). The concavity of $f$ then requires that $f(x) = l(x)$ for all $\overline{x} \le x \le x_{max}$. There thus must then be must be some $\epsilon > 0$ such that $(x', y' - \alpha) \in C$ for all $\alpha < \epsilon$. Let $\epsilon$ be small enough so that $y' - \epsilon > \overline{y}$. The triangle

$$C_1 = C \cap \{(x,y) : x \ge x', y > y' - \epsilon\}$$

is nonempty. After translation and positive scaling this triangle is also in exactly the form addressed by part (i) above so $E\left(M_n^{C_1}\right) = \Omega\left(\sqrt{n}\right)$ and from Lemma 3 we find that $\mathbf{E}\left(M_n^C\right) = \Omega\left(\sqrt{n}\right)$.

We have thus shown that $\mathbf{E}\left(M_n^C\right) = \Omega\left(\sqrt{n}\right)$ for all convex regions $C$ that do not have an upper-right-hand-corner and have finished this part of the proof.

*(b) The case that $C$ does have an upper-right-hand-corner $p$:*

We must show that for $C$ of this type $\mathbf{E}\left(M_n^C\right) = O(\log n)$. The proof is divided into three stages.

(i) Let $0 \le s \le 1$, $0 \le r \le 1$ be constants and let $C$ be the region whose boundary is composed of the following curves: the line segment connecting $(1,1)$ and $(0,0)$; the horizontal line segment connecting $(0,0)$ and $(s,0)$; the vertical line segment connecting $(1,1)$ and $(1,r)$; the curve $(x, f(x))$, $s \le x \le 1$ where $f : [s,1] \to [r,1]$ is a convex monotonically increasing function such that $f(s) = 0$, $f(1) = r$, and $f'_-(1) \le \infty$. See Figure 3 (d). We show that for $C$ of this type $\mathbf{E}\left(M_n^C\right) = O(\log n)$.

Recall that the set $S = \{p_1, \ldots, p_n\}$ is composed of points chosen I.I.D. uniformly from $C$. We use the notation $q = (q.x, q.y)$ to denote the components of a two-dimensional point $q$. Let $x' = \max\{q.x : q \in S\}$. Let $y' = \min\{y : (x', y) \in C\}$ : if $x' < s$ then $y' = 0$ while if $x' \ge s$ then $y' = f(x')$. We define two new regions and a line segment

$$\begin{aligned} C_1 &= C \cap \{(x,y) : x < x', y \geq y'\} \\ C_2 &= C \cap \{(x,y) : x < x', y < y'\} \\ l &= C \cap \{(x,y) : x = x'\}. \end{aligned}$$

Our definitions ensure that $S \subset C_1 \cup C_2 \cup l$ and that there is at least one point $\overline{q} \in S$ with $\overline{q}.x \in l$.

Now $l$ is a vertical line so the highest point in $l$ dominates every other point in $l$ and $|\mathrm{MAX}(S) \cap l| = 1$. Let $\overline{q}$ be this highest point. From the definition of $y'$ we also know that $\overline{q}.y \geq y'$. Therefore $\overline{q}$ dominates all of the points in $S \cap C_2$ and $|\mathrm{MAX}(S) \cap C_2| = 0$.

We have already seen, in Example 6 that $E\left(M_n^{C_1}\right) = O(\log n)$. The points in $S \cap C_1$ have the same distribution as $|S \cap C_1|$ points chosen I.I.D. uniformly from $C_1$. Thus

$$\begin{aligned} &\mathbf{E}(|\mathrm{MAX}(S) \cap C_1|) \leq \mathbf{E}(|\mathrm{MAX}(S \cap C_1)|) \\ &\quad = O\left(\mathbf{E}(\log |S \cap C_1|)\right) = O(\log n). \end{aligned}$$

Using Lemma 2 to combine the results of the preceeding three paragraphs we find that

$$\mathbf{E}\left(M_n^C\right) = O(\log n)$$

and have finished part (i).

(ii) We now slightly generalize the $C$'s dealt with in the previous case. Let $C$ be a convex planar region with upper-right-hand-corner $p$. Recall the definition of $l$ and $d$ given in Section 2. We further assume that, in $l$ and $d$'s domain of definition $[0, \epsilon]$ the function $l$ is the straight line $l(\alpha) = \alpha$. This last condition simply means that "near" $p$ the left tangent is a line segment with slope 1. See Figure 3(e). We show that for $C$'s of this type $\mathbf{E}\left(M_n^C\right) = O(\log n)$.

Without loss of generality we may assume that $p = (1,1)$ and $\epsilon = 1$; otherwise we can translate and positively scale the region using Lemma 1 so that this is true. We define three regions in $C$:

$$\begin{aligned} T &= C \cap \{(x,y) : y \geq 0, x \leq 1 - d(1)\} \\ U &= C \cap \{(x,y) : y \geq 0\} \\ V &= C \cap \{(x,y) : y \leq 0\} \end{aligned}$$

Note that $U$ is in exactly the form required by part (i) and thus from Lemma 2

$$\begin{aligned} &\mathbf{E}(|\mathrm{MAX}(S) \cap U|) \leq \mathbf{E}(|\mathrm{MAX}(S \cap U)|) \\ (4.6) \qquad &\quad = O\left(\mathbf{E}(\log |S \cap U|\right) = O(\log n). \end{aligned}$$

Note too that $V$ and $T$ are in exactly the form required by Lemma 4 so if $q_1, \ldots, q_m$ were $m$ points chosen I.I.D. uniformly from $V \cup T$ then $\mathbf{E}(|\mathrm{MAX}(q_1, \ldots, q_m) \cap V|) = O(1)$. Thus

$$\mathbf{E}(|\mathrm{MAX}(S) \cap V|) \leq \mathbf{E}(|\mathrm{MAX}(S \cap (V \cup T)) \cap V|) = O(1).$$

Since $C = V \cup U$ we can combine this last inequality with equation (4.6) and apply Lemma 2 to derive $\mathbf{E}\left(M_n^C\right) = O(\log n)$.

(iii) We are now ready to prove the lemma in its full generality. Let $C$ be a region which satisfies the condition of the lemma. Let $l$ be the line which bisects the angle formed by the left and down tangents at the upper-right-hand-corner $p$. We may assume that $l$ has slope 1. Otherwise we may use Lemma 1 to positively scale $C$ so that $l$ does have slope 1 without changing the values of $\mathbf{E}\left(M_n^C\right)$. We may also assume that $p = (1,1)$. Thus $l$ is the line $x = y$. We split $C$ into two parts; that above the line and that below it:

$$C_1 = \{(x,y) \in C : x \geq y\}, \quad C_1 = \{(x,y) \in C : x \leq y\}.$$

See Figure 3(f). $C_1$ is in exactly the form analyzed in part (ii) so $E\left(M_n^C\right)_1 = O(\log n)$. Let $T(x,y) = (y,x)$ be the transformation that reflects over the line $x = y$. The reflection $TC_2$ of $C_2$, has the form analyzed in part (ii) so $E\left(M_n^{TC_2}\right) = O(\log n)$. From Lemma 1 $E\left(M_n^{C_2}\right) = E\left(M_n^{TC_2}\right) = O(\log n)$. Using the fact that $C = C_1 \cup C_2$ and Lemma 2 we conclude by establishing that $\mathbf{E}\left(M_n^C\right) = O(\log n)$ and are finished. ■

## 5 Conclusion And Some Open Problems

In this note we studied the asymptotic behavior of $\mathbf{E}\left(M_n^C\right)$, the expected number of maxima in a set of $n$ points chosen I.I.D. uniformly from a bounded convex set $C$. We also discussed algorithmic appplications, and the asymptotics of the higher moments of $M_n^C$. Many problems in this area remain unanswered. To list just two:

- When does $M_n^C$ obey a central limit theorem? It is known that it does when $C$ is the unit square. For other $C$ the question seems to be open.
- What can be said about $\mathbf{E}\left(M_n^C\right)$ when $C$ is a $d$-dimensional convex region, $d > 2$? As discussed in section 2 the results in this note can be somewhat generalized to higher dimensions but there are many cases still left unanalyzed.

**Acknowledgements**: The author thanks Weiping Shi for bringing these problems to his attention, Luc Devroye and Rex Dwyer for their emailed comments concerning the expected number of maxima, and Jon Bentley for the idea which led to the proof of Theorem 1, part (i).

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ISSN 0249 - 6399

*RR_1832*