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SUR DES SOLUTIONS NON BORNÉES DE L'ÉQUATION DE BELLMAN ASSOCIÉE AUX PROBLÈMES DE COMMANDE OPTIMALE AVEC CONTRAINTES SUR L'ÉTAT

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ABSTRACT

We study a QVI system with unbounded solutions. It represents the Bellman equation associated to an optimal switching control problem with state constraints arising from production engineering. We show that the optimal cost is the unique viscosity solution of the system.

RÉSUMÉ

On étudie un système d'inéquations quasi-variationnelles avec solutions non bornées. Il représente les équations de Bellman associées à un problème de contrôle optimal de commutation avec contraintes sur l'etat, lié à un système de production. On montre que le coût optimal est l'unique solution de viscosité du système.

0. INTRODUCTION

We study in this paper a quasi-variational inequality system (QVI) arising from the application of Dynamic Programming methodology to an optimal switching control problem. Our central result is the characterization of the optimal cost function as the unique solution, the maximum subsolution and the minimum supersolution of this QVI system.

The optimal cost function U must verify in this problem, special boundary conditions resulting from the state restrictions imposed to the state $x \in Q$ of the dynamical system. In a part of ∂Q, U must tend to $+\infty$, while in the remaining points of ∂Q, a regular condition of Dirichlet type holds. Both boundary conditions are an essential part of the definition of viscosity solutions of the QVI and of the proof of the uniqueness of its solution.

Being the function U unbounded, the associated regularity results are only local. For the special dynamical system here analyzed, U is locally Lipschitz-continuous, being the Lipschitz-coefficient independent of the discount factor λ. This property plays a key-rôle in the analysis of the ergodic case and in the establishment of convergence rates in the numerical solution of QVI ([6], [7]).

In the QVI associated to this problem, the behavior of ∇U appears explicitly only in the interior of Q, while in Soner [9], the state restrictions impose conditions on the values of ∇U on ∂Q. Let us also mention, between others related papers, that Capuzzo Dolcetta-Evans studied in [3] the case of functions U bounded and uniformly Hölder-continuous defined in all $\Re^m$; that Capuzzo Dolcetta-Lions analyzed in [4] problems with state restrictions and deterministic controls, while in Lasry-Lions [8], the use of stochastic controls originates non linear elliptic second order equations with highly singular boundary conditions.

1. THE CONTROL PROBLEM

1.1 The set Q of admissible states

We will denote the state of the system with

$$y(t) = (y_1(t),..., y_i(t),..., y_m(t)), \;\; t \geq 0, \;\; y(t) \in \Re^m \tag{1}$$

while the initial state will be denoted

$$y(0) = x \tag{2}$$

The inequalities

$$0 \leq y_i \leq M_i, \quad i = 1,..., m, \tag{3}$$

are the constraints on the state components.

We will use special sets $\Gamma \subset \Re^m$, defined through the use of the following notation:

$$(a_1,..., a_i,..., a_m) \in D^m, \quad D = \{0, 1,...., m\} \tag{4}$$

$$\Gamma(a_1,..., a_i,..., a_m) = \left\{ y = (y_1,..., y_m) \in \Re^m \;/\; (5) \right\}$$

$$\left| \begin{array}{l} a_i = 0 \Rightarrow y_i = 0 \\ a_i = 1 \Rightarrow 0 < y_i < M_i \\ a_i = 2 \Rightarrow y_i = M_i \end{array} \right. \tag{5}$$

We define the set Q of admissible states

$$Q = \bigcup_a \left\{ \Gamma(a_1,..., a_i,..., a_m) \,/\, \text{at most one component } a_i = 0 \right\}, \tag{6}$$

and the corresponding interior Ω:

$$\mathring{Q} = \Omega = \Gamma(1,..., 1,..., 1) \tag{7}$$

The boundary ∂Q will be splitted in the following way

$$\partial Q = \partial Q_e \bigcup \partial Q^+ \qquad \partial Q_e \bigcap \partial Q^+ = \emptyset, \tag{8}$$

where

$$\partial Q_e = \bigcup_{i=1}^{m} (\gamma_i^+ \cup \gamma_i^-), \tag{9}$$

being

$$\gamma_i^+ = \bigcup_a \Gamma(a_1,..., a_i,..., a_m) \cap Q, \; a_i = 2; \qquad \gamma_i^- = \bigcup_a \Gamma(a_1,..., a_i,..., a_m) \cap Q, \; a_i = 0, \tag{10}$$

and

$$\partial Q^+ = \bigcup_a \left\{ \Gamma(a_1, \ldots, a_i, \ldots, a_m) / \text{ at least two components } a_i = 0 \right\}. \tag{11}$$

1.2 Description of the dynamical system and the admissible controls.

An admissible control $\alpha(\cdot)$ is a step function with values in the finite set D. The control will be characterized by a sequence of pairs (θ_ν, d_ν), where

$$0=\theta_0 \leq \theta_1 < \dots < \theta_\nu < \theta_{\nu+1} < \dots \tag{12}$$

are the switching times, and the step function verifies:

$$\alpha(t) = d_\nu, \quad \forall\, t \in (\theta_\nu, \theta_{\nu+1}],\ d_\nu \in D,\ d_\nu \neq d_{\nu+1},\ \nu = 0, 1, \dots . \tag{13}$$

The state of the controlled system is the solution of the following differential equation

$$\frac{dy}{dt} = g(\alpha(t)),\ y \in \Re^m,\ y(0) = x \in Q. \tag{14}$$

For any $x \in Q$, $d \in D$, the set of admissible controls $\mathcal{A}_x^d$ is defined in the following way

$$\mathcal{A}_x^d = \left\{ \alpha(\cdot) = (\theta_\nu, d_\nu)_{\nu=0}^{\infty} \ /\ \forall t \in \Re^+,\ y(t) \in Q,\ d_0 = d \right\}. \tag{15}$$

In other words, we will consider sequences (θ_ν, d_ν) such that the associated trajectory remains in Q, $\forall\, t \geq 0$.

1.3 The optimal cost function U

The following cost functional will associate, to any admissible control policy $\alpha(\cdot)$, the value:

$$J(\alpha(\cdot)) = \sum_{\nu=1}^{\infty} \left(\int_{\theta_{\nu-1}}^{\theta_\nu} f(y(s), \alpha(s))\, e^{-\lambda s} ds + q(d_{\nu-1}, d_\nu)\, e^{-\lambda \theta_\nu} \right), \tag{16}$$

where $f(\cdot, d)$ is Lipschitz-continuous in $\bar{Q}$ and $q(d_{\nu-1}, d_\nu)$ represents the switching cost between $d_{\nu-1}$ and d_ν. It verifies

* $\exists\ q_o > 0$, such that for any closed loop $(d_0, d_1, \dots, d_p, d_{p+1})$, with $d_0 = d_{p+1}$, $p \leq m$, we have (non-zero cost loop condition)

$$\sum_{i=0}^{p} q(d_i, d_{i+1}) \geq q_o. \tag{17}$$

* $q(d, \tilde{d}) \geq 0 \quad \forall\, d \neq \tilde{d};\ \ q(d, d) = 0,\ \forall\, d$ (18)

* $q(d, \hat{d}) \leq q(d, \tilde{d}) + q(\tilde{d}, \hat{d}) \quad \forall\, d \neq \tilde{d} \neq \hat{d}.$ (19)

For any $d \in D$ and $x \in Q$, we define the optimal cost function:

$$U_d(x) \equiv \inf\left\{ J(\alpha(\cdot)) \,/\, \alpha(\cdot) \in \mathcal{A}_x^d \right\} \tag{20}$$

The optimization problem consists in finding, for any $x \in Q$, $d \in D$, an optimal control policy $\bar{\alpha}_x^d(\cdot) \in \mathcal{A}_x^d$, satisfying:

$$J(\bar{\alpha}_x^d(\cdot)) = U_d(x). \tag{21}$$

1.4 Application to a production system.

The model previously described can be applied to the optimization of a production system that comprises a multi-item production machine. In that case, m is the number of items that the machine can produce.

We denote with $d = 0$ the idle state, while $d=1,\ldots, m$ indicates the production of d^{th} item.

$\forall\, d \neq 0$, the evolution of each component is given by:

$$g_d(\alpha) = \left| \begin{array}{ll} -r_d & \text{if } \alpha \neq d \\ & \\ p_d - r_d & \text{if } \alpha = d; \end{array} \right. \tag{22}$$

where $r_d > 0$, $p_d > 0$ are respectively the instantaneous demand and the instantaneous production rates of item d.

We will suppose that switching times are negligible and that the following condition is verified:

$$\sum_{d=1}^{m} \frac{r_d}{p_d} < 1. \tag{23}$$

This condition implies the existence of admissible production schedules.

For this production system, the state $y=(y_1,\ldots, y_i,\ldots, y_m)$ represents the stock levels of items $1,\ldots, i,\ldots, m$ and restriction (15) means that backlogging is not allowed.

In this case we can see that the set ∂Q^+ is the set of points that cannot be the initial point of any admissible trajectory of the system.

The results of the following sections are established for the production model here described; nevertheless, those results continue to hold in more general models. Some of those extensions are described in §4.

2. PROPERTIES OF THE OPTIMAL COST FUNCTION

2.1 Properties of local boundness

Theorem 2.1: *U is locally bounded, i.e.* $\exists\ C > 0$ *verifying the following inequality:*

$$0 \leq U_i(x) \leq \frac{C}{\lambda} - C\ \log\Big(d(x,\ \partial Q^+)\Big), \qquad \forall\ x \in Q\ ,\ \forall\ i \in D. \tag{24}$$

Theorem 2.2: *U is unbounded in neighborhoods of* ∂Q^+*; moreover,* $\exists\ C > 0$ *verifying the following inequality:*

$$U_i(x) \geq -C\ \log\Big(d(x,\ \partial Q^+)\Big) - C, \qquad \forall\ x \in Q\ ,\ \forall\ i \in D. \tag{25}$$

2.2 Property of local Lipschitz-continuity

Theorem 2.3: *U is locally Lipschitz-continuous, i.e.* $\exists\ L(\cdot)$ *non-increasing /* $\forall\ x \in Q$, $\forall\ x' \in Q$, $\forall\ i \in D$

$$\left| U_i(x) - U_i(x') \right| \leq\ L(\eta)\ \| x - x' \| \tag{26}$$

being

$$\eta = \min\ (\ d(x,\ \partial Q^+),\ d(x',\ \partial Q^+)). \tag{27}$$

Remark: From Theorem 2.3 it follows that $U(\cdot)$ is uniformly Lipschitz-continuous in the complement of any neighborhood of ∂Q^+.

Note: The proof of theorems 2.1, 2.2, 2.3 are contained in [7].

3. THE QUASI-VARIATIONAL INEQUALITY SYSTEM (QVI)

3.1 Definition of the system QVI

We will say that $v \in (C(Q))^{m+1}$ is a viscosity solution of the IQV system if a), b), c) and d) are satisfied:

a) v verifies (28)-(29)-(30)

$$v_d(x) \leq (S^d(v))(x) \qquad \forall\, x \in \Omega \tag{28}$$

$$v_{\tilde{d}}(x) = (S^{\tilde{d}}(v))(x) \qquad \forall\, x \in \gamma_{\tilde{d}}^{-},\ \forall\, \tilde{d} \neq d \tag{29}$$

$$v_d(x) = (S^d(v))(x) \qquad \forall\, d \neq 0,\ \forall\, x \in \gamma_d^{+} \tag{30}$$

where

$$(S^d(v))(x) = \min_{\tilde{d}} \left(q(d, \tilde{d}) + v_{\tilde{d}}(x) \right) \tag{31}$$

b) $\displaystyle \lim_{x \to \partial Q^{+}} v_d(x) = +\infty \qquad \forall\, d$ (32)

c) $\displaystyle \frac{\partial v_d}{\partial x}\, g(\cdot\,, d) + f(\cdot\,, d) - \lambda\, v_d \geq 0$ (33)

in Ω in the viscosity sense, i. e. the following condition must be verified

$\forall\, \phi \in C^1(\Omega)$, $\forall\, \zeta \in \Omega$ / $v_d - \phi$ has a local maximum in ζ, then

$$\left(\frac{\partial \phi}{\partial x}\right)(\zeta)\; g(\zeta, d) + f(\zeta, d) - \lambda\, v_d(\zeta) \geq 0 \tag{34}$$

d) $\displaystyle \frac{\partial v_d}{\partial x}\, g(\cdot\,, d) + f(\cdot\,, d) - \lambda\, v_d = 0$ (35)

in the viscosity sense in the set $C_v^d = \left\{ x \in \Omega \ / \ v_d(x) < (S^d(v))(x) \right\}$,

i. e. $\forall\, \phi \in C^1(C_v^d)$, must be verified

d_1) if $v_d - \phi$ has a local maximum in $\zeta \in C_v^d$, then

$$\left(\frac{\partial \phi}{\partial x}\right)(\zeta)\; g(\zeta, d) + f(\zeta, d) - \lambda\, v_d(\zeta) \geq 0 \tag{36}$$

d_2) if $v_d - \phi$ has a local minimum in $\zeta \in C_v^d$, then

$$\left(\frac{\partial \phi}{\partial x}\right)(\zeta)\; g(\zeta, d) + f(\zeta, d) - \lambda\, v_d(\zeta) \leq 0. \tag{37}$$

3.2 The optimal cost function U as solution of QVI

Theorem 3.1: *The optimal cost function U is a viscosity solution of the system QVI.*

Proof: By virtue of Theorem 2.3, the function U_d is locally Lipschitz-continuous in Q; then, condition $U \in (C(Q))^{m+1}$ is verified.

Condition (28) follows from the definition of optimal cost; in fact, if $x \in \Omega$, it is admissible to switch to any control $\tilde{d} \in D$, and after that, to follow the optimal control policy starting from the new initial condition; in consequence, we have

$$U_d(x) \le q(d, \tilde{d}) + U_{\tilde{d}}(x) \qquad \forall\, \tilde{d} \ne d \tag{38}$$

and immediately we get

$$U_d(x) \le \min_{\tilde{d}} \left(q(d, \tilde{d}) + U_{\tilde{d}}(x) \right) = (S^d(U))(x)\,. \tag{39}$$

If $x \in \gamma_d^-$, any admissible policy comprises an initial period with control d; in consequence, starting from any other discrete state, the system must switch to the state d without delay; then, from the dynamic programming principle it holds:

$$U_{\tilde{d}}(x) = U_d(x) + q(\tilde{d}, d) \quad \forall\, \tilde{d} \ne d\,. \tag{40}$$

By virtue of (19), (31) and (39), we obtain

$$U_{\tilde{d}}(x) = \left(S^d(U)\right)(x) \quad \forall\, \tilde{d} \ne d\,. \tag{41}$$

If $x \in \gamma_d^+$, there is not an initial period with control d for any admissible control policy. Then, only initial periods with control $\tilde{d} \ne d$ are admissible. From the state d, the system must switch instantaneously to any other state $\tilde{d}$; then, by the dynamic programming principle, we have

$$U_d(x) = \min_{\tilde{d}} \left\{ q(d, \tilde{d}) + U_{\tilde{d}}(x) \right\} = (S^d(U))(x)\,. \tag{42}$$

In consequence, U verifies condition (a).

Condition (b) holds, because by virtue of (24) we get $\lim_{x \to \partial Q^+} U_d(x) = +\infty$, $\forall$ d.

To prove condition (c), we consider $\zeta \in \Omega$, such that $U_d - \phi$ has a local maximum in ζ, (w. l. g. we consider that this maximum is zero), i. e., $\exists$ B (open set), $\zeta \in B \subset \Omega$, such that

$$U_d(y) - \phi(y) \le U_d(\zeta) - \phi(\zeta) = 0 \qquad \forall\, y \in B. \tag{43}$$

By the dynamic programming principle, we have, for δ small enough,

$$U_d(\zeta) \le \int_0^\delta e^{-\lambda s} f(y(s), d)\, ds + e^{-\lambda\delta}\, U_d(y(\delta)) \le \int_0^\delta e^{-\lambda s} f(y(s), d)\, ds + e^{-\lambda\delta}\, \phi(y(\delta))\,; \tag{44}$$

then

$$\frac{U_d(\zeta) - e^{-\lambda\delta}\, U_d(\zeta)}{\delta} \le \frac{1}{\delta}\int_0^\delta e^{-\lambda s}\, f(y(s),\, d)\, ds + e^{-\lambda\delta}\left(\frac{\phi(y(\delta)) - \phi(\zeta)}{\delta} \right) \tag{45}$$

and taking limit for $\delta \to 0$ we get:

$$\lambda\, U_d(\zeta) \le f(\zeta,\, d) + \left(\frac{\partial\phi}{\partial x}\right)(\zeta)\; g(\zeta,\, d) \tag{46}$$

and in consequence, (c) is verified.

To prove condition (d), we consider that by virtue of the dynamic programming principle we have:
$\forall\, x \in C_U^d,\ \exists\, \delta(x) > 0\ /$

$$U_d(x) = \int_0^\delta e^{-\lambda s}\, f(y(s),\, d)\, ds + e^{-\lambda\delta}\, U_d(y(\delta)) \qquad \forall\, \delta \le \delta(x)\,. \tag{47}$$

Then, if $\phi \in C^1(C_U^d)$ verifies that $U_d - \phi$ has a local maximum in $\zeta \in C_U^d$ (with value zero); we have, for δ small enough,

$$\phi(\zeta) = U_d(\zeta) \le \int_0^\delta e^{-\lambda s}\, f(y(s),\, d)\, ds + e^{-\lambda\delta}\, \phi(y(\delta)) \tag{48}$$

and in consequence we get

$$\left(\frac{\partial\phi}{\partial x}\right)(\zeta)\; g(\zeta,\, d) + f(\zeta,\, d) - \lambda\, U_d(\zeta) \ge 0\,. \tag{49}$$

In a similar way, if we suppose that $U_d - \phi$ has a local minimum in $\zeta \in C_U^d$ (also with zero value), we have

$$\phi(\zeta) = U_d(\zeta) \ge \int_0^\delta e^{-\lambda s}\, f(y(s),\, d)\, ds + e^{-\lambda\delta}\, \phi(y(\delta)) \tag{50}$$

from here, we get

$$\left(\frac{\partial\phi}{\partial x}\right)(\zeta)\; g(\zeta,\, d) + f(\zeta,\, d) - \lambda\, U_d(\zeta) \le 0\,. \tag{51}$$

so, (d) is satisfied and we have proven that the optimal cost function U is a viscosity solution of the QVI system.

□

3.3 The set of subsolutions W

3.3.1 Definition of subsolution

We will say that $w \in (C(Q))^{m+1}$ is a viscosity subsolution of the system QVI if:

- $w_d(x) \leq (S^d(w))(x) \quad \forall\, x \in \Omega$ (52)

- $\frac{\partial w_d}{\partial x}\, g(d) + f(\cdot\,,\, d) - \lambda\, w_d(\cdot) \geq 0$ (53)

  in Ω in the viscosity sense, i. e.:
  $\forall\, \varphi \in C^1(\Omega)$ such that $w_d - \varphi$ has a local maximum in $\zeta \in \Omega$, it is satisfied

$$\left(\frac{\partial \phi}{\partial x}\right)(\zeta) \cdot g(d) + f(\zeta,\, d) - \lambda\, w_d(\zeta) \geq 0. \tag{54}$$

We will denote with W the set of subsolutions.

3.3.2 Properties of W.

We introduce here a technical result which proof can be found in the Appendix:

Lemma 3.1: *Let O be an open domain and $w \in C(O)$ a subsolution in the viscosity sense of the equation:*

$$\frac{\partial w}{\partial x}\, g(\cdot\,,\, d) + f(\cdot\,,\, d) - \lambda\, w(\cdot) = 0, \tag{55}$$

i. e., $\forall\, \phi \in C^1(O)$ such that $w - \phi$ has a local maximum in ζ, it is verified:

$$\frac{\partial \phi}{\partial x}(\zeta)\, g(\zeta,\, d) + f(\zeta,\, d) - \lambda\, w(\zeta) \geq 0. \tag{56}$$

Then, $\forall\, x,\, x' \in O$ / $$x' = x + g(d)\, t \qquad (t > 0), \tag{57}$$

$$y(s) = x + g(d)\, s \in O \qquad \forall\; 0 \leq s \leq t \tag{58}$$

it is verified that:

$$w(x) \leq \int_0^t e^{-\lambda s}\, f(y(s),\, d)\, ds + e^{-\lambda t}\, w(x'). \tag{59}$$

We can state now the following property

Proposition 3.1: *If $w \in W$, then for any admissible control policy $\alpha(\cdot)$, it is verified*

$$w_d(x) \leq J(x, \alpha(\cdot)). \tag{60}$$

Proof: Let $\alpha(\cdot)$ be an admissible policy and $y(\cdot)$ the corresponding trajectory.

By virtue of (53) and (59) it is verified, for any $t \leq t' \,/\, \theta_i < t \leq t' < \theta_{i+1}$

$$w_{\alpha(t)}(y(t)) \leq w_{\alpha(t')}(y(t'))\, e^{-\lambda(t'-t)} + \int_t^{t'} e^{-\lambda(s-t)} f(y(s), \alpha(s))\, ds. \tag{61}$$

Then, as w is continuous by virtue of (52), we have

$$w_d(x) \leq \int_0^\tau e^{-\lambda s} f(y(s), \alpha(s))\, ds + \sum_{i=0}^{\theta_{i+1} \leq \tau} q(d_i, d_{i+1})\, e^{-\lambda\theta_i} + e^{-\lambda\tau}\, w_{\alpha(\tau)}(y(\tau)). \tag{62}$$

In [7] it has been proved that, $\exists\ \rho > 0$ and a sequence $\tau_n \to \infty$ satisfying $d(y(\tau_n), \partial Q^+) > \rho$. This implies, by virtue of (24), that $\exists\ C > 0\ /$

$$\overline{\lim_{n\to\infty}}\ w_{\alpha(\tau_n)}(y(\tau_n)) \leq C \tag{63}$$

and, in consequence,

$$\overline{\lim_{n\to\infty}}\ w_{\alpha(\tau_n)}(y(\tau_n))\, e^{-\lambda\tau_n} = 0. \tag{64}$$

Now, taking limit for $n\to\infty$ en (62), we obtain

$$w_d(x) \leq J(x, \alpha(\cdot)) \quad \forall\ \alpha(\cdot) \in \mathcal{A}_x^d. \tag{65}$$

□

Theorem 3.2: *The optimal cost function U is the maximum element of W (the maximum subsolution), i.e.*

$$w_d(x) \leq U_d(x) \quad \forall\ d,\ \forall\ x \in Q. \tag{66}$$

Proof: It follows from (20) and (65).

□

3.4 The set of supersolutions S.

3.4.1 Definition of supersolution.

We will say that $s \in (C(Q))^{m+1}$ is a supersolution of the system QVI if:

- $s_{\tilde{d}}(x) \geq (S^{\tilde{d}}(s))(x) \qquad \forall\, x \in \gamma_d^-,\ \forall\, \tilde{d} \neq d,\ \forall\, d \neq 0.$ (67)

- $s_d(x) \geq (S^d(s))(x) \qquad \forall\, x \in \gamma_d^+,\ \forall\, d \neq 0.$ (68)

- $\lim\limits_{x \to \partial Q^+} s_d(x) \to +\infty \quad \forall\, d \in D.$ (69)

- If $s_d(x) < (S^d(s))(x)$, $x \in \Omega$ then, there exists a neighborhood $B_{\epsilon(x)}(x)$ of x, such that

$$\frac{\partial s_d}{\partial x}\, g(d) + f(\cdot\,, d) - \lambda\, s_d(\cdot) \leq 0 \tag{70}$$

  in $B_{\epsilon(x)}(x)$ in the viscosity sense, i. e., if $\varphi \in C^1(B_{\epsilon(x)}(x))$ such that $s_d - \varphi$ has a local minimum in $\zeta \in B_{\epsilon(x)}(x)$, then:

$$\left(\frac{\partial \phi}{\partial x}\right)(\zeta)\; g(d) + f(\zeta, d) - \lambda\, s_d(\zeta) \leq 0\,. \tag{71}$$

We will denote with S the set of supersolutions.

3.4.2 Construction of admissible control polices using supersolutions.

Lemma 3.2: *Let* **O** *be an open domain such that, for any $s \in C(\mathbf{O})$,*

$$\frac{\partial s}{\partial x} g(d) + f(\cdot\,, d) - \lambda\, s(\cdot) \leq 0, \tag{72}$$

in the viscosity sense.

Let be $t < t'$ such that

$$x + g(d)\delta \in \mathbf{O} \quad \forall\, \delta \in [0,\, t' - t], \tag{73}$$

then:

$$s(x) \geq e^{-\lambda\delta}\, s(x + g(d)\delta) + \int_0^\delta e^{-\lambda\rho}\, f(x + g(d)\rho,\, d)\, d\rho \qquad \forall\, \delta \in [0,\, t' - t]. \tag{74}$$

Proof: (74) follows taking $s(x) = -w(x)$ and applying that lemma 3.1.

□

We can state now:

Proposition 3.2: *For any supersolution s, it is possible to find, for any $x \in Q$ and $d \in D$ an admissible control policy $\alpha(\cdot) \in \mathcal{A}_x^d$ such that*

$$s_d(x) \geq J(x, \alpha(\cdot)). \tag{75}$$

Proof: Let be $x \in Q$, $d \in D$. The construction of the control is based in the analysis of the difference $s_d(x) - (S^d(s))(x)$. There are two possible cases:

- $s_d(x) < (S^d(s))(x)$
- $s_d(x) \geq (S^d(s))(x)$.

We define an admissible control policy, starting at the initial state (x, d), and proceeding iteratively, as described below.

We define $\theta_0 = 0$, $d_0 = d$, $y(\theta_0) = x$; and, given θ_ν, d_ν, $y(\theta_\nu)$, we compute the values $\theta_{\nu+1}$, $d_{\nu+1}$, $y(\theta_{\nu+1})$ in the following way:

(A): If $s_{d_\nu}(y(\theta_\nu)) < (S^{d_\nu}(s))(y(\theta_\nu))$,

by virtue of conditions (67) and (68) it follows immediately

$$\exists\, \bar{\delta}_{d_\nu}(y(\theta_\nu)) > 0 \ / \ y(\theta_\nu) + g(d_\nu)\,\delta \in Q \quad \forall\, \delta \in [0, \bar{\delta}_{d_\nu}(y(\theta_\nu))]\,. \tag{76}$$

Let C_d be defined as

$$C_d = \left\{\varsigma \in Q \ / \ s_d(\varsigma) < \left(S^d(s)\right)(\varsigma)\right\}.$$

In consequence, $y(\theta_\nu) \in C_{d_\nu}$, and in this set

$$\frac{\partial s_{d_\nu}}{\partial x}\, g(d_\nu) + f(\cdot\,, d_\nu) - \lambda\, s_{d_\nu}(\cdot) \leq 0 \tag{77}$$

in the viscosity sense.

We define

$$\theta_{\nu+1} = \theta_\nu + \inf\left\{\tau \ / \ y(\theta_\nu) + g(d_\nu)\ \tau \notin C_{d_\nu}\right\}; \tag{78}$$

(the definition if correct because C_{d_ν} is bounded).
We have, in consequence, by property (74)

$$s_{d_\nu}(y(\theta_\nu)) \geq e^{-\lambda\delta}\, s_{d_\nu}(y(\theta_\nu) + g(d_\nu)\,\delta) + \int_0^\delta e^{-\lambda\rho}\, f(y(\theta_\nu) + g(d_\nu)\,\rho,\, d_\nu)\, d\rho \qquad \forall\, \delta \leq \theta_{\nu+1} - \theta_\nu. \tag{79}$$

This majoration of $s_{d_\nu}(y(\theta_\nu) + g(d_\nu)\,\delta)$ implies, by virtue of condition (69), that

$$y(\theta_\nu) + g(d_\nu)\,(\theta_{\nu+1} - \theta_\nu) \notin \partial Q^+. \tag{80}$$

Also, by the continuity of s, we have

$$s_{d_\nu}(y(\theta_\nu)) \geq e^{-\lambda(\theta_{\nu+1} - \theta_\nu)}\, s_{d_\nu}(y(\theta_\nu) + g(d_\nu)\,(\theta_{\nu+1} - \theta_\nu)) + \int_{\theta_\nu}^{\theta_{\nu+1}} e^{-\lambda\rho}\, f(y(\theta_\nu) + g(d_\nu)\,\rho,\, d_\nu)\, d\rho. \tag{81}$$

For the state of the system at time $\theta_{\nu+1}$:

$$y(\theta_{\nu+1}) = y(\theta_\nu) + g(d_\nu)\,(\theta_{\nu+1} - \theta_\nu),$$

it is satisfied that

$$s_{d_\nu}(y(\theta_{\nu+1})) = \left(S^{d_\nu}(s)\right)(y(\theta_{\nu+1}))\,. \tag{82}$$

and this property leads us to analyze the second case:

(B): $s_{d_\nu}(y(\theta_\nu)) \geq (S^{d_\nu}(s))(y(\theta_\nu)).$ (83)

In this case we must define the value $d_{\nu+1}$ of the control in the interval $[\theta_{\nu+1}\,,\,\theta_{\nu+2}\,]$.

That value $d_{\nu+1}$ is chosen verifying the inequality

$$s_{d_\nu}(y(\theta_\nu)) \geq q(d_\nu\,,\, d_{\nu+1}) + s_{d_{\nu+1}}(y(\theta_{\nu+1}))\,. \tag{84}$$

and that is possible by virtue of (83).

After having done that choice, we can re-start the analysis and in that form, we get a sequence of values

$$(\theta_{\nu+r}\,,\, d_{\nu+r}\,,\, r = 1,\ldots,\, p)$$

such that

$$s_{d_\nu}(y(\theta_\nu)) \geq \sum_{r=1}^{p} q(d_{\nu+r-1},\, d_{\nu+r}) + s_{d_{\nu+p}}((y(\theta_{\nu+p}))\,. \tag{85}$$

As the set of values $s_{\hat{d}}(x)$, $\hat{d} \in D$ is finite, we have, by virtue of condition (17), that this sequence (verifying $\theta_{\nu+r} = \theta_\nu$, $r = 1,\ldots,\, p$), can have at most, m elements, and the last element

$$(\theta_{\nu+p}\,,\, y(\theta_{\nu+p}),\, d_{\nu+p})$$

must verify

$$s_{d_{\nu+p}}(y(\theta_{\nu+p})) < \left(S^{d_{\nu+p}}(s)\right)(y(\theta_{\nu+p}))$$

In that case, the construction of the control restarts at A.

The iterative use of this procedure enables us to get a sequence (θ_ν, d_ν) with the property

$$s_d(x) \geq e^{-\lambda\theta_\nu} s_{d_\nu}(y(\theta_\nu)) + \int_0^{\theta_\nu} e^{-\lambda\rho} f(y(\rho), d(\rho))\, d\rho + \sum_{\mu=1}^{\nu} e^{-\lambda\theta_\mu} q(d_{\mu-1}, d_\mu). \tag{86}$$

From this last inequality it follows that the sequence $\{\theta_\nu : \nu=1, 2,...\}$ cannot have a finite cluster point τ. In fact; if such point τ exists, we would have, by virtue of (19), that

$$\sum_{\mu=1}^{\nu} e^{-\lambda\theta_\mu} q(d_{\mu-1}, d_\mu) \to +\infty, \text{ when } \nu \to \infty \tag{87}$$

and that contradicts (86); then, we have that $\theta_\nu \to +\infty$.

Then, as

$$\overline{\lim_{\nu\to\infty}}\, e^{-\lambda\theta_\nu} s_{d(\theta_\nu)}(y(\theta_\nu)) \geq 0, \tag{88}$$

from (86) it results

$$s_d(x) \geq \lim_{\nu\to\infty} \Big(\int_0^{\theta_\nu} e^{-\lambda\rho} f(y(\rho), d(\rho))\, d\rho + \sum_{\mu=1}^{\nu} e^{-\lambda\theta_\mu} q(d_{\mu-1}, d_\mu) \Big) = J(x, \alpha(\cdot)). \tag{89}$$

This inequality proves that $\alpha(\cdot)$ is admissible and that (74) holds. □

Theorem 3.3: *The optimal cost function U is the minimum supersolution.*

Proof: U is a supersolution by virtue of Theorem 3.1 and the definition of supersolution. Finally, by (89) we have that $\forall$ s $\in$ S, $\exists$ $\alpha(\cdot)$ verifying:

$$s_d(x) \geq J(x, \alpha(\cdot)) \tag{90}$$

then

$$U_d(x) = \inf_{\alpha(\cdot)} J(x, \alpha(\cdot)) \leq J(x, \alpha(\cdot)) \leq s_d(x) \tag{91}$$

□

Theorem 3.4: *U is the unique viscosity solution of QVI.*

Proof: If we suppose that v is solution, that implies $v \in W$, then, $v \leq U$. Inversely, if v is solution, that implies $v \in S$, then, $v \geq U$. Both inequalities imply $v = U$, and we get the uniqueness of the viscosity solution. □

4. COMMENTS

Final remarks and generalizations

1) In Theorem 2.3, the key assumption is (23). It implies a controllability property which is a sufficient condition to prove that the optimal cost function is locally Lipschitz-continuous with the Lipschitz-coefficients independent of the discount factor λ. When this condition does not hold, it is only possible to prove that U is locally Hölder-continuous, but now the coefficients are not longer independent of λ. Nevertheless, the problem in that case also has an associated QVI system with unique solution in the viscosity sense.

2) The specific problem above analyzed can be generalized to other models where the domain Q is not longer restricted to be an hypercube.

For those cases, the points of the singular boundary are characterized in the following way:

a) $x \in \partial Q^+ \Leftrightarrow \forall\, d \in D,\ x+\rho\, g(d) \notin Q,\ \forall\, \rho / \ 0 < \rho \le \bar{\rho}$.

b) $\exists\, \bar{\delta},\ \lambda_0,\ \lambda_1, \dots, \lambda_m \in \Re^+ \ / \ \sum_{i=1}^{m} \lambda_i = 1,\ x+s \sum_{i=1}^{m} \lambda_i\, g(i) \in \mathring{Q},\ \forall\, 0 < s \le \bar{\delta}$.

These conditions imply:

- The optimal cost function $U_d(x) \to +\infty$ si $x \to \partial Q^+$.
- The problem is similar to the above analyzed, i. e. the optimal cost function is well defined in Q and it is the unique solution of the associated QVI system.

3) Numerical approximation methods ([6], [7]), has been applied to the production system described in (1.4). The set Q was approximated with sets Q_k, verifying $\bar{Q}_k \cap \partial Q^+ = \emptyset$; this choice avoids the explicit presence of condition $\lim_{x \to \partial Q^+} U_d(x) = +\infty$. The rate of convergence of the method is $\sqrt{k}$. In [2] we get a rate of convergence of order k using a special ad-hoc triangularization.

4) Extensions to the ergodic case.

For the production model above considered, we can analyze the asymptotic case $\lambda \to 0$, related to the optimization of the long term average cost:

$$\mathring{J}(x, \alpha(\cdot)) = \overline{\lim_{\nu \to \infty}} \frac{1}{\theta_\nu} \left(\int_0^{\theta_\nu} f(y(s), \alpha(s))\, ds + \sum_{\mu=1}^{\nu} q(d_{\mu-1}, d_\mu) \right)$$

The study of this problem has been done in [1], where some issues are analysed; between them the existence of optimal policies, the QVI associated system, the set of solutions (because in general there is not uniqueness of viscosity solutions), the numerical approximation of its solutions, the development of fast computational algorithms and the establishment of the rate of convergence.

APPENDIX

Proof of Lemma 3.1:

The result will be established by the absurd. Then, let be $x^1, x^2 \in O$, such that

- $x^2 = x^1 + g(d)\, t$

- $x^1 + g(d)\, r \in O \qquad \forall\ 0 \leq r \leq t$ (92)

- $w(x^1) > \int_0^t e^{-\lambda s} f(x^1 + g(d)\, s,\ d)\, ds + e^{-\lambda t}\, w(x^2).$ (93)

In consequence, the function

$$z(s) = w(x^1 + g(d)\, s) - \int_s^t e^{\lambda(s-r)} f(x^1 + g(d)\, r,\ d)\, dr - e^{-\lambda(t-s)}\, w(x^2) \tag{94}$$

satisfies

$$z(t) = 0 \ \text{ and } \ z(0) > 0. \tag{95}$$

We define the parameters h, k and functions φ, $\hat{\varphi}$ y ψ in the following form:

$$k = z(0) \tag{96}$$

$$h = \max_{0 \leq r \leq t} z(t); \tag{97}$$

$$\varphi(\xi) = (\tfrac{1}{2} - \tfrac{1}{\pi} \operatorname{arctg}(2\xi - 1)) \tag{98}$$

$$\hat{\varphi}(x, \epsilon, \rho) = \varphi\Big(1 + \tfrac{1}{\epsilon}\Big(\zeta(x) + \rho - 1 + \sqrt{\epsilon}\Big)\Big), \tag{99}$$

where, if $\langle \cdot, \cdot \rangle$ is the inner product in $\Re^m$,

$$\zeta(x) = \frac{\langle x - x^1, g(d)\rangle}{\langle x^2 - x^1, g(d)\rangle} \tag{100}$$

and

$$\psi(x, \epsilon, \rho, \eta) = \int_{\zeta(x)}^{1} e^{-\lambda(s - \zeta(x))t} f\Big(x^1 + g(d)\ t\ s,\ d\Big)\, t\, ds + e^{-\lambda t(1-\zeta(x))}\, w(x^2) +$$

$$+ \frac{k}{2} + h\ \hat{\varphi}(x, \epsilon, \rho) + \frac{1}{\eta} \left\| \Big(x^1 - \frac{\langle x^1, g(d)\rangle}{\|g(d)\|^2}\, g(d)\Big) - x + \frac{\langle x, g(d)\rangle}{\|g(d)\|^2}\, g(d)\Big) \right\|^2 \tag{101}$$

We can see now that $\psi \in C^1$ verifies, $\forall\ 0 \leq \delta \leq t$

$$\psi(x^1 + g(d)\,\delta,\, \epsilon,\, \rho,\, \eta) = \int_\delta^t e^{-\lambda(r-\delta)}\, f\Big(x^1 + g(d)\, r,\, d\Big)\, dr + e^{-\lambda(t-\delta)}\, w(x^2) +$$

$$+ \frac{k}{2} + h\, \hat{\varphi}\Big(x + g(d)\,\delta,\, \epsilon,\, \rho\Big). \tag{102}$$

Then, for $\rho = 0$, if ϵ and η are small enough, we get, by virtue of (93), that $\forall$ x belonging to a neighborhood of the segment $x^1 + g(d)\, s$, $s \in [0,t]$,

$$\psi(x,\, \epsilon,\, 0,\, \eta) > w(x)\,.$$

while, form the other hand, for ρ=1, we have

$$\psi(x^1,\, \epsilon,\, 1,\, \eta) = w(x^1) - \frac{k}{2} < w(x^1)\,. \tag{103}$$

From here we get, by virtue of the continuity of ψ, that there will exist $\overline{\rho}$, $\overline{x}$ such that

- $\overline{x} \neq x^1$, $\overline{x} \neq x^2$
- $\overline{x}$ belongs to a neighborhood of the segment $x^1 + g(d)\, s$, $s \in [0,t]$,
- $w(x) - \psi(x,\, \epsilon,\, \overline{\rho},\, \eta)$ has a local maximum, with value 0, in $\overline{x}$,

from here, by (56) it follows

$$\frac{\partial \psi(\overline{x})}{\partial x}\, g(d) + f(\overline{x},\, d) - \lambda\, w(\overline{x}) \geq 0\,. \tag{104}$$

By definition of ψ, we have:

$$\frac{\partial \psi(x)}{\partial x}\, g(d) + f(x^1 + g(d)t\zeta(x),\, d) - \lambda\, \psi(x) < 0 \tag{105}$$

$\forall$ x belonging to a neighborhood of the segment $x^1 + g(d)\, s$, $s \in [0,t]$.

Then, for η small enough,

$$\frac{\partial \psi}{\partial x}(\overline{x})\, g(d) + f(\overline{x}) - \lambda w(\overline{x}) < 0\,. \tag{106}$$

As (106) contradicts (104) and (59) is proved.

□

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