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## *Programme 1*

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## ANALYSIS OF SOME NETWORKS WITH LOCAL INTERACTION

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## Analyse de certains réseaux à interaction locale

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### Résumé.

Nous étudions un modèle stochastique de réseau introduit récemment pour l'analyse de certains réseaux de neurones. Pour ce modèle l'interaction entre les nœuds est locale. À chaque nœud est associé un nombre positif, son inhibition, celle-ci décroît linéairement au cours du temps. Quand l'inhibition atteint la valeur 0, le nœud envoie une décharge sur ses voisins qui augmente leurs inhibitions respectives d'une certaine quantité aléatoire. L'inhibition du nœud qui a déchargé prend alors une nouvelle valeur aléatoire. Nous nous intéressons aux conditions pour lesquelles chaque nœud est assuré de décharger une infinité de fois (stabilité) et, quand ces conditions ne sont pas remplies, quels sont les nœuds qui vont mourir, i.e. ne plus décharger du tout à partir d'un certain moment. Nous étudions principalement deux topologies pour le graphe du réseau, la topologie complètement connectée et le réseau linéaire où les nœuds sont situés sur une droite. Une quantité  $\rho$  est associée à chacun de ces réseaux et on montre que les propriétés de stabilité ne dépendent que de ce paramètre.

Pour le réseau complètement connecté, nous donnons la condition nécessaire et suffisante de stabilité ( $\rho < 1$ ) de même que l'expression explicite de la transformée de Laplace de la mesure invariante du processus de Markov associé. Dans le cas du réseau linéaire, la situation est plus intéressante. Nous montrons que la valeur critique de  $\rho$  pour la stabilité est  $1/2$  si le nombre de nœuds est impair et  $\frac{1}{2 \cos \pi / (N+1)}$  sinon. Quand ce réseau n'est pas stable, nous étudions les différents états asymptotiques possibles.

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## Analysis of some networks with local interaction

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### Abstract.

In this paper we study a stochastic network model introduced recently in the analysis of neural networks. In this model the interaction between the nodes of the network is local: to each node is associated some real number (the *inhibition* in the language of neural networks) which is decreasing linearly with time. When this number reaches 0, it sends out some random input to its neighbors (a *spike*) and restarts with some random value. We are interested in the asymptotic behavior of the network, that is, under which conditions the nodes do not *die* (i.e. their inhibition is not converging to infinity). When these conditions are not satisfied, we analyze the set of nodes which are likely to die. We consider networks with a finite number of nodes and two kind of topologies, the *fully connected* network and related graphs and the linear network where the nodes are located on a line. A quantity  $\rho$  is associated to this network and the stability properties of the network depend only on it.

For the fully connected network, we give the necessary and sufficient condition for the stability of the network ( $\rho < 1$ ) as well as the explicit expression for the invariant measure of the Markov process associated to this model. For the stability of the linear network of size  $N$ , we prove that the critical value for  $\rho$  is  $1/2$  if  $N$  is odd, otherwise it is the constant  $\frac{1}{2 \cos \pi/(N+1)}$ . When the network is not stable, the set of possible asymptotic states is analyzed.

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## Abstract

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# 1 Introduction, description of the model

We consider a locally interacting process on a finite network. This mathematical model has been introduced in [3] to analyze the behavior of some neural cells in the cortex. A non negative real number is associated to every node: its *inhibition*. The inhibition represents the duration of time after which the node will modify the state of other nodes if no interaction takes place meanwhile. As long as the inhibition of a node is strictly positive, it decreases linearly with time and the node does not modify the behavior of its neighbors. When its inhibition is 0, the node fires, i.e. the inhibition of its neighbors is increased by some random quantity (the node is also said to *send out a spike*). Throughout this paper, we will use the picturesque terminology of neural networks in the analysis of our model.

Our network is a finite graph  $\mathcal{G}$  with  $N$  nodes. The stochastic process  $(X_t(i))_{1 \leq i \leq N} \in \mathbb{R}_+^N, t \in \mathbb{R}_+$  describing the state of our network evolves as follows :

From an initial configuration  $(X_0(i), 1 \leq i \leq N)$ , every coordinate decreases linearly in time with slope 1 until the first time  $T_1$  when the state of some node  $i_1$  reaches 0. At this moment, the component with index  $i_1$  restarts with a new value, an exponentially distributed random variable  $G_{i_1}$  (with parameter  $\lambda_{i_1}$ ). At the same time, the node  $i_1$  sends out a spike to all its neighbors, i.e. increases the value of their components by the same amount, a random variable with distribution  $F_{i_1}$ . Starting from this new configuration  $(X_{T_1}(i), 1 \leq i \leq N)$ , we iterate this procedure until time  $T_2$  of the second spike, and so on... A right-continuous Markov process  $(\underline{X}_t)_{t \geq 0}$  is thus defined in this way. Since its evolution between two successive spikes is deterministic (linear decay with rate 1), there is a natural embedded Markov chain at the instants of spikes:  $(\underline{X}_n)_{n \in \mathbb{N}} = (X_{T_n}(i), 1 \leq i \leq N)_{n \in \mathbb{N}}$ .

The first results on this model were obtained in [3]. It was assumed that the  $F_i$ 's were Dirac measure at some  $\theta > 0$ . It was proved that the Markov chain  $(\underline{X}_n)_{n \in \mathbb{N}}$  is Lebesgue-irreducible, aperiodic and that if  $\theta < \min_i \frac{E(G_i)}{V_i}$ , it is positive recurrent ( $V_i$  is the number of neighbors of the node  $i$  and  $\bar{E}(\cdot)$  denotes the expected value). As we will see in the following, this condition is in general not necessary for the stability of the network.

We begin by the case of a completely connected graph (i.e. every pair of nodes is linked by an edge of the graph). For this topology, the necessary and sufficient condition for the stability of the network is obtained:  $\min_i \rho_i =$



$\min_i E(\theta_i)/E(G_i) < 1$ , where  $\theta_i$  is some random variable with distribution  $F_i$ . Under this stability condition, we give the explicit expression for the Laplace transform of the invariant measure of the Markov process associated to this model. When the network is not stable, its asymptotic behavior is fairly simple to describe: only one node does not *die*, i.e. after some time all the other nodes do not fire anymore.

For the case of a linear network of size  $N$ , the behavior is quite interesting. We slightly modify the interaction between nodes: at the moment of a spike, instead of sending the same random variable to all its neighbors, a node sends out independent identically distributed random variables. We assume that the  $F_i$ 's [resp  $G_i$ ] are exponentially distributed with parameter  $\mu$  [resp.  $\lambda$ ]. We prove the following results:

- a) If  $N$  is odd, the network is stable if  $\rho = \frac{\lambda}{\mu} < 1/2$  and non stable if  $\rho > 1/2$ .
- b) If  $N$  is even, then the network is stable if  $\rho < \frac{1}{2 \cos \pi/(N+1)}$  and non stable if  $\rho > \frac{1}{2 \cos \pi/(N+1)}$ .

The proof of ergodicity involves the *second vector field* associated to Markov processes. It has been introduced by V. Malyshev and M. Menshikov in [6] in the analysis of the ergodicity of multi-dimensional random walks on  $\mathbb{N}^n$ . To our knowledge, this is one of the first applications of the second vector field to study the ergodicity of Markov processes in state spaces of dimension  $\geq 5$ .

When the network is not stable, it has different possible asymptotic distributions, depending on its initial state. We call these distributions the stable states of the network. We give a list of possible stable states. It depends mainly on the position of  $\rho$  among the  $\frac{1}{2 \cos \pi/(2p+1)}$ 's for  $p \in \mathbb{N}$ .

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## 2 The completely connected network

We assume in this section that the graph  $\mathcal{G}$  is *fully connected*, i.e. every pair of vertices is connected. For  $1 \leq i \leq N$ , we define  $\rho_i = \lambda_i E(\theta_i)$ ,  $\theta_i$  being some generic random variable with distribution  $F_i$ .

**Theorem 1** *If  $\max_{1 \leq i \leq N} \rho_i < 1$ , the Markov chain  $(\underline{X}_n)_{n \in \mathbb{N}}$  admits the invariant probability measure given by*

$$\pi_N = \sum_{i=1}^N p_i \mu_i$$

where, for  $1 \leq i \leq N$ ,

$$p_i = \frac{\lambda_i}{1 - \rho_i} \left( \sum_{j=1}^N \frac{\lambda_j}{1 - \rho_j} \right)^{-1},$$

and  $\mu_i$  is the distribution of the random vector  $\underline{G} + (W_i + \theta_i)\underline{e}_i$ . The vector  $\underline{e}_i$  has all components 1 except the  $i^{\text{th}}$  one which is 0, and the independent random variables  $\underline{G}, W_i, \theta_i$  satisfy

- a)  $\underline{G} = (G_1, \dots, G_N)$  is a vector of independent exponentially distributed random variables with parameters  $\lambda_1, \dots, \lambda_N$ ,
- b)  $W_i$  has the same distribution as the stationary waiting time of an  $M/G/1$  queue with arrival rate  $\lambda_i$  and service distribution  $F_i$ , i.e.

$$W_i \stackrel{(d)}{=} \max(W_i + \theta_i - G_i, 0), \quad (1)$$

where  $\stackrel{(d)}{=}$  stands for equality of distributions.

### Remark

The Laplace transform of  $\pi_N$  can be explicitated since for  $i \leq N$ , it is well known (see [1] for example) that the Laplace transform of  $W_i$  is given by

$$\tilde{W}_i(\xi) = \frac{1 - \rho_i}{1 - \lambda_i \frac{1 - \tilde{F}_i(\xi)}{\xi}}, \quad \xi \in \mathbb{R}_+, \quad (2)$$

denoting by  $\tilde{D}$  the Laplace transform of a distribution  $D$ .

**Proof:** If  $P$  denotes the transition probabilities of the Markov chain, all we have to prove is that  $\pi_N P = \pi_N$ . We will show the equality of the Laplace transforms,  $\widetilde{\pi_N P} = \widetilde{\pi_N}$ . We fix  $1 \leq i \leq N$ , and we compute  $\widetilde{\mu_i P}$ , i.e. the Laplace transform of the distribution of  $\underline{X}_1$  if the initial distribution of  $\underline{X}_0$  is  $\mu_i$ . Thus the initial inhibition state is  $X_0(i) = G_i$  for node  $i$  and  $X_0(j) = G_j + W_i + \theta_i$  for  $j \neq i$ , where  $W_i, \theta_i, G_k, 1 \leq k \leq N$  are independent. The node with the smallest inhibition state will thus send a spike first. There are two possibilities:

- . If  $G_i < W_i + \theta_i$ , node  $i$  sends out a spike first and  $X_1(i) = G'_i$ ,  $X_1(j) = G_j + (W_i + \theta_i - G_i) + \theta'_i$  for  $j \neq i$ , where  $G'_i$  and  $\theta'_i$  are other independent random variables with respective distributions  $G_i$  and  $F_i$ .
- . If  $G_i \geq W_i + \theta_i$ ,  $X_0(j) \geq W_i + \theta_i$  for every  $1 \leq j \leq N$ . Due to the deterministic evolution of  $(X_t)_{t \in \mathbb{R}_+}$  between spikes,  $P(\underline{x}, \cdot) = P(\underline{y}, \cdot)$  if  $y_j = x_j - a$  ( $1 \leq j \leq N$ ) with  $a \leq \min_{1 \leq j \leq N} x_j$ . Therefore  $\underline{X}_1$  has the same distribution as if the initial state was  $\underline{X}'_0$  with  $X'_0(j) = G_j$  for  $j \neq i$  and  $X'_0(i) = G_i - (W_i + \theta_i)$ . Because of the usual properties of exponential distributions, the distribution of  $\underline{X}'_0$  conditioned on  $\{G_i \geq W_i + \theta_i\}$  is simply  $\nu$ , the distribution of  $\underline{G}$ .

Thus, for  $\underline{\xi} = (\xi_1, \dots, \xi_N) \in \mathbb{R}_+^N$ ,

$$\begin{aligned} \widetilde{\mu_i P}(\underline{\xi}) &= E \left( 1_{\{G_i < W_i + \theta_i\}} e^{-\xi_i G'_i - \sum_{j \neq i} \xi_j (G_j + (W_i + \theta_i - G_i) + \theta'_i)} \right) \\ &\quad + P(G_i \geq W_i + \theta_i) \widetilde{\nu P}(\underline{\xi}). \end{aligned} \tag{3}$$

We are left with the computation of  $\widetilde{\nu P}(\underline{\xi})$ . In this case, we assume that  $\underline{X}_0 = \underline{G}$ , hence if  $G_j = \min_{1 \leq k \leq N} \{G_k\}$ ,  $X_1(j) = G'_j$  and  $X_1(k) = G_k - G_j + \theta'_j$  for  $k \neq j$ . We obtain

$$\widetilde{\nu P}(\underline{\xi}) = \sum_{j=1}^N \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}'_j(\underline{\xi}),$$

where  $\mu'_j$  is the distribution of the random vector  $\underline{G}' + \theta'_j e_j$ . Because of (1) and (2) we have  $P(W_i + \theta_i \leq G_i) = P(W_i = 0) = 1 - \rho_i$ . Thus by (1), we

can rewrite equation (3) as

$$\widetilde{\mu}_i P = \tilde{\mu}_i - (1 - \rho_i) \tilde{\mu}'_i + (1 - \rho_i) \sum_{j=1}^N \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}'_j$$

At this point it is clear that a sufficient condition for the stationarity of  $\pi_N = \sum_{i=1}^N p_i \mu_i$  is that  $p_1, \dots, p_N$  satisfy

$$-\sum_{i=1}^N p_i (1 - \rho_i) \tilde{\mu}'_i + \sum_{i=1}^N p_i (1 - \rho_i) \left( \sum_{j=1}^N \frac{\lambda_j}{\sum_{k=1}^N \lambda_k} \tilde{\mu}'_j \right) = 0$$

or that for every  $i$ ,  $1 \leq i \leq N$ ,

$$p_i (1 - \rho_i) = \frac{\lambda_i}{\sum_{k=1}^N \lambda_k} \sum_{j=1}^N p_j (1 - \rho_j).$$

Hence,  $p_i = C \frac{\lambda_i}{1 - \rho_i}$ . The normalization relation  $\sum_{i=1}^N p_i = 1$  ( $\pi_N$  is a probability) gives the value of  $C$ . Our theorem is proved. ■

We can now state our stability result for the fully connected network.

**Proposition 2** *When  $\max_{1 \leq j \leq N} \rho_j < 1$ , the Markov chain  $(\underline{X}_n)_{n \in \mathbb{N}}$  is Harris ergodic.*

### Proof

We obtain this proposition by proving the coupling property for the Markov process  $(\underline{X}_t)_{t \geq 0}$ . Recall ([1]) that a Markov process  $(Y_s)_{s \in S}$  (where  $S$  is either  $\mathbb{N}$  or  $\mathbb{R}_+$ ) has the coupling property if for any probability distributions  $\mu_0^1$  and  $\mu_0^2$ , one can construct a probability space and two stochastic processes  $(Y_s^1)_{s \in S}$  and  $(Y_s^2)_{s \in S}$  on it such that

- a) the process  $(Y_s^i)_{s \in S}$  is Markov with the same transition probabilities as  $(Y_s)_{s \in S}$  for  $i = 1, 2$
- b) the distribution of  $Y_0^i$  is  $\mu_0^i$  for  $i = 1, 2$
- c) there exists some random time after which the two processes are identical.

Coupling for the Markov process  $(\underline{X}_t)_{t \geq 0}$  clearly implies coupling for the embedded Markov chain  $(\underline{X}_n)_{n \in \mathbb{N}}$ . The coupling property and the existence of an invariant measure are sufficient to prove Harris ergodicity (see [1], Proposition 3.13 p. 157). We prove a stronger result, i.e. that starting from any initial condition, there exists some random time when the distribution of the process is a product of independent exponential distributions. Hence at some random time, the components of  $\underline{X}$  are independent. Notice nevertheless that for the stationary distribution  $\pi_N$ , the components are not independent.

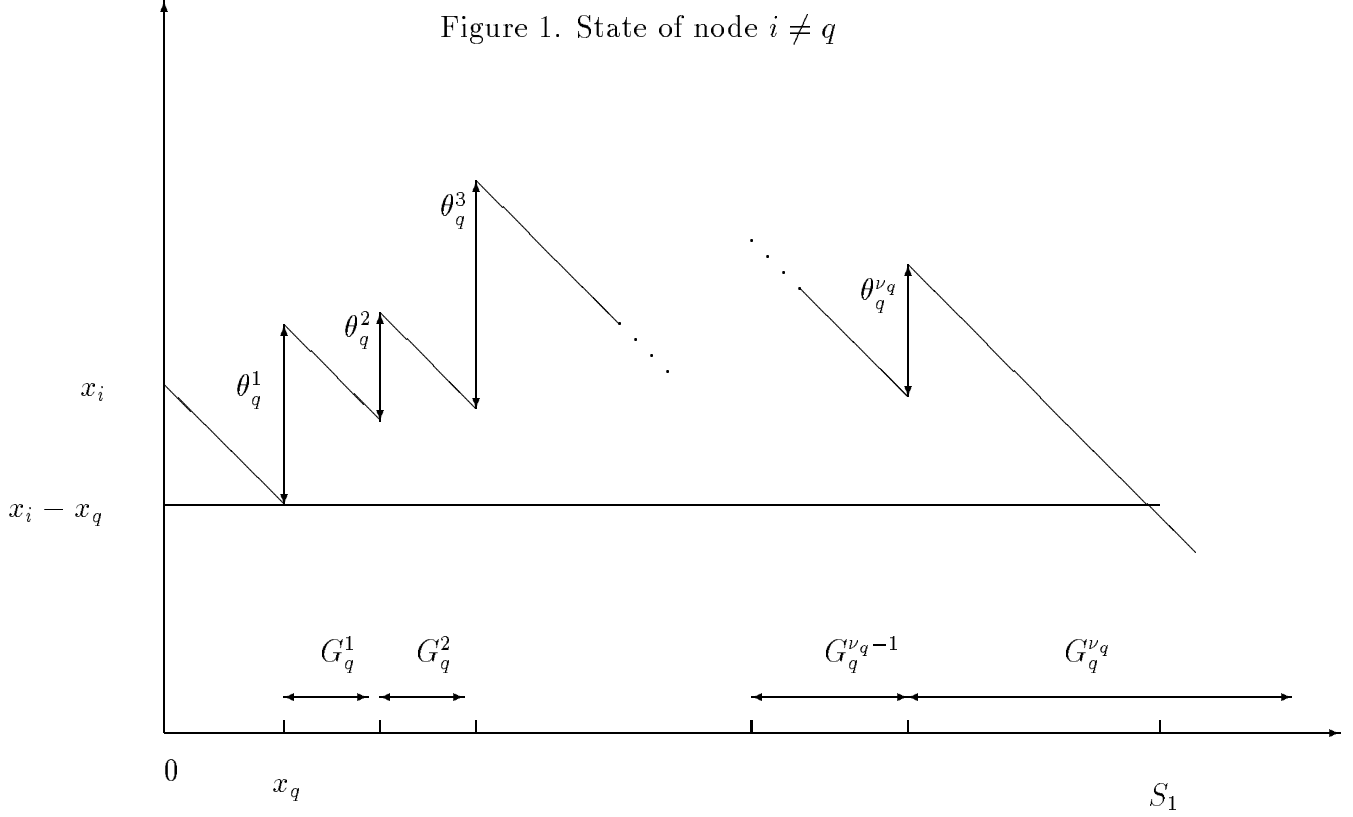
If  $(x_1, \dots, x_N)$  is the initial state of the network, let us assume that  $x_q$  is the smallest of the  $x_i$ 's so that node  $q$  will be the first to send out a spike to all the other nodes. Until time  $x_q$ , all the coordinates decrease at rate 1. Let  $(\theta_q^i)_i$  be the sequence of spikes sent out by node  $q$  and  $(G_q^i)_i$  the sequence of new values of node  $q$  after sending spikes (independent exponentially distributed random variables with parameter  $\lambda_q$ ). If we define

$$\nu^q = \inf \left\{ k \geq 1 / \sum_{j=1}^k (\theta_q^j - G_q^j) < 0 \right\},$$

$\nu^q$  is finite a.s because  $\rho_q < 1$ . As long as  $t \leq S_1 = x_q + \sum_{j=1}^{\nu^q} \theta_q^j$ ,

$$X_i(t) = x_i + \sum_{j=1}^k \theta_q^j - t \quad \text{for } i \neq q,$$

where  $k$  is such that  $t \in [x_q + \sum_{j=1}^{k-1} G_q^j, x_q + \sum_{j=1}^k G_q^j[$ . Up to time  $S_1$ , node  $q$  is the only one to fire. At that time,  $X_q(S_1)$  is exponentially distributed and for  $j \neq q$ ,  $X_j(S_1) = x_j - x_q$  (see Figure 1).



We repeat this procedure and construct a non decreasing sequence  $(S_k)_{k \in \mathbb{N}}$ . For  $k \geq 1$ , the state of the network at time  $S_{k+1}$  is defined by

$$X_i(S_{k+1}) = X_i(S_k) - \min\{X_j(S_k), 1 \leq j \leq N\}, i \neq i_0,$$

where  $i_0$  is the index which achieves the minimum of the coordinates of  $X_{S_k}$  and  $X_{i_0}(S_{k+1})$  is an independent exponential variable ( $\lambda_{i_0}$ ). We will say that  $i_0$  is at the origin of the  $k$ -th period. By induction on  $k$  we remark that if node  $i$  has fired at least once before  $S_k$ ,  $X_i(S_k)$  is an exponential random variable, independent of the other components. If a node  $j$  never fired up to time  $S_{(k+1)N}$ , then its state at that time is  $\leq x_j$  minus a sum of  $k$  independent exponential variables with parameter  $\lambda = \min_i \lambda_i$  (there exists a node which is at the origin of at least  $k + 1$  periods before time  $S_{(k+1)N}$ ). Hence after a while, all the nodes will have fired at least once, thus there exists some random time  $S_p$  when the coordinates of the Markov process

are independent exponentially distributed random variables. The coupling property is proved. ■

**Theorem 3** *If we order the  $\rho$ 's so that*

$$\rho_1 \leq \rho_2 \leq \dots \leq \rho_N,$$

*and if for some  $p \leq N - 1$ ,  $\rho_p \leq 1 < \rho_{p+1}$ , then almost surely, there exists some  $i > p$  such that the inhibitions of all nodes different from  $i$  converge to  $+\infty$  with probability 1. Hence, with probability 1,  $N - 1$  nodes are “dead” after a while.*

**Proof**

We keep the same notation as in the proof of Proposition 2. When node  $q$  begins to fire, variable  $\nu^q$  is the number of consecutive spikes sent out by node  $q$  before being possibly interrupted by the other nodes.

On the event  $A_q = \{\nu^q = +\infty\} = \{\min_k \sum_{j=1}^k (\theta_q^j - G_q^j) > 0\}$ , node  $q$  will continuously send out spikes to the other nodes. In this case the inhibition of node  $i \neq q$  just before the  $k$ -th spike of node  $q$  will be

$$x_i - x_q + \sum_{j=1}^{k-1} (\theta_q^j - G_q^j)$$

and thus will converge to  $+\infty$  with probability 1.

If  $\rho_q > 1$ , then  $P(A_q) > 0$  (see Lemma 1 of appendix 3 of [2] for example), thus with some positive probability  $\nu^q$  can be infinite. Hence if a node with index  $> p$  begins to fire, with positive probability it will kill all the other nodes.

To prove that, let us assume that at least two nodes fire infinitely often. We can suppose that one node  $q$  with index  $> p$ . Indeed we have  $\rho_i < 1$  for  $i \leq p$ , hence with the same argument as in the proof of Proposition 2, at least one node with index  $> p$  will fire after a period initiated by a node of index less than  $p$ . Because of our assumption, node  $q$  does not fire continuously but it fires infinitely often. It implies that it restarts firing after every period of the other node. But at these restarting times, we have a sequence of independent events  $(A_q^n)_n$  with the same probability as the event  $A_q$  and none of them is realized (because node  $q$  will be interrupted by the other node). This is impossible with probability 1. We conclude that with

probability 1 one node with index  $> p$  fires continuously. Our theorem is proved. ■

### Remarks

- a) The sufficient stability condition of [3] is in this case  $\max_{1 \leq j \leq N} \rho_j < \frac{1}{N-1}$  which is quite strong for this topology.
- b) The assumption of exponentially distributed variables is not essential for stability properties (see [4]).
- c) The above results for the stability can be generalized to a larger class of graphs, the *N-partite complete graphs* (see [4]). A graph is N-partite and complete if its vertex set can be decomposed into a partition  $\bigcup_1^N H_k$  such that
  - . two nodes of  $H_k$  are not connected for  $k \in \{1, \dots, N\}$ ,
  - . two nodes from different  $H$ 's are connected.

## 3 The linear network

In this section we study the one dimensional topology for the network where every node has two neighbors except at both ends. We assume that the  $\theta_i$ 's [resp.  $G_i$ 's] are identically distributed [resp. exponentially distributed with parameter  $\lambda$ ]. We know that if  $\rho = \lambda E(\theta_1) < 1/2$  then it is stable ([3]), the following proposition shows that the condition  $\rho < 1/2$  is indeed necessary for the stability when  $N$  is odd (if we forget the case  $\rho = 1/2$ ).

**Proposition 4** *If  $\rho > 1/2$  and if the number of nodes is odd, then the linear network is unstable.*

### Proof

Let  $N = 2k + 1$ , and for  $i = 1, \dots, k + 1$ , denote by  $Q_i(t) = \sum_{t_j^i \leq t} \theta_i^j$  where  $P_i = (t_j^i)_{j \geq 0}$  is a Poisson process with parameter  $\lambda$ . We assume that all the  $P_i$ 's and  $\theta_j^i$ 's are independent. For  $i = 1, \dots, k + 1$ ,  $(\theta_j^i)_{j \geq 0}$  is the sequence of spikes sent by node  $2i - 1$  and  $(t_{j+1}^i - t_j^i)_j$  are the exponential variables



associated to this node. Using that  $2\rho > 1$  and the law of large numbers we get that  $R_i = \inf\{Q_i(t) + Q_{i+1}(t) - t, t \geq 0\}$  is finite almost surely for  $i = 1, \dots, k$ , hence there exists an  $x_0 > 0$  such that  $P(R_i > -x_0, 1 \leq i \leq k+1) > 0$ .

Now take the initial state,  $x_{2i+1}(0) = 0$  and  $x_{2i}(0) = x_0$  for  $i = 1, \dots, k$ , then, as long as  $x_{2i}(s) > 0$  for all  $s \leq t$  and  $i \leq k$ ,  $x_{2i}(t) = x_0 + Q_i(t) + Q_{i+1}(t) - t$ . The last equality will be true for all  $t$  if  $x_{2i}(0) > -R_i$  for all  $i$  and in this case  $x_{2i}(t)$  converges to  $+\infty$ . Hence, on a set of positive probability, the components of  $(\underline{Q}(t))_{t \geq 0}$  with an even index converge to infinity. In other words, for all  $i$  odd, nodes  $i$  and  $i+2$  “kill” node  $i+1$  with positive probability. Our proposition is proved. ■

As we will see in Theorem 5, the condition  $\rho < 1/2$  is not necessary when the number of nodes is even. From now on, we consider the linear network with an even number of nodes and assume that  $1 > \rho > 1/2$ , so that for some  $\theta \in ]0, \pi/3[$ ,  $\rho = \frac{1}{2 \cos \theta}$ .

For technical convenience, we will slightly change the dynamics of our process: at the moment of a spike, we assume that, instead of sending the same spike to its neighbors, a node sends out independent exponentially distributed ( $\mu$ ) spikes to its neighbors, hence  $\rho = \frac{\lambda}{\mu}$ . We can then describe the state of the network as a Markov process  $(\underline{X}(t))_{t \geq 0}$  on the countable state space  $\mathbb{N}^N$ . The dynamics of the process  $(\underline{X}(t))_{t \geq 0}$  are described by the following transitions:

$$\begin{aligned} x &\longrightarrow x - \underline{\delta}_i & \mu & \text{if } x_i \neq 0 \\ x &\longrightarrow x + \underline{\delta}_{i+1} + \underline{\delta}_{i-1} & \lambda & \text{if } x_i = 0, \end{aligned}$$

for  $i = 1, \dots, N$ , assuming  $x_0 = x_{N+1} = +\infty$  for convenience, if  $N$  is the number of nodes of the network. The vector  $\underline{\delta}_i$  is the  $i$ -th unit vector:  $\underline{\delta}_i(j) = 1_{\{j=i\}}$ .

With this description, the value of the inhibition of the  $i$ -th node will be the sum of an exponential r.v ( $\lambda$ ) and of  $X_i(t)$  exponential r.v.’s ( $\mu$ ). All the exponential r.v.’s used to express the inhibition of the nodes are independent. Hence we kept the basic features of our model: the inhibition of every coordinate is still decreased at rate 1 and when there is a spike, r.v. with independent exponential distribution ( $\mu$ ) are sent out to the neighbors. The main result of this section is the following theorem :

**Theorem 5** For a linear network with  $2N$  nodes, the Markov process  $(\underline{X}(t))_{t \geq 0}$  is ergodic if  $\rho < \rho_c(N) = \frac{1}{2 \cos \pi/(2N+1)}$  and transient if  $\rho > \rho_c(N)$ .

**Remark**

The sequence of critical constants  $(\rho_c(N))_N$ , is non increasing and converging to  $1/2$  which is the critical constant for the “odd” case.

We begin by the following simple lemma (see also [3]).

**Lemma 6** If the network with  $N$  nodes is ergodic and for  $1 \leq i \leq N$   $y_i^t$  denotes the sum of the spikes sent out by node  $i$  to node  $i - 1$  up to time  $t$ , then  $y_i(N) = \lim_{t \rightarrow +\infty} \frac{y_i^t}{t}$  satisfies

$$\rho y_{i-1}(N) + y_i(N) + \rho y_{i+1}(N) = \rho, i = 1, \dots, N \text{ with } y_0(N) = y_{N+1}(N) = 0. \tag{4}$$

The useful properties of  $y_1(N)$  are given in Proposition 10 of the appendix.

**Proof**

The existence of a limit is simply the consequence of an ergodic theorem for ergodic Markov processes. Clearly,  $y_i^t$  has the same distribution as  $\bar{y}_i^t$ , the sum of the inhibitions sent out by node  $i$  to node  $i + 1$ , hence we have also  $y_i(N) = \lim_{t \rightarrow +\infty} \frac{\bar{y}_i^t}{t}$ . If a node spends time  $s$  in state 0, it will send in the limit, for  $s$  large, the inhibition  $\rho s$  to its neighbors. Now if we remark that, up to an asymptotically negligible term, the time spent by  $i$  in state 0 up to time  $t$  is  $t - y_{i+1}^t - \bar{y}_{i-1}^t$ , then our lemma is proved. ■

**Proof of Theorem 5**

We proceed by induction on  $N$  to prove the following property:

If  $\rho_c(N) < \rho < \rho_c(N - 1)$ , all the networks with a even number of nodes larger than  $2N$  are transient and for  $\rho < \rho_c(N)$ , the network with  $2N$  nodes is ergodic.

For  $N = 1$ , the property is trivially true since

- . If  $\rho < 1$  the network with 2 nodes is stable according to the previous section: This network is indeed completely connected and the change we made on the dynamics of the process does not affect the model since there is only one neighbor for every node.

- When  $\rho > 1$ , the network with  $2k$  nodes,  $k \geq 1$ , is unstable by the same argument as in Proposition 4: if the inhibition of nodes  $1, 3, \dots, 2k - 1$ , is sufficiently large then each of them will be killed by his right neighbor because of  $\rho > 1$  (in this case one node is sufficient to kill another one).

Now let us assume that our property is satisfied up to rank  $N$ .

a)  $\rho_c(N + 1) < \rho < \rho_c(N)$

Let  $k \geq N + 1$ , we want to prove that a network with  $2k$  nodes is unstable. According to the induction hypothesis, the network with  $2N$  nodes is ergodic. Using Lemma 6, the vector  $(y_i(2N))_{1 \leq i \leq 2N}$  is solution of (6) and according to the hypothesis on  $\rho$ , ( $\theta \in ]\frac{\pi}{2N+3}, \frac{\pi}{2N+1}[$ ) and Proposition 10, we have  $y_1(2N) + \rho > 1$ .

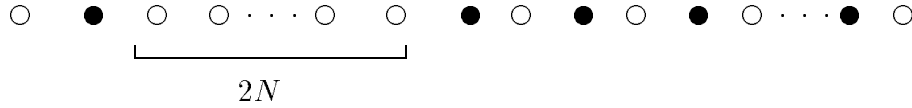


Figure 2.

If in the picture of the network of  $2k$  nodes (Figure 2), the inhibition of black nodes is sufficiently large, then with some positive probability none of them will fire anymore. For the black nodes n° 2 and  $2N + 3$ , it is a consequence of the relation  $y_1(2N) + \rho > 1$  and for the others, it is simply because  $\rho > 1/2$ . The proof is similar to that of Proposition 4 and therefore will be omitted. Hence the network with  $2k$  nodes,  $k \geq N + 1$  is transient when  $\rho_c(N + 1) < \rho < \rho_c(N)$ .

b)  $\rho < \rho_c(N + 1)$

We want to prove that  $(\underline{X}(t))_{t \geq 0}$  is an ergodic Markov process in  $\mathbb{N}^{2N+2}$ . Our main tool in the proof of ergodicity is the ergodicity criteria of [6]. The results of [6] are stated in the context of Markov chains. Although our study fit in that context (by looking at the embedded Markov chain  $(\underline{X}_n)_{n \in \mathbb{N}}$  at the instants of jumps for example), we found more convenient to work with processes and adjust the notations. We translate some of the notations of [6] to our case.

### Definition

A *face*  $\Delta$  is a subset of  $\{1, \dots, 2N+2\}$ . For each face  $\Delta$  we consider the same network model but with the nodes in  $\Delta$  (whose index is in  $\Delta$ ) are frozen, i.e. do not fire anymore. The induced Markov process on  $\mathbb{N}^{2N+2}$  associated with  $\Delta$  is denoted by  $(X_\Delta(t))_{t \geq 0}$ .

If we remove the components of  $X_\Delta(t)$  which are in  $\Delta$  then we still have a Markov process. We sill say that the face is ergodic [resp. transient] if this Markov process is.

### The second vector field

For some face  $\Delta$ , we can write  $\{1, \dots, 2N+2\} - \Delta$  as  $\cup_{i \in \Delta} L_i = \cup_{i \in \Delta} R_i$ , where the  $L_i$ 's and  $R_i$ 's are (possibly empty) intervals such that if  $L_i \neq \emptyset$  [resp.  $R_i \neq \emptyset$ ] then  $i-1 \in L_i$  [resp.  $i+1 \in R_i$ ]. For convenience, we assume that the fictitious nodes at both ends of the network, 0 and  $2N+3$  are always in  $\Delta$ . The Markov process associated with  $\Delta$  is basically the concatenation of the independent processes  $(\underline{X}_{L_i})_{i \in \Delta}$ , where  $\underline{X}_J$  is the Markov process associated with a network of size  $|J|$ .

We define  $y_1(J) = y_1(|J|)$  if the interval  $J$  has the cardinality  $|J|$  even,  $y_1(J) = 0$  if  $J = \emptyset$  and  $\rho$  if  $|J|$  is odd. When  $|J|$  is even,  $y_1(J)$  is simply the output at equilibrium of the first node of a network of size  $|J|$  (cf. Lemma 6). This leads us to the following definition for the drift vector  $v^\Delta$ :

- a) For  $i \in \Delta$ ,  $v_i^\Delta = y_1(L_i) + y_1(R_i) - 1$ .
- b) If  $|L_i|$  is even or 1,  $v_j^\Delta = 0$  for all  $j \in L_i$ .
- c) If  $L_i = \{a_1, a_1+1, \dots, a_1+2k\}$ ,  $v_{a_1+2j}^\Delta = 0$ ,  $j = 0, \dots, k$  and  $v_{a_1+2j+1}^\Delta = 2\rho - 1$ ,  $j = 0, \dots, k-1$ .

If  $\Delta$  is some ergodic face, then for every  $i \in \Delta$ ,  $\underline{X}_{L_i}$  is ergodic, hence, according to Proposition 4 and the induction hypothesis,  $|L_i| = 0, 1$  or  $2k$ , with  $k \leq N$ . In this case, the only coordinates of  $v^\Delta$  which are non zero are those with indices in  $\Delta$ . According to our assumption on  $\rho$  ( $\theta \in [0, \frac{\pi}{2N+3}]$ ) and Proposition 10 of the appendix, it is easily verified that for  $i \in \Delta$  such that  $|L_i|$  or  $|R_i| \neq 1$  then  $v_i^\Delta < 0$  and that  $v_i^\Delta = 2\rho - 1 > 0$  otherwise. In this case, the vector  $v^\Delta$  has the following intuitive meaning: If we freeze the

nodes of  $\Delta$  and let the other nodes reach a stationary state, then for  $i \in \Delta$ ,  $v_i^\Delta$  is simply the stationary drift of the  $i$ -th component, i.e.

$$\lim_{t \rightarrow +\infty} \frac{1}{t} ((\text{sum of the inputs received by } i \text{ up to time } t) - t)$$

For the embedded Markov chain of  $(\underline{X}_\Delta(n))_{n \in \mathbb{N}}$  at the instants of jumps of  $(X_\Delta(t))_{t \geq 0}$ , the  $i$ -th component of the stationary drift can be expressed as

$$\lim_{n \rightarrow +\infty} \frac{1}{n} (\text{nb of upward jumps of } i - \text{nb of downward jumps of } i, \text{ up to the } n^{\text{th}} \text{ jump of } (X_\Delta(t))_{t \geq 0}),$$

it is easy to see the stationary drift for the discrete case is simply  $\frac{\mu}{\lambda^\Delta} v^\Delta$ , where  $\lambda^\Delta$  is the intensity of jumps of  $\underline{X}_\Delta(t)$ .

If  $\Delta$  is non ergodic, then at least one of the intervals among the  $L_i$ 's or  $R_i$ 's,  $J$  say, has an odd cardinality. Here, the components  $n^\circ 2, 4, \dots$  of  $J$  will have the value  $2\rho - 1 > 0$ .

Let

$$B^\Delta = \{x \in \mathbb{R}_+^{2N+2} / x_i = 0 \text{ if } i \notin \Delta \text{ and } x_i > 0 \text{ otherwise}\}.$$

The second vector field  $(v(x))_{x \in \mathbb{R}^{2N+2}}$  of the Markov process  $(\underline{X}(t))_{t \geq 0}$  (see [6] and [5]) is defined by  $v(x) = v^\Delta$  for  $x \in B^\Delta$  and  $v(\underline{0}) = \underline{0}$ .

When  $\Delta = \{1, \dots, 2N + 2\}$ ,  $v^\Delta$  is simply the usual drift vector  $v_i^\Delta = -1$  for all  $i \leq 2N + 2$  and  $(v(x))_{x \in B^\Delta}$  is the first vector field.

### The associated dynamical system

We can now define the deterministic dynamical system  $(\Gamma^x(t))_{t \geq 0}$  associated to the second vector field, for  $x \in \mathbb{R}^{2N+2}$ :

$$\begin{aligned} \Gamma^x(0) &= x \\ \frac{d\Gamma^x}{dt}(t) &= v(\Gamma^x(t)), \end{aligned} \tag{5}$$

and  $\tau(x)$  the hitting time of  $\underline{0}$  by  $\Gamma^x$ ,

$$\tau(x) = \inf\{t / \Gamma^x(t) = \underline{0}\} \quad (\text{with } \inf \emptyset = +\infty).$$

From the definition of the second vector field, it is easy to remark that  $\Gamma^x(t)$  is non-negative for all  $t \geq 0$  and that its rate of growth is bounded by 1 in absolute value. Denote by  $\Delta_t$  the face such that  $\Gamma^x(t) \in B^{\Delta_t}$  with  $\Delta_0 = \Delta$ . The travel of  $\Gamma^x$  on the faces can then be decomposed as follows:

Step 1. Let us call an island a subset of  $\Delta \{i, i + 1, \dots, i + p\}$  with  $p > 0$ . The number of nodes in islands of  $\Delta_t$  is decreasing to 0 (simply because  $\rho < 1$ ). Consequently, if there is no more island at time  $t$  this will be true for all  $s \geq t$ . Moreover, after a while the nodes at both ends do not belong to  $\Delta_t$  anymore.

Step 2. We assume that step 1 is achieved. An additional interval  $L_i$  with an even cardinality cannot be created by one of the only possible operations:

- splitting, when a transient face is reached,
- merging, when a node disappears between two  $L_i$ 's whose cardinalities have distinct parities ( $1, 2k$  or vice versa).

Hence the number of  $i$  such that  $|L_i|$  is even is non increasing.

Step 3. There exists at least some  $L_i$  whose cardinality is  $2k$  with  $k > 0$ . Let  $S$  be the set of  $L_i$ 's with an even cardinality, it is easy to see that the cardinality of  $S$  is odd. Between two consecutive elements  $L$  and  $L'$  of  $S$  (i.e. there is no element of  $S$  between them), the  $L_i$ 's have cardinality 1. According to our hypothesis on  $\rho$ , if  $L$  and  $L'$  do not vanish, these  $L_i$ 's are swallowed by  $L$  or  $L'$  and when the last node of  $\Delta$  in the middle finally collapses, the whole space created then has an odd cardinality and thus is split into  $L_i$ 's of size one. Hence at the end, there will remain only one element of  $S$  which will swallow all the remaining  $L_i$ 's with cardinality 1. Point  $\underline{0}$  is finally reached, hence  $\tau(x)$  is finite for all  $x \in \mathbb{R}_+^{2N+2}$ .

We defined the second vector field and its associated dynamical system when the dimension of the state space is even ( $2N + 2$ , here). When the dimension is odd, ( $2N + 1$  say), we can define in the same way the second vector field and the dynamical system  $\Gamma'$  without any change. The corresponding hitting time  $\tau'$  will be in this case

$$\tau'(x) = \inf \{t / \Gamma'^x(t) \in B^{\{2,4,6,\dots,2N\}}\}$$

To describe the travel of  $\Gamma'$  on the faces, steps 1 and 2 do not change but step 3 has to be modified in the following way:

Step 3'. With the notations of Step 3, we remark that the cardinality of  $S$  is necessarily even (possibly 0). As before two consecutive elements of  $S$  finally collapse and split into  $L_i$ 's of size one if they are not destroyed before. Hence all the elements of  $S$  will disappear, all the remaining  $L_i$ 's will have a cardinality 1 and the nodes at both ends are not in  $\Delta$  (we assume that Step 1 is achieved). We conclude that  $\tau'(x)$  is finite for all  $x \in \mathbb{R}_+^{2N+1}$ .

**Lemma 7** *The dynamical systems  $(\Gamma^x(t))_{t \geq 0}$ ,  $(\Gamma'^x(t))_{t \geq 0}$  do not visit the same face twice and there exists some constant  $K > 2$  such that  $|\tau(x)| < K\|x\|$  for  $x \in \mathbb{R}_+^{2N+2}$  and  $|\tau'(x)| < K\|x\|$  for  $x \in \mathbb{R}_+^{2N+1}$ .*

**Proof**

For the first part of the lemma we can proceed by induction. The case  $N = 0$  is trivial. We assume that all the dynamical systems  $(\Gamma_{2k}^x(t))_{t \geq 0}$  and  $(\Gamma'_{2k+1}(t))_{t \geq 0}$ ,  $k \leq N$  do not visit the same face twice (the subscript  $2k$  or  $2k + 1$  refers to the number of nodes of the dynamical system). We prove that this is also the case for  $(\Gamma_{2N+2}^x(t))_{t \geq 0}$  and  $(\Gamma'_{2N+3}(t))_{t \geq 0}$ .

Assume that some face  $\Delta$  is visited twice by  $(\Gamma_{2N+2}^x(t))_{t \geq 0}$ . This face cannot have some island. Otherwise this island must remain between the two visits to  $\Delta$  since no island can be created. But the complement of this island is made of 1 or 2 intervals of size  $\leq 2N$ . According to the induction hypothesis, the dynamical system cannot visit the same face twice on this (or these) interval(s). Applying the same reasoning,  $\Delta$  cannot contain neither node 1 or  $2N + 2$  nor some  $|L_i|$  with an even cardinality (since following steps 2 and 3, the elements of  $S$  can only grow or disappear as time goes on). Thus the result is proved for the case  $2N + 2$ . The method is strictly the same for  $(\Gamma'_{2N+3}(t))_{t \geq 0}$ . The first part of our lemma is proved.

To complete the proof of the lemma, we just have to remark that the duration of any of the face transitions described in one of the steps 1, 2, 3, and 3' above is proportional to the values of the components of the starting point and that there is only a bounded number of them. ■

We can now state the continuity property of  $\tau$ .

**Proposition 8** *The function  $x \rightarrow \tau(x) = \inf\{t/\Gamma^x(t) = \underline{0}\}$  is finite and Lipschitz.*

**Proof**

If  $x, y \in \mathbb{R}_+^{2N+2}$  are on the same face, as long as the two dynamical systems  $\Gamma^x(t)$  and  $\Gamma^y(t)$  remain on this face, the difference  $\|\Gamma^x(t) - \Gamma^y(t)\|$  does not change.

Hence we can assume that  $x$  and  $y$  are not on the same face and denote by  $\alpha = \|x - y\|$ . There exists some  $p \leq 4N + 4$ , such that the interval  $]\alpha K^p, \alpha K^{p+1}]$  does not contain any of the  $4N + 4$  coordinates of  $x$  and  $y$ . If  $T$  is the subset of all  $i \leq 2N + 2$  such that  $x_i > \alpha K^{p+1}$  or equivalently (since  $K > 2$ ),  $y_i > \alpha K^{p+1}$ , we can write  $T^c = \{1, \dots, 2N + 2\} - T$  as a union of intervals  $\cup_{J \in \mathcal{J}} J$ . For all  $t < K^{p+1}\alpha$ , the coordinates of  $\Gamma^x(t)$  and  $\Gamma^y(t)$  with index in  $T$  never reach zero since the drift of the coordinates is always larger than  $-1$ . Using this property it is easy to see that on the time interval  $[0, \alpha K^{p+1}[$ , the  $T^c$ -components of the dynamical system  $\Gamma^x(t)$  [resp.  $\Gamma^y(t)$ ] behave as a concatenation of the dynamical sub-systems  $\Gamma_J^x(t)$  [resp.  $\Gamma_J^y(t)$ ],  $J \in \mathcal{J}$ . According to Lemma 7, for any  $J \in \mathcal{J}$ , we know that  $\Gamma_J^x(t)$  and  $\Gamma_J^y(t)$  will have reached the same face at time  $\alpha K^{p+1}$ , either the empty face or the face with the even components of  $J$ , depending on the parity of  $|J|$ . Hence at some time  $t_0 \leq \|x - y\| K^{4N+5}$ ,  $\Gamma^x(t)$  and  $\Gamma^y(t)$  will have reached the same face and  $\|\Gamma^x(t_0) - \Gamma^y(t_0)\| < 3\|x - y\| K^{4N+5}$  (remember that the rate of growth of  $\Gamma$  is bounded by 1 in absolute value). Now we use the fact that there is only a bounded number of changes of faces to get the desired Lipschitz property. ■

The function  $\tau$  thus satisfies Condition B of [6], p. 17,

- a)  $\tau \geq 0$ .
- b)  $\tau(x) - \tau(y) \leq K_1 \|x - y\|$ ,  $x, y \in \mathbb{N}^{2N+2}$ .
- c) For any ergodic face  $\Delta$  and all  $x \in B^\Delta \cap \{x_i > C, i \in \Delta\}$ .

$$\tau(x + v(x)) - \tau(x) \leq -\delta$$

If we choose  $C$  such that  $C > \|v^\Delta\|$  for all ergodic faces  $\Delta$ , then c) is satisfied with  $\delta = 1$  because  $\Gamma^x(1) = x + v(x)$  for  $x \in B^\Delta \cap \{x_i > C, i \in \Delta\}$ .

As we said before, we worked with the process rather than the embedded Markov chain for which the results of [6] may apply. But as we have already remarked, the second vector field for the Markov chain is proportional to



the vector field for the Markov process (with the proportionality constant depending only on the face considered). Hence the variable  $\tau$  can be modified so that condition B is satisfied for the embedded Markov chain. It implies that  $(\underline{X}_n)_n$  is ergodic, hence our Markov process is also ergodic. Our proof by induction is finished as well as the proof of Theorem 5. ■

### Remarks

- a) The second vector field of the Markov process is essential in our proof of ergodicity. It is not the case for the stability proof of [3], for which the first vector field, the usual drift vector, is sufficient. There, the sum of the components of the vector is in fact a natural Lyapounov function. The stability result of [3] can be obtained using this remark and results relating Lyapounov functions and ergodicity of Markov processes (see Theorem 3.7 and Proposition 5.10 of [7] for example).
- b) Our second vector field is not multi-valued on the transient faces. In general the second vector field is generally multi-valued in order to get the appropriate continuity properties of the associated dynamical system. On the transient faces, its values are all the possible drift vectors pointing to an ergodic face (see [5]).
- c) The behavior of the network depends on the parity of its size. A similar phenomenon has been analyzed for a queueing network with local interaction in [8], but in this case the critical  $\rho$ 's are not explicitly known.

### Stable states of the linear network

Our purpose here is the asymptotic behavior of the linear network with  $N$  nodes. We enlarge our state space to allow the value  $+\infty$  for the components of  $(\underline{X}(t))_{t \geq 0}$  so that we deal with measures defined on  $(\mathbb{N} \cup \{+\infty\})^N$ . We do not consider the special cases where  $\rho$  is one of the  $\frac{1}{2 \cos \pi / (2N+1)}$ ,  $N \in \mathbb{N}$ . As in the proof of Theorem 5, for every subset  $\Delta$  of  $\{1, \dots, N\}$ , we define the intervals  $L_i$ 's such that

$$\{1, \dots, N\} - \Delta = \bigcup_{i \in \Delta} L_i,$$

we set  $L_0 = \{1, \dots, N\}$  when  $\Delta = \emptyset$ . To every  $\Delta$  such that  $|L_i| = 0, 1, 2k$   $k \geq 1$  for all  $i$ , we define the distribution  $\pi^\Delta$  on  $(\mathbb{N} \cup \{+\infty\})^N$  by

$$\pi^\Delta = \prod_{i \in \Delta} \delta_{+\infty} \prod_{L_i \neq \emptyset} \pi_{|L_i|}$$

where  $\pi_k$  is the invariant measure of the linear network with  $k$  nodes and  $\delta_{+\infty}$  the Dirac measure at infinity. Notice that this definition is meaningful only on some range of  $\rho$ . The distribution  $\pi^\Delta$  is a concatenation of linear subnetworks in equilibrium and  $\Delta$  is simply the set of dead nodes for  $\pi^\Delta$ .

**Proposition 9** *Any distribution of  $\mathcal{E} = \{\pi^\Delta / \Delta \in \mathcal{S}\}$  is a possible limiting distribution for  $(\underline{X}(t))_{t \geq 0}$ . The set  $\mathcal{S}$  of admissible  $\Delta$ 's is defined by*

$$\underline{\rho < 1/2}$$

$$\mathcal{S} = \emptyset.$$

$$\underline{\rho > 1}$$

$\mathcal{S}$  is the set of all  $\Delta$ 's such that if  $i \in \Delta$  then  $i - 1$  or  $i + 1 \notin \Delta$  and for all  $i < N$  then either  $i \in \Delta$  or  $i + 1 \in \Delta$ .

$$\underline{1/2 < \rho < 1}$$

We have  $\rho_c(K + 1) < \rho < \rho_c(K)$  for some  $K \geq 1$ .

i)  $N$  is even and  $\leq 2K$

$$\mathcal{S} = \{\emptyset\}.$$

ii)  $N$  is odd and  $\leq 2K + 1$

$$\mathcal{S} = \{\Delta_0\} \text{ with } \Delta_0 = \{1, 3, 5, \dots, N\}.$$

iii)  $N \geq 2K + 2$

$\mathcal{S}$  is the set of all  $\Delta$ 's such that

-  $1, N \notin \Delta$ .

- For all  $i \in \Delta$ ,  $i - 1 \in L_i$  and  $i + 1 \in L_{i+1}$  and  $|L_i| = 1$  or  $2K$ .

- If  $|L_i| = 2K$  then  $|L_{i-1}| = |L_{i+1}| = 1$ .

### Proof

As in the proof of Proposition 4, the method consists in using ergodic theorems for ergodic Markov processes. ■

### Remarks

- a) We conjecture that every limiting distribution of  $(\underline{X}(t))_{t \geq 0}$  is a convex combination of the  $\pi^\Delta$  with  $\Delta \in \mathcal{S}$ , i.e.  $\mathcal{E}$  is the set of extremal measures of the set of limiting distributions of  $(\underline{X}(t))_{t \geq 0}$ .
- b) When  $\rho < 1$ , the dead nodes (if any) are isolated.
- c) In the definition of  $\mathcal{S}$  in *iii*), notice that if there is a linear subnetwork of size  $2k$ , then the neighboring subnetworks must be isolated nodes. The reason is the following: two subnetworks of respective sizes  $2k, k'$  can be neighbors only if their output can “kill” the node between them. If  $\rho_c(K+1) < \rho < \rho_c(K)$ , this is possible only if  $y_1(2k) + y_1(k') > 1$ . The case  $k < K$  is impossible because  $y_1(k') < \rho$  and  $y_1(2k) + \rho < 1$  ( $\rho < \rho_c(k)$ ), hence  $k = K$ . If  $k' > 1$  then  $k' = 2K$  for the same reason, but according to Proposition 10 of the appendix,  $2y_1(2K) < 1$ , hence  $k' = 1$ .
- d) Following c), Figure 3 is an example of a stable configuration of our network (when  $\rho < 1$ ), but that cannot be reached from any initial state with finite components (the components represented by black nodes have the value  $+\infty$ ).



Figure 3.

## 4 Appendix

Let  $(y_i(N))_{1 \leq i \leq N}$  be a solution of the following linear system,

$$\begin{aligned}
2y_1 \cos \theta + y_2 &= 1 \\
y_1 + 2y_2 \cos \theta + y_3 &= 1 \\
y_2 + 2y_3 \cos \theta + y_4 &= 1 \\
&\dots \\
y_{i-1} + 2y_i \cos \theta + y_{i+1} &= 1 \\
&\dots \\
y_{N-1} + 2y_N \cos \theta &= 1,
\end{aligned} \tag{6}$$

**Proposition 10** *If  $N$  is an even integer, then*

- a) *if  $f(\theta) = y_1(N) + \frac{1}{2\cos\theta} - 1$  then  $f(0) = \frac{-1}{2(N+1)}$ ,  $f$  is negative on the interval  $[0, \frac{\pi}{N+3}]$  and positive on  $]\frac{\pi}{N+3}, \frac{\pi}{N+1}[$ .*
- b) *For  $\theta \in [0, \frac{\pi}{N+1}[$ ,  $y_1(N) < 1/2$ .*

**Proof**

Standard calculations with the above recurrence give easily that for  $0 < \theta < \frac{\pi}{N+1}$ , we have

$$y_1(N) = \frac{1}{2(1 + \cos \theta)} \left( 1 + \frac{\sin N\theta - \sin \theta}{\sin(N+1)\theta} \right),$$

$f$  can thus be written as

$$f(\theta) = \frac{((1 - 2 \cos^2 \theta) \sin(N+1)\theta + \cos \theta (\sin N\theta - \sin \theta))}{2(1 + \cos \theta) \cos \theta \sin(N+1)\theta}.$$

If  $p(\theta)$  is the numerator of  $f$ , then

$$p(\theta) = \sin(N+1)\theta - \sin(N+3)\theta - \sin 2\theta.$$

We remark that  $p(\theta_0) = 0$ , for  $\theta_0 = k\pi/N+3$ ,  $k$  odd  $\leq N+3$  or  $\theta_0 = k\pi/N+1$ ,  $k$  odd  $\leq N+1$  and

$$\frac{p(\theta)}{\sin \theta} = U_N(x) - U_{N+2}(x) - U_1(x)$$

where  $x = \cos \theta$  and  $U_N$  is the  $N$ -th Chebyshev polynomial of the second kind. The right hand side of the above expression is a polynomial of degree

$\leq N + 2$ , hence we have got all the zeroes of  $p$  in  $[0, \pi]$ . It is now easy to complete the proof of a).

To prove b) notice that  $y_1(N)$  can be written as

$$\begin{aligned} y_1(N) &= \frac{1}{2(1+\cos\theta)} \left( 1 + \frac{\sin(N+1)\theta \cos\theta - \cos(N+1)\theta \sin\theta - \sin\theta}{\sin(N+1)\theta} \right) \\ &= \frac{1}{2} - \frac{\sin\theta}{2(1+\cos\theta)} \frac{\cos(N+1)\theta/2}{\sin(N+1)\theta/2}. \end{aligned}$$

Hence our proof of b) is complete. ■

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