Linear probing with lazy deletions, parking problem with exponential sojourn time, receiver release Cambridge ring and other related problems

Philippe Jacquet

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél.: (1) 39 63 55 11

Rapports de Recherche

1992
25ème anniversaire

N° 1816

Programme 2
Calcul Symbolique, Programmation et Génie logiciel

LINEAR PROBING WITH LAZY DELETIONS, PARKING PROBLEM WITH EXPONENTIAL SOJOURN TIME, RECEIVER RELEASE CAMBRIDGE RING
and other related problems

Philippe JACQUET

Décembre 1992

RR-1816

LINEAR PROBING WITH LAZY DELETIONS, PARKING PROBLEM WITH EXPONENTIAL SOJOURN TIME, RECEIVER RELEASE CAMBRIDGE RING and other related problems

Philippe Jacquet

Abstract. In this paper we intend to give an exact analytical evaluation of some parameters arising in a general bin-packing problem. There is an infinite sequence of cells ranked from the left to the right. A cell is either empty, either busy, *i.e* contains an item. Items are generated over the cells according to independent Poisson processes of λ item per cell and per time unit. When an item is generated on an empty cell it enters the cell which therefore becomes busy. When an item is generated on a busy cell, it tries the next cell on the right, and so forth, until it finds an empty cell which it enters. Each stored item has a sojourn time in the cell which is supposed to be exponential of mean one. When the item leaves a cell, it disappears completely, *i.e* quits the system, the cell immediately becomes empty and available for a next item. We give an exact evaluation of the steady state of the system in terms of a Taylor expansion with respect to variable λ.

HACHAGE LINÉAIRE AVEC RETRAITS SIMPLES, PROBLÈME DU PARKING AVEC UN TEMPS DE SÉJOUR EXPONENTIEL, ANNEAU DE CAMBRIDGE AVEC RETRAIT À LA DESTINATION et autres problèmes apparentés

Résumé. Dans ce papier nous tentons de donner une évaluation exacte de certains paramètres qui apparaissent dans un problème d'allocation de ressources. Il y a une infinité de cellules rangées de la gauche vers la droite. Une cellule est soit libre, soit occupée, c'est à dire elle contient un objet. Les objets sont créés au dessus des cellules suivant des processus de Poisson indépendants correspondant à λ objet par cellule. Quand un objet est créé sur une cellule libre il entre dedans, la cellule devient occupée. Si il est créé sur une cellule occupée, il cherche sur sa droite jusqu'à ce qu'il trouve une cellule libre dans laquelle il entre. Les objets stokés en cellule ont un temps de séjour fini qui suit une loi exponentielle de moyenne 1. Quand un objet quitte une cellule, il disparaît définitivement aux yeux du système, et sa cellule redevient instantanément libre. Nous donnons une évaluation exacte de la distribution stationnaire du système sous les termes d'un développement limité de Taylor en la variable λ.

LINEAR PROBING WITH LAZY DELETIONS, PARKING PROBLEM WITH EXPONENTIAL SOJOURN TIME, RECEIVER RELEASE CAMBRIDGE RING and other related problems

Philippe Jacquet

June 19, 1992

Abstract

In this paper we intend to give an exact analytical evaluation of some parameters arising in a general bin-packing problem. There is an infinite sequence of cells ranked from the left to the right. A cell is either empty, either busy, *i.e* contains an item. Items are generated according to an independent Poisson process of λ item per cell and per time unit. When an item is generated on an empty cell it enters the cell which therefore becomes busy. When an item is generated on a busy cell, it tries the next cell on the right, and so forth, until it finds an empty cell which it enters. Each stored item has a sojourn time in the cell which is supposed to be exponential of mean one. When the item leaves a cell, it disappears completely, *i.e* quits the system, the cell immediately becomes empty and available for a next item. We give an exact evaluation of the steady state of the system in terms of a Taylor expansion with respect to variable λ.

1 Introduction and motivations

We consider the following simple and general bin-packing problem. We consider an infinite row of cells ranked from the left to the right. The row is infinite in both way. A cell is either empty, either busy. it is busy when it contains an item. Items are generated according to an independent Poison process of parameter λ per cell and per time unit. When an item is generated on an empty cell it enters the cell which therefore becomes busy. When an item is generated on a busy cell, it tries the next cell on the right, and so forth, until it finds an empty cell which it enters. When the item is finally stored in a cell, it stays during a sojourn time which is supposed to be independent and exponential of mean one. When the sojourn time expires the item leaves the cell and completely diappears, *i.e* quits the system; the cell immediately becomes empty and available for a next item.

This simple problem has connection with several problems of practical interest. The first one is a refinement of the parking problem of D. Knuth (exercise 29, p. 545 in *The Art of Computer Programning*, Vol 3: Sorting and Searching). "A certain one-way street has m parking places in a row, numbered 1 through m. A man and his dozing wife drive by, and suddenly she wakes up and orders him to park immediately. He dutifully parks at the first available space; but if there are no places left that he can get to without backing up (*i.e* his wife awoke when the car approached space k, but spaces k, $k+1, \ldots, m$ are all full), he expresses his regrets and drives on." In our problem we consider that: (i) cars enter the street as a Poisson stream of rate $m\lambda$ per time unit, (ii) the cars speed are exactly one space per time

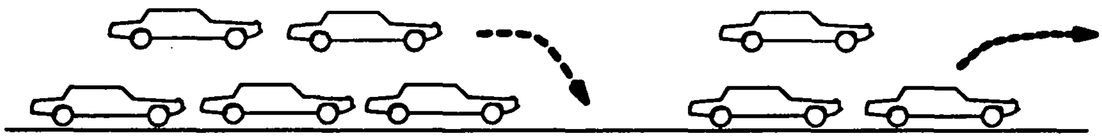

Figure 1: Parking problem with finite sojourn time

unit, (iii) the cars parks during an exponential sojourn time of mean one time unit. We also consider the limiting case where $m = \infty$. Point (iii) is the essential difference between our problem and exercise 29 of Knuth. See the nice figure 1 for an illustration.

The second connection can be found in the fact that the parking problem is highly connected to the linear probing analysis. but in our case the connection to our problem must be done about linear probing *with lazy deletions, i.e*, items are deleted in place without global reconfiguration of the hashing table [1]. Linear hashing with lazy deletions is not an efficient way of hashing in a general sense (unsuccessful searches are costly since one has to scan all the table); but it may find restricted applications, for example for databases with periodic or random data refreshment.

More anecdotal is the model of circular conveyors belts collecting luggages in airports. More serious is the modelization of Slotted Ring network with receiver slot-release ([2]). The ring is circular with m slots rotating in the same way. Each station is connected to the ring *via* a single tap. Stations are distributed around the ring. A transmitter with a pending packet, place it in the first slot which is detected empty when passing through the transmitter's tap, the receiver releases the slot by removing the packet from the ring *via* its tap. Of course, we suppose that $m = \infty$ and that the distance between transmitter and receiver is an exponential random variable.

There are also connections to other problems in physics dealing with particles (see Malyshev and Robert [3]).

2 Mathematical modelization

2.1 Notations

When we look at the infinite row of cells at a given time, we may see it like a succession of portions of consecutive busy cells separated by portions filled with empty cells. In the sequel, we define a *block* as a sequence of cells such that the leftmost one is empty and the others are busy. The *degenerate* or *zero* block is simply an empty cell (followed by zero busy cells). The size k of a block is the number of busy cells inside. At any time the system can be viewed as an infinite sequence of blocks. Note that portion of ℓ consecutive empty cells is equivalent to a sequence of $\ell - 1$ zero blocks.

We model our system *via* an infinite vector of integers $(., k_{-1}, k_0, k_1, k_2, .)$. Notation "$z_2, .)$" (respectively "$(., z_{-1}$") expresses that the sequence has continuation to $+\infty$ (respectively $-\infty$); it can be seen as an abbreviation of "$z_2, \ldots)$" (respectively $(\ldots, z_{-1}$. Each integer k_i, $i \in Z$ is the size of the ith block. We arbitrarily choose block number 0. We introduce the formal generating function

$$g(., z_{-1}, z_0, z_1, z_2, .) = \sum_{(.,k_{-1},k_0,k_1,k_2,.)} p(., k_{-1}, k_0, k_1, k_2, .) \prod_{i \in Z} z_i^{k_i} , \tag{1}$$

Where $p(., k_{-1}, k_0, k_1, k_2, .)$ is the probability weight of the occurrence of the corresponding vector $(., k_{-1}, k_0, k_1, k_2, .)$ when the system is at equilibrium. Since we consider that block number 0 is arbitrary, it comes that probability generating function g is invariant regarding any circular permutation, *i.e* :

$$g(., z_{-1}, z_0, z_1, z_2, .) = g(., z_0, z_1, z_2, z_3, .) . \tag{2}$$

2.2 Breaks and recombinations

Suppose that a new item is generated on block number 0 between time t and time $t + dt$, note that such an event occurs with probability $(k_0 + 1)\lambda dt$. The item will find its place at the end of the block, on the empty cell of block number 1. This leads to merging block number 0 and block number 1 into a new block of size $k_0 + k_1 + 1$, by making busy the empty cell in between. We call such an event a *recombination*.

Suppose now that an item stored in block number 0 quits the system between time t and time $t + dt$, such an event occurs with probability $k_0 dt$. In the following we consider $k_0 > 0$. Suppose that the removed item was in the ℓth busy cell of the block ($\ell \leq k_0$), the ℓth cell becomes empty. Doing so, block number 0 is therefore broken into two consecutive blocks, the first one being of size $\ell - 1$ and the second one, of size $k_0 - \ell$. We call such an event a *break*.

2.3 The recombination operator

Let g be the *p.g.f* of the system at time t and g' be the *p.g.f* at time $t + dt$. Our aim is to consider the differential discrepancy between g and g' when a recombination has the possibility to occur between block number 0 and block number 1 and between time t and time $t + dt$. An item is generated on block number 0 with probability $(1 + k_0)\lambda dt$. A recombination on block number 0 consists in mapping vector $(., k_{-1}, k_0, k_1, k_2, .)$ in vector $(., k'_{-1}, k'_0, k'_1, k'_2, .)$, with the following rules:

$$\begin{array}{rcl} k'_i &=& k_i, \text{ when } i < 0, \\ k'_0 &=& k_0 + k_1 + 1, \\ k'_i &=& k_{i+1} \text{ when } i > 0. \end{array} \tag{3}$$

In short, vector $(., k'_{-1}, k'_0, k'_1, k'_2, .)$ is identical to vector $(., k_{-1}, k_0+k_1+1, k_2, k_3, .)$. When considering generating functions, we note the following identity:

$$\prod_{i \in Z} z_i^{k'_i} = \prod_{i<0} z_i^{k_i} \times z_0 z_0^{k_0} z_0^{k_1} \times \prod_{i>0} z_i^{k_{i+1}} . \tag{4}$$

Summing the above identity on all possible vectors $(., k_{-1}, k_0, k_1, k_2, .)$, with respect to their probability weights and taking into account the probability $(1 + k_0)\lambda dt$, we obtain the identity:

$$\begin{array}{rcl} g'(., z_{-1}, z_0, z_1, z_2, .) &=& g(., z_{-1}, z_0, z_1, z_2, .) + \lambda dt[z_0 g(., z_{-1}, z_0, z_0, z_1, .) + \\ && + z_0^2 D_0 g(., z_{-1}, z_0, z_0, z_1, .) - g(., z_{-1}, z_0, z_1, z_2, .) - \\ && - z_0 D_0 g(., z_{-1}, z_0, z_1, z_2, .)] , \end{array} \tag{5}$$

where D_i denotes the operator derivation with respect to variable at position i in the vector $(., z_0, z_1, .)$. In other words $g' = g + \lambda dt R_0 g$, defining recombination operator R_i using the identity below:

$$\begin{array}{l} R_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) = z_i g(., z_{i-1}, z_i, z_i, z_{i+1}, .) + z_i^2 D_i g(., z_{i-1}, z_i, z_i, z_{i+1}, .) - \\ - g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) - z_i D_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) . \end{array} \tag{6}$$

Let us define the r_i operator by

$$r_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) = g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) + z_i D_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) . \tag{7}$$

We have

$$R_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) = z_i r_i g(., z_{i-1}, z_i, z_i, z_{i+1}, .) - r_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) . \tag{8}$$

2.4 The break operator

Let g'' be the $p.g.f$ of the system at time $t + dt$. We want to estimate the differential discrepancy between g and g'' when a break has the possibility to occur in block number 0 between time t and time $t + dt$. The break occurs with probability weight $k_0 dt$. Let us consider that the item in the ℓth busy cell in block number zero leaves the system. We have $0 < \ell \le k_0$. The new vector $(., k''_{-1}, k''_0, k''_1, k''_2, .)$ satisfies the following.

$$\begin{aligned} k''_i &= k_i \text{ when } i < 0, \\ k''_0 &= \ell - 1 , \\ k''_1 &= k_0 - \ell , \\ k''_i &= k_{i-1} \text{ when } i > 1 . \end{aligned} \tag{9}$$

Using generating functions and the fact that

$$\sum_{\ell=0}^{\ell=k_0-1} z_0^\ell z_1^{k_0-\ell} = \frac{z_0^{k_0} - z_1^{k_0}}{z_0 - z_1} , \tag{10}$$

we obtain the identity

$$\begin{aligned} g''(., z_{-1}, z_0, z_1, z_2, .) &= g(., z_{-1}, z_0, z_1, z_2, .) + dt[\tfrac{g(., z_{-1}, z_0, z_2, z_3, .) - g(., z_{-1}, z_1, z_2, z_3, .)}{z_0 - z_1} - \\ &\qquad - z_0 D_0 g(., z_{-1}, z_0, z_1, z_2, .)] . \end{aligned} \tag{11}$$

Similarly as about the recombination operator, we define the break operator C_i as following

$$\begin{aligned} C_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) &= \tfrac{g(., z_{i-1}, z_i, z_{i+2}, z_{i+3}, .) - g(., z_{i-1}, z_{i+1}, z_{i+2}, z_{i+3}, .)}{z_i - z_{i+1}} - \\ &\qquad - z_i D_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) . \end{aligned} \tag{12}$$

In other words, $g'' = g + dt\, C_0 g$. Note that when $z_i = z_{i+1}$ we simply have

$$\begin{aligned} & C_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) = \\ & D_i g(., z_{i-1}, z_i, z_{i+2}, z_{i+3}, .) - z_i D_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) . \end{aligned} \tag{13}$$

We can define operator c_i as

$$c_i g(., z_{i-1}, z_i, z_{i+1}, z_{i+2}, .) = \frac{g(., z_{i-1}, z_i, z_{i+2}, z_{i+3}, .) - g(., z_{i-1}, z_{i+1}, z_{i+2}, z_{i+3}, .)}{z_i - z_{i+1}} . \tag{14}$$

Thus $C_i g(., z_i, .) = c_i g(., z_i, .) - z_i D_i g(., z_i, .)$.

2.5 The fundamental equation

Since breaks or recombinations may indifferently occur on any block (their respective probability weights are already included in the operators), the fundamental equation for the steady state of the system is therefore:

$$\sum_{i \in Z} (C_i + \lambda R_i) g(., z_{-1}, z_0, z_1, z_2, .) = 0 . \tag{15}$$

3 Elementary recursions

Let us define restriction of general function g. Let g_n be the n-variable function defined by $g_n(z_1, \ldots, z_n) = g(., 1, z_1, \ldots, z_n, 1, .)$, for integer $n > 0$; symbols "1, .)" (respectively "(., 1") means that the vector is filled with 1's up to infinity (respectively down to $-\infty$). In other words

$$g_n(z_1, \ldots, z_n) = \mathrm{E}[z_1^{k_1} \cdots z_n^{k_n}] , \tag{16}$$

where E$[X]$ denotes the probabilistic expectation of a random variable X. Let $n > 1$ and let us consider the fundamental equation over g_n:

$$\sum_{i \in Z}(C_i + \lambda R_i)g(.,1,z_1,\ldots,z_n,1,.) = 0 . \tag{17}$$

Note that $R_i g(.,1,z_1,\ldots,z_n,1,.) = 0$ and $C_i g(.,1,z_1,\ldots,z_n,1,.) = 0$ for $i > n$. By extension, we define the operators ρ_i for $i \in [1,n)$ by

$$\begin{aligned} \rho_i g_{n+1}(z_1,\ldots,z_n) &= z_i g_{n+1}(z_1,\ldots,z_{i-1},z_i,z_i,z_{i+1},\ldots,z_n) + \\ &+ (z_i)^2 D_i g_{n+1}(z_1,\ldots,z_{i-1},z_i,z_i,z_{i+1},\ldots,z_n) . \end{aligned} \tag{18}$$

Thus $R_i g(.,1,z_1,\ldots,z_n,1,.) := \rho_i g_{n+1}(z_1,\ldots,z_n) - r_i g_n(z_1,\ldots,z_n)$ We also have for $i \leq 0$

$$R_i g(.,1,z_1,\ldots,z_n,1,.) = D_{i-1}g(.,1,z_1,\ldots,z_n,1,.) - D_i g(.,1,z_1,\ldots,z_n,1,.) \tag{19}$$

We also extend operators C_i for $i \in [1,n-1)$ by

$$\begin{aligned} C_i g_n(z_1,\ldots,z_n) &= \frac{g_{n-1}(z_1,\ldots,z_{i-1},z_i,z_{i+2},\ldots,z_n) - g_{n-1}(z_1,\ldots,z_{i-1},z_{i+1},z_{i+2},\ldots,z_n)}{z_i - z_{i+1}} - \\ &\quad - z_i D_i g_n(z_1,\ldots,z_n) . \end{aligned} \tag{20}$$

For the special case $i = n$, we have

$$C_n g_n(z_1,\ldots,z_n) = \frac{g_n(z_1,\ldots,z_n) - g_{n-1}(z_1,\ldots,z_{n-1})}{z_n - 1} - z_n D_n g_n(z_1,\ldots,z_n) . \tag{21}$$

When $i = 0$ we note that

$$C_0 g(.,1,z_1,\ldots,z_n,1,.) = \frac{g_n(z_1,\ldots,z_n) - g_{n-1}(z_2,\ldots,z_n)}{z_1 - 1} - D_0 g(.,1,z_1,\ldots,z_n,1,.) . \tag{22}$$

And when $i < 0$ we have

$$C_i g(.,1,z_1,\ldots,z_n,1,.) = D_{i+1}g(.,1,z_1,\ldots,z_n,1,.) - D_i g(.,1,z_1,\ldots,z_n,1,.) . \tag{23}$$

The mixture condition, proved in [3], states that far blocks or groups of blocks tend to behave independently when the distance tends to infinity. It is equivalent to the following convergence, true for any integers n and ℓ ($\ell \leq n$):

$$\lim_{k \to \infty} g(.1,z_1,\ldots,z_\ell,\underbrace{1,\ldots,1}_{k \text{ times}},z_{\ell+1},\ldots,z_n,1,.) = g_\ell(z_1,\ldots,z_\ell)g_{n-\ell}(z_{\ell+1},\ldots,z_n) . \tag{24}$$

Thus

$$\lim_{i \to -\infty} D_i g(.,1,z_1,\ldots,z_n,1,.) = g_1'(1)g_n(z_1,\ldots,z_n) , \tag{25}$$

where $g_1'(1)$ is the first derivative of $g_1(z_1)$ at $z_1 = 1$. Thus we again obtain the fundamental equation

$$\sum_{i=0}^{i=n}(C_i + \lambda R_i)g_n(z_1,\ldots,z_n) = 0 \tag{26}$$

but the simple corrections in operators C_0 and R_0:

$$C_0 g_n(z_1,\ldots,z_n) = \frac{g_n(z_1,\ldots,z_n) - g_{n-1}(z_2,\ldots,z_n)}{z_1 - 1} - g_1'(1)g_n(z_1,\ldots,z_n) , \tag{27}$$

and

$$R_0 g_n(z_1, \ldots, z_n) = -D_1 g_{n+1}(1, z_1, \ldots, z_n) + g_1'(1) g_n(z_1, \ldots, z_n)\,. \tag{28}$$

By extension we can set $g_0 = 1$ and makes the previous formula valid for $n = 1$. Since $g_1'(1) = \lambda/(1-\lambda)$, the general equations are therefore

$$\begin{array}{l} \sum_{i=1}^{i=n} z_i D_i g_n(z_1, \ldots, z_n) - \sum_{i=1}^{i=n-1} c_i g_{n-1}(z_1, \ldots, z_n) - \\ - \frac{g_n(z_1,\ldots,z_n) - g_{n-1}(z_2,\ldots,z_n)}{z_1 - 1} - \frac{g_n(z_1,\ldots,z_n) - g_{n-1}(z_1,\ldots,z_{n-1})}{z_n - 1} = \\ \lambda[\sum_{i=1}^{i=n} \rho_i g_{n+1}(z_1, \ldots, z_n) - r_i g_n(z_1, \ldots, z_n) - \\ -g_n(z_1, \ldots, z_n) - D_1 g_{n+1}(1, z_1, \ldots, z_n)] \end{array} \tag{29}$$

Note that the part before the equal sign of the above concerns with g_n and g_{n-1} and decreases degrees of monomials. The part after the equal sign concerns with g_{n+1} and g_n and increases degrees of monomials.

4 Resolution of equations

4.1 How miracles simply do not occur

A tempting hypothesis is to suppose a product form for $g(., z_1, z_2, .) = \prod_{i \in Z} h(z_i)$. Indeed this form, mapped in equation (29), divided by $g(., z_1, z_2, .)$ gives

$$\begin{array}{l} \sum_{i=1}^{i=n} h'(z_i)/h(z_i) - \sum_{i=1}^{i=n-1} \frac{1/h(z_i) - 1/h(z_{i+1}}{z_i - z_{i+1}} - \\ - \frac{1 - 1/h(z_1)}{1 - z_1} - \frac{1/h(z_n) - 1}{z_n - 1} = \\ \lambda[-1 - h'(1) + \sum_{i=1}^{i=n} z_i h(z_i) + z_i^2 h'(z_i) - 1 - z_i h'(z_i)/h(z_i)]\,, \end{array} \tag{30}$$

where h' denotes the first derivative of function h. Let us try the above formula for the special case where the z_i are all equal to a single variable x:

$$\begin{array}{c} nh'(x)/h(x) - (n-1)h'(x)/h^2(x) - 2\frac{1/h(x) - 1}{x - 1} = \\ \lambda[-1/(1-\lambda) + n(xh(x) + x^2 h'(x) - 1 - xh'(x)/h(x))]\,. \end{array} \tag{31}$$

It is clear that such equation can be satisfied if and only if function h verifies both differential equations:

$$h'(x)/h^2(x) - 2\frac{1/h(x) - 1}{x - 1} = -\frac{\lambda}{1 - \lambda}\,, \tag{32}$$

and

$$h'(x)/h(x) = \lambda[-\frac{1}{1-\lambda} + xh(x) + x^2 h'(x) - 1 - xh'(x)/h(x)]\,. \tag{33}$$

Such coincidence is impossible. The product form does not hold.

4.2 Taylor expansion of the generating function

4.2.1 Polynomials of degree k

The functions g_n's are in relation according to a simple recursion:

$$g_n(z_1, \ldots, z_n) = g_{n+1}(1, z_1, \ldots, z_n) = g_{n+1}(z_1, \ldots, z_n, 1)\,. \tag{34}$$

In the sequel we adopt the change of variable $z_i = 1 + s_i$. Let us consider the Taylor expansion of the g_n's with respect to the parameter λ:

$$g_n(z_1, \ldots, z_n) = \sum_{k=0}^{\infty} \lambda^k P_n^k(s_1, \ldots, s_n)\,. \tag{35}$$

It is clear that $P_n^k(0,\ldots,0) = 0$ except when $k = 0$, where $P_n^0(s_1,\ldots,s_n) = 1$. We also have $P_{n+1}^k(0, s_1,\ldots,s_n) = P_{n+1}^k(s_1,\ldots,s_n,0) = P_n^k(s_1,\ldots,s_n)$ because the property stil holds for the g_n's.

We have the elementary recursion derived from (29):

$$\begin{array}{l} \sum_{i=1}^{i=n}(1+s_i)D_iP_n^{k+1}(s_1,\ldots,s_n) - \sum_{i=1}^{i=n-1} c_iP_{n-1}^{k+1}(s_1,\ldots,s_n) - \\ -\frac{P_n^{k+1}(s_1,\ldots,s_n)-P_{n-1}^{k+1}(s_2,\ldots,s_n)}{s_1} - \frac{P_n^{k+1}(s_1,\ldots,s_n)-P_{n-1}^{k+1}(s_1,\ldots,s_{n-1})}{s_n} = \\ \sum_{i=1}^{i=n}(\rho_i+\rho_i'+\rho_i'')P_{n+1}^k(s_1,\ldots,s_n) - [1+(1+s_i)D_i]P_n^k(s_1,\ldots,s_n) \\ -P_n^k(s_1,\ldots,s_n) - D_1P_{n+1}^k(0,s_1,\ldots,s_n)\,, \end{array} \tag{36}$$

With ρ_i, ρ_i' and ρ_i'' be defined according to the change of variable: if f_{n+1} is a function of $n+1$ variables we have

$$\begin{aligned} \rho_i f_{n+1}(s_1,\ldots,s_n) \;=\; & s_i f_{n+1}(s_1,\ldots,s_{i-1},s_i,s_i,s_{i+1},\ldots,s_n)+ \\ & +(s_i)^2 D_i f_{n+1}(s_1,\ldots,s_{i-1},s_i,s_i,s_{i+1},\ldots,s_n)\,, \end{aligned} \tag{37}$$

$$\begin{aligned} \rho_i' f_{n+1}(s_1,\ldots,s_n) \;=\; & f_{n+1}(s_1,\ldots,s_{i-1},s_i,s_i,s_{i+1},\ldots,s_n)+ \\ & +2s_iD_i f_{n+1}(s_1,\ldots,s_{i-1},s_i,s_i,s_{i+1},\ldots,s_n)\,, \end{aligned} \tag{38}$$

and

$$\rho_i'' f_{n+1}(s_1,\ldots,s_n) = D_i f_{n+1}(s_1,\ldots,s_{i-1},s_i,s_i,s_{i+1},\ldots,s_n)\,. \tag{39}$$

Note that this equation is not valid for $n = 0$ without appropriate interpretations.

We will first establish an easy proposition, namely the fact that the P_n^k's are polynomials of degrees less than or equal to k. For this purpose, we note that $P_n^0 = 1$ (the proposition is still valid for $k = 0$) and we make use of the above formula (36), taking care of the fact that operators c_i's, ρ_i'' and D_i decrease degrees of polynomial of one, operators s_iD_i and ρ_i' let degrees steady and operators ρ_i increase degrees by one. For instance we have $P_n^1(s_1,\ldots,s_n) = s_1 + \cdots + s_n$.

It is interesting to cut polynomials P_n^k in slices. Let $Q_n^k[m]$ be the polynomial formed by the monomials of P_n^k of degree m. We know that $Q_n^k[m] = 0$ when $m > k$. Using equation (36) and translating it term by term we get the following recursion:

$$\begin{array}{l} mQ_n^{k+1}[m](s_1,\ldots,s_n) + \sum_{i=1}^{i=n} D_iQ_n^{k+1}[m+1](s_1,\ldots,s_n) - \\ -\sum_{i=1}^{i=n-1} c_iQ_{n-1}^{k+1}[m+1](s_1,\ldots,s_n) - \\ -1/s_1(Q_n^{k+1}[m+1](s_1,\ldots,s_n) - Q_{n-1}^{k+1}[m+1](s_2,\ldots,s_n)) \\ -1/s_n(Q_n^{k+1}[m+1](s_1,\ldots,s_n) - Q_{n-1}^{k+1}[m+1](s_1,\ldots,s_{n-1})) = \\ \sum_{i=1}^{i=n}(\rho_iQ_{n+1}^k[m-1] + \rho_i'Q_{n+1}^k[m] + \rho_i''Q_{n+1}^k[m+1])(s_1,\ldots,s_n) - \\ -Q_n^k[m](s_1,\ldots,s_n) - D_1Q_{n+1}^k[m+1](0,s_1,\ldots,s_n)\,. \end{array} \tag{40}$$

This construction allows to express $Q_n^{k+1}[m]$ as function of $Q_n^{k+1}[m+1]$ and $Q_{n+1}^k[m-1]$, $Q_{n+1}^k[m+1]$ and $Q_{n+1}^k[m]$. The later leads to exact evaluation of the $Q_n^k[m]$'s. and furthermore an accurate Taylor expansion with respect to λ of the $g_n(z_1,\ldots,z_n)$, with $z_i = 1 + s_i$. Below we give examples of such result.

$$\begin{aligned} g_1(z) \;=\; & 1-\lambda+1/2\lambda^2-\tfrac{1}{2}\lambda^3+\tfrac{5}{12}\lambda^4-\tfrac{91}{144}\lambda^5+\tfrac{3173}{2880}\lambda^6-\tfrac{694171}{259200}\lambda^7+ \\ & +z(\lambda-2\lambda^2+\tfrac{8}{3}\lambda^3-\tfrac{59}{18}\lambda^4+\tfrac{31}{8}\lambda^5-\tfrac{10879}{2160}\lambda^6+\tfrac{137953}{17280}\lambda^7)+ \\ & +z^2(\tfrac{3}{2}\lambda^2-\tfrac{29}{6}\lambda^3+\tfrac{685}{72}\lambda^4-\tfrac{1409}{96}\lambda^5+\tfrac{82903}{4320}\lambda^6-\tfrac{4691641}{207360}\lambda^7)+ \\ & +z^3(\tfrac{8}{3}\lambda^3-\tfrac{427}{36}\lambda^4+\tfrac{21641}{720}\lambda^5-\tfrac{1203289}{21600}\lambda^6+\tfrac{42754117}{518400}\lambda^7)+ \\ & +z^4(\tfrac{125}{24}\lambda^4-\tfrac{14123}{480}\lambda^5+\tfrac{435959}{4800}\lambda^6-\tfrac{41239477}{207360}\lambda^7)+ \\ & +z^5(\tfrac{54}{5}\lambda^5-\tfrac{530753}{7200}\lambda^6+\tfrac{162089117}{604800}\lambda^7)+ \\ & +z^6(\tfrac{16807}{720}\lambda^6-\tfrac{18778177}{100800}\lambda^7)+ \\ & +\tfrac{16384}{315}z^7\lambda^7+O(\lambda^8)\,. \end{aligned} \tag{41}$$

$$
\begin{aligned}
g_1(z) = {} & 1 + \frac{531441\,\lambda^8 z^8}{4480} + \lambda\, z - \lambda - \frac{24688039297\,\lambda^9 z^8}{20321280} - \frac{10879\,\lambda^6 z}{2160} + \frac{3\,\lambda^2 z^2}{2} \\
& -2\,\lambda^2 z + \frac{82903\,\lambda^6 z^2}{4320} + \frac{8\,\lambda^3 z^3}{3} + \frac{435959\,\lambda^6 z^4}{4800} - \frac{29\,\lambda^3 z^2}{6} + \frac{8\,\lambda^3 z}{3} + \frac{125\,\lambda^4 z^4}{24} \\
& - \frac{18778177\,\lambda^7 z^6}{100800} - \frac{1409\,\lambda^5 z^2}{96} + \frac{31\,\lambda^5 z}{8} - \frac{427\,\lambda^4 z^3}{36} + \frac{685\,\lambda^4 z^2}{72} - \frac{59\,\lambda^4 z}{18} + \frac{21641\,\lambda^5 z^3}{720} \\
& + \frac{54\,\lambda^5 z^5}{5} - \frac{530753\,\lambda^6 z^5}{7200} + \frac{16807\,\lambda^6 z^6}{720} - \frac{1203289\,\lambda^6 z^3}{21600} - \frac{14123\,\lambda^5 z^4}{480} \\
& - \frac{34521705241\,\lambda^8 z^5}{50803200} - \frac{441765970009\,\lambda^9}{182891520 0} - \frac{17127333310379\,\lambda^9 z^6}{7620480000} + \frac{16384\,\lambda^7 z^7}{315} - \frac{4691641\,\lambda^7 z^2}{207360} \\
& + \frac{137953\,\lambda^7 z}{17280} + \frac{162089117\,\lambda^7 z^5}{604800} + \frac{42754117\,\lambda^7 z^3}{518400} - \frac{41239477\,\lambda^7 z^4}{207360} + \frac{156250\,\lambda^9 z^9}{567} \\
& + \frac{98943238789\,\lambda^8 z^6}{127008000} + \frac{65731033\,\lambda^8}{8709120} - \frac{1003914803\,\lambda^8 z^7}{2116800} + \frac{\lambda^2}{2} - \frac{\lambda^3}{2} \\
& + \frac{5\,\lambda^4}{12} + \frac{3173\,\lambda^6}{2880} - \frac{91\,\lambda^5}{144} - \frac{694171\,\lambda^7}{259200} + \frac{8778415733\,\lambda^8 z^2}{326592000} \\
& - \frac{2803202123\,\lambda^8 z}{163296000} - \frac{33699113101\,\lambda^8 z^3}{326592000} + \frac{74473085497\,\lambda^8 z^4}{217728000} + \frac{569277689609\,\lambda^9 z^7}{254016000} \\
& - \frac{32341455519467\,\lambda^9 z^4}{65318400000} - \frac{1647101993479\,\lambda^9 z^2}{39191040000} + \frac{470099407487\,\lambda^9 z}{9797760000} \\
& + \frac{3187239000131\,\lambda^9 z^5}{2381400000} + \frac{11825080488629\,\lambda^9 z^3}{97977600000} + O(\lambda^{10})\,.
\end{aligned}
\tag{42}
$$

And for two variables

$$
\begin{aligned}
g_2(z_1, z_2) = {} & 1 + \lambda\, z_1 - 2\,\lambda + \frac{3\,\lambda^2 z_1^2}{2} - 3\,\lambda^2 z_1 + \frac{8\,\lambda^3 z_1^3}{3} - \frac{19\,\lambda^3 z_1^2}{3} + \frac{29\,\lambda^3 z_1}{6} \\
& + \frac{125\,\lambda^4 z_1^4}{24} - \frac{1046\,\lambda^5 z_1^2}{45} + \frac{1309\,\lambda^5 z_1}{288} - \frac{523\,\lambda^4 z_1^3}{36} + \frac{259\,\lambda^4 z_1^2}{18} \\
& - \frac{403\,\lambda^4 z_1}{72} + \frac{30127\,\lambda^5 z_1^3}{720} + \frac{54\,\lambda^5 z_1^5}{5} - \frac{5541\,\lambda^5 z_1^4}{160} + 2\,\lambda^2 - \frac{5\,\lambda^3}{3} \\
& + \frac{17\,\lambda^4}{18} + \frac{\lambda^5}{18} + \frac{1105\,\lambda^5 z_2}{288} - \frac{15911\,\lambda^5 z_2^2}{720} + \frac{7453\,\lambda^5 z_2^3}{180} \\
& - \frac{5541\,\lambda^5 z_2^4}{160} + \frac{54\,\lambda^5 z_2^5}{5} - \frac{9547\,\lambda^5 z_1 z_2}{1440} + \frac{8\,\lambda^3 z_2^3}{3} - \frac{395\,\lambda^4 z_2}{72} \\
& + \frac{257\,\lambda^4 z_2^2}{18} - \frac{523\,\lambda^4 z_2^3}{36} + \frac{125\,\lambda^4 z_2^4}{24} + \frac{3\,\lambda^2 z_2^2}{2} + \frac{29\,\lambda^3 z_2}{6} \\
& - \frac{19\,\lambda^3 z_2^2}{3} + \lambda\, z_2 - 3\,\lambda^2 z_2 + \lambda^2 z_1 z_2 + \frac{3\,\lambda^3 z_1^2 z_2}{2} - \frac{11\,\lambda^3 z_1 z_2}{3} \\
& + \frac{3\,\lambda^3 z_1 z_2^2}{2} + \frac{20\,\lambda^4 z_1 z_2}{3} - \frac{57\,\lambda^4 z_1^2 z_2}{8} - \frac{505\,\lambda^4 z_1 z_2^2}{72} + \frac{8\,\lambda^4 z_1^3 z_2}{3} \\
& + \frac{9\,\lambda^4 z_1^2 z_2^2}{4} + \frac{8\,\lambda^4 z_1 z_2^3}{3} + \frac{4139\,\lambda^5 z_1^2 z_2}{240} + \frac{161\,\lambda^5 z_1 z_2^2}{10} - \frac{5683\,\lambda^5 z_1^3 z_2}{360} \\
& - \frac{18257\,\lambda^5 z_1^2 z_2^2}{1440} - \frac{11051\,\lambda^5 z_1 z_2^3}{720} + \frac{125\,\lambda^5 z_1^4 z_2}{24} + 4\,\lambda^5 z_1^3 z_2^2 + 4\,\lambda^5 z_1^2 z_2^3 + \frac{125\,\lambda^5 z_1 z_2^4}{24} + O(\lambda^6)\,.
\end{aligned}
\tag{43}
$$

We also have

$$
\begin{aligned}
g_1(0) &= 1 - \lambda + \tfrac{1}{2}\lambda^2 - \tfrac{1}{2}\lambda^3 + \tfrac{5}{12}\lambda^4 - \tfrac{91}{144}\lambda^5 + O(\lambda^6) \\
g_2(0,0) &= 1 - 2\lambda + 2\lambda^2 - \tfrac{5}{3}\lambda^3 + \tfrac{17}{18}\lambda^4 + \tfrac{1}{18}\lambda^5 + O(\lambda^6) \\
g_3(0,0,0) &= 1 - 3\lambda + \tfrac{9}{2}\lambda^2 - \tfrac{29}{6}\lambda^3 + \tfrac{287}{72}\lambda^4 - \tfrac{833}{288}\lambda^5 + O(\lambda^6) \\
g_4(0,\ldots,0) &= 1 - 4\lambda + 8\lambda^2 - 11\lambda^3 + \tfrac{209}{18}\lambda^4 - \tfrac{14657}{1440}\lambda^5 + O(\lambda^6) \\
g_5(0,\ldots,0) &= 1 - 5\lambda + \tfrac{25}{2}\lambda^2 - \tfrac{127}{6} + \tfrac{1967}{72}\lambda^4 + O(\lambda^5)\,.
\end{aligned}
\tag{44}
$$

and we may derive

$$
g_n(0,\ldots,0) = 1 - n\lambda + \frac{n^2\lambda^2}{2} + \left(-\frac{n^3}{6} - 1/3\right)\lambda^3 + O(\lambda^4)\,, \tag{45}
$$

4.2.2 Global weight of monomials of degree k

Our aim is to have an exact estimate of $Q_n^k[k]$, in words the global weight of monomials of degree k in P_n^k. For notational convenience let us introduce the polynomial $\sigma_n^k(s_1,\ldots,s_n) = Q_n^k[k](s_1,\ldots,s_n)$. Retaining only monomials of degree $k+1$ in equation (36), or letting $m = k+1$ in (40) we get the simple recursion:

$$
(k+1)\sigma_n^{k+1}(s_1,\ldots,s_n) = \sum_{i=1}^{i=n} s_i r_i \sigma_{n+1}^k(s_1,\ldots,s_i,s_i,\ldots,s_n)\,. \tag{46}
$$

From this we can easily state that for all n and k $\sigma_n^k(s_1,\ldots,s_n)$ is invariant with respect to any permutation over variables $s_1,\ldots,s_n$. For this purpose, we note that the proposition is still valid for $k=0$ and $k=1$, and we make use of the fact that if $\sigma_{n+1}^k(s_1,\ldots,s_{n+1}$ is invariant with respect to any permutation of its variables, so is $\sum_{i=1}^{i=n} s_i r_i \sigma_{n+1}(s_1,\ldots,s_i,s_i,\ldots,s_n)$. We note that $\sigma_n^0=1$, $\sigma_n^1(s_1,\ldots,s_n)=S_1$, $\sigma_n^2(s_1,\ldots,s_n)=1/2(2S_2+(S_1)^2)$ and $\sigma_n^3(s_1,\ldots,s_n)=1/6(9S_3+6(S_2)(S_1)+(S_1)^3)$, with $S_\ell = s_1^\ell+\cdots+s_n^\ell$.

Let $\sigma_n^k(1)$ be a short hand notation for $\sigma_n^k(1,\ldots,1)$. It is clear that $\sigma_n^k(1)$ is the global weight of all the monomials of $P_n^k(s_1,\ldots,s_n)$ of degree k. From the above property we can derive the fact that $\sigma_n^k(1)=n(n+k)^{k-1}/k!$. The proof follows. We have

$$(k+1)\sigma_n^{k+1}(1)=n\sigma_{n+1}^k(1)+\sum_{i=1}^{n} D_i\sigma_{n+1}^k(1,\ldots,1)\,. \tag{47}$$

Since $\sum_{i=1}^{n+1} D_i\sigma_{n+1}^k(1,\ldots,1)=k$, we have $D_i\sigma_{n+1}^k(1,\ldots,1)=k/(n+1)$ because of the invariance with respect to variable permutation. Therefore

$$(k+1)\sigma_n^{k+1}(1)=(1-\frac{1}{n+1})(n+1+k)\sigma_{n+1}^k(1)\,. \tag{48}$$

The fact that $\sigma_n^0(1)=1$ terminates the proof. We note that the invariance to variable permutation was not completely necessary to the proof, since we simply have the invariance to circular permutation: $\sigma_{n+1}^k(s_1,\ldots,s_n,1)=\sigma_{n+1}(1,s_1,\ldots,s_n)$.

Remark: Note that $\sigma_k^n(1)$ only depends on the recombination operator, since succession of k recombinations, each of one costing $O(\lambda)$, is the only way obtain blocks of global size k starting from scratch with a probability cost of $O(\lambda^k)$. If a break occur somewhere we should need more than $k+1$ recombinations for the same result and therefore a probability cost of $O(\lambda^{k+1})$. Thus our result can be extended to cases where sojourn time has more sophisticated distribution with mean one.

A more involved analysis should have given

$$\sigma_n^k(s_1,\ldots,s_n)=\sum_{k_1+\cdots+k_n=k}\prod_{i=1}^{i=n}\frac{(k_i+1)^{k_i}}{(k_i+1)!}s_i^{k_i}\,, \tag{49}$$

which leads to interesting combinatorial identities.

4.2.3 One more conjecture down

The identity $\sigma_1^k(1)=(k+1)^k/(k+1)!$ makes $g_1(z)$ a good candidate to be identical to the *p.g.f* $h(z)$ of block size in a hashing table of occupation rate λ without deletion. We have the equation

$$h(z)=\exp\{\lambda(zh(z)-1)\} \tag{50}$$

and the expansion

$$h(z)=\sum_{k=0}^{\infty}\frac{(k+1)^k e^{-(k+1)\lambda}}{(k+1)!}\lambda^k z^k\,. \tag{51}$$

But the proposition is not valid because it makes $h(0)=e^{-\lambda}=1-\lambda+1/2\lambda^2-1/6\lambda^3+O(\lambda^4)$ which is not satisfied by $g_1(0)=1-\lambda+1/2\lambda^2-1/2\lambda^3+O(\lambda^4)$.

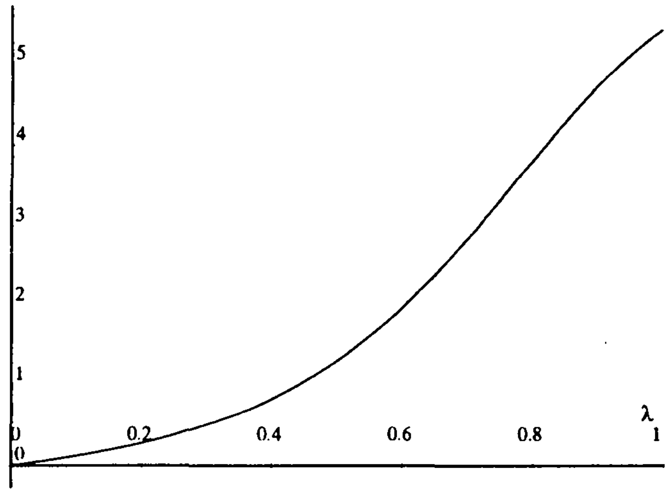

Figure 2: Mean access delay *versus* λ

4.2.4 Exploitation of results: distribution of access delay

Considering a random item, an interesting parameter is the number of busy cells it has to visit before finding a free cell. This random variable can be seen as a random access delay in the Ring network, or the time the woman has to wait before the car stops in the parking problem. The mean number of visited cells by a random item is

$$\frac{1/2g_1''(1) + g_1'(1)}{1 + g_1'(1)}, \tag{52}$$

g_1' and g_1'' are respectively first and second derivative of g_1. The distribution of the number of cells visited by a random item has the following probability generating function

$$\frac{zg_1(z) - 1}{(z-1)(1+g_1'(1))}. \tag{53}$$

We already know that $g'(1) = \lambda/(1-\lambda) = \lambda+\lambda^2+\lambda^3+\lambda^4+\lambda^5+\lambda^5+\lambda^6+\lambda^7+\lambda^8+\lambda^9+O(\lambda^{10})$, and $g_1''(1) = 3\lambda^2 + \frac{19}{3}\lambda^3 + \frac{373}{36}\lambda^4 + \frac{1113}{80}\lambda^5 + \frac{216101}{10800}\lambda^6 + \frac{68672749}{3628800}\lambda^7 + \frac{50132888261}{1143072000}\lambda^8 - \frac{17626151640329}{685843200000}\lambda^9 + O(\lambda^{10})$, and so forth. In figure 2 and 3 we respectively plot mean and variance of access delays.

References

[1] D. Aldous, M. Hofri, W. Szpankowski "Maximum size of a dynamic data structure: Hashing with Lazy deletion revisited," SIAM J. Computing, to appear.

[2] R. Falconer, J. Adams, "Orwell: a protocol for an integrated services local network," British Telecom. Technology Jour., Vol 3., pp. 27-35, 1985.

[3] V. Malyshev and P. Robert "Phase transition in a loss load sharing model," INRIA Research Report 1602, 1992.

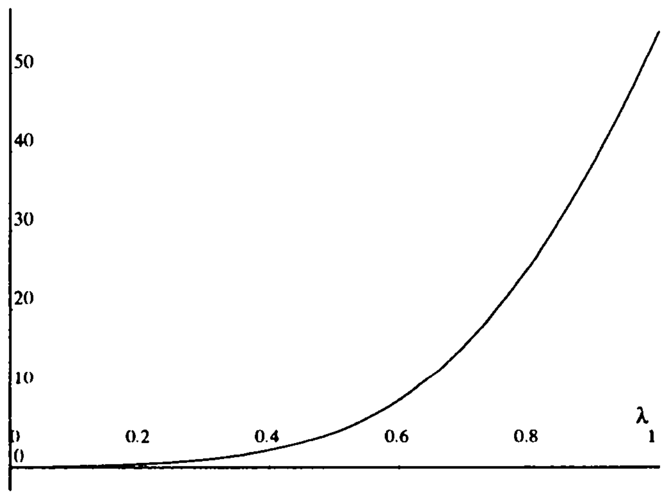

Figure 3: Variance of access delay *versus* λ

ISSN 0249-6399