Application of fictitious domain method to the solution of the Helmholtz equation in unbounded domain

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Rapports de Recherche

1992

25ème anniversaire

N° 1797

Programme 6
Calcul Scientifique, Modélisation et Logiciels numériques

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Novembre 1992

RR_1797

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Abstract

A specific variant of fictitious domain method for solving the Helmholtz wave equation is considered in the paper. Numerical experiments carried out for test 2D problems presented this variant in a very good light. For example, a problem has been solved for airplane-like obstacles with 90 wavelengthes on its body and 1 400 000 mesh nodes.

Key words : Helmholtz wave equation, unbounded domain, fictitious domain method.

Application de la Méthode de Domaines Fictifs à la Résolution de l'Equation d'Helmholtz en Domaine Non Borné

Résumé

Dans cet article, on propose une variante de la méthode des domaines fictifs pour les équations des ondes en formulation fréquentielle. Des tests numériques 2D sont présentés sur de très gros problèmes, jusqu'à 1 400 000 points et 90 longueurs d'onde par longueur d'obstacle.

Mots-clés : Domaine fictif, domaine non borné, équation de Helmholtz, solveurs rapides.

0 Introduction

Computation of a wave (acoustic or electromagnetic) scattered by an obstacle is a problem of great practical importance. Mathematically it consists in the solution of the Helmholtz wave equation with some boundary condition at the obstacle and the Sommerfeld condition at infinity.

Many numerical methods for solving this problem have been suggested (see [2, 3, 7, 6, 8, 12, 13, 17] and others) but their design is still an heavy task because a lot of actual problems cannot be solved by means of the known methods (especially when the wavelength is much less than the obstacle).

Fictitious domain (or fictitious components) approach for solving partial differential equations has been proposed and investigated in [1, 3, 5, 9, 10, 11, 12, 13, 14, 15] and others and presented itself in a good light. Application of this method to solving the Helmholtz wave equation has been already considered in [2, 3, 8, 12, 13].

This article continues these investigations especially begun in [13]. The author applied to solving the problem a variant of fictitious domain method which was known earlier [9, 10, 11, 15] but as far as he knows had never been used for it before. This one is called "the variant with a non-symmetric upper enlargement".

Numerical experiments carried out for test 2D problems showed that this variant works much well. It solved the problem faster than other known methods including other variants of fictitious domain method. Moreover, some considered problems could be solved only with this variant. For example, a problem has been solved for an airplane-like obstacle with 90 wavelengthes and 1 400 000 mesh nodes.

The plan of the paper is:

1) the problem;

2) method of its finite element approximation in coupling with fictitious domain approach;

3) approximation of the Sommerfeld condition and its transformation to an absorbing boundary condition for a bounded domain;

4) the used variant of fictitious domain method and iterative procedure;

5) numerical experiments.

The author is grateful to Prof. Yu.A. Kuznetsov (Institute of Numerical Mathematics of Russian Academy of Sciences) and to Prof. O. Pironneau (University Paris 6 & INRIA) for formulation of the problem and permanent support of the work.

1 The problem

Scattering of the 2D flat stationary wave $v = e^{i\vec{\omega}\vec{x}}$ by an obstacle Ω in a homogeneous medium is described by the Helmholtz wave equation with the Sommerfeld condition at infinity:

$$\begin{aligned} \Delta u + \omega^2 u &= 0 \quad \text{in} \quad R^2 \setminus \bar{\Omega}, \\ u &= -v \quad \text{on} \quad \partial\Omega, \\ \lim_{r\to\infty} r^{1/2}\left(\frac{\partial u}{\partial r} - i\omega u\right) &= 0. \end{aligned} \tag{1.1}$$

Here, $\vec{x} = (x, y)^T$ is a point of R^2 with Cartesian coordinates x, y, $\vec{\omega} = (\omega\cos\alpha, \omega\sin\alpha)^T$ is a wavevector of the incident wave v where α is an angle of incidence, the complex-valued function $u = u(x,y)$ is the reflected wave, Δ is the Laplas operator, r is the distance to some fixed point.

It is known [2, 16] that this problem is correct and has an unique solution.

Perhaps, the boundary condition on $\partial\Omega$ in (1.1) is not completely realistic in many cases but it can be used in the test problem.

2 Variational formulation and approximation

We will use the usual Galerkin's formulation of the problem for its finite element approximation, i.e we replace (1.1) by:

$$\int_{R^2\setminus\bar{\Omega}} (\nabla u \cdot \nabla w - \omega^2 uw)\, d\Omega = 0 \quad \forall w \in \overset{\circ}{H}{}^1 \,(R^2 \setminus \bar{\Omega}), \tag{2.2}$$

where $\overset{\circ}{H}{}^1\,(R^2 \setminus \bar{\Omega})$ is the Sobolev's space of functions in $R^2 \setminus \bar{\Omega}$ equaled zero on $\partial\Omega$ and the function u satisfies the boundary condition on $\partial\Omega$ from (1.1). Hereafter we will use for u the following equivalent form of the Sommerfeld condition at infinity [3]:

$$u(r,\theta) \to G(\theta)\frac{e^{i\omega r}}{r^{1/2}} \quad \text{for} \quad r \to \infty, \tag{2.3}$$

where (r,θ) is some polar system of coordinates.

To solve (2.2)-(2.3) we apply the finite element method, i.e. we build a mesh $(R^2\setminus\bar{\Omega})_h$ and consider a subspace $\overset{\circ}{H}{}^1_h\,((R^2 \setminus \bar{\Omega})_h)$ of piecewise-polynomial functions in $(R^2 \setminus \bar{\Omega})_h$ equaled zero on $\partial\Omega_h$. Let us introduce a basis w_j^h, $j = 1, 2, ..., \infty$, of this subspace and some function $g^h \in H_h^1((R^2\setminus\bar{\Omega})_h)$ such that its value on $\partial\Omega_h$ approximates the boundary condition in (1.1). Let us find the approximate solution u^h in the form:

$$u^h = g^h + \sum_{j=1}^{\infty} u_j w_j^h. \tag{2.4}$$

Thus we get the infinite-dimensional system of grid equations:

$$\int_{(R^2\setminus\bar{\Omega})_h} (\nabla u^h \cdot \nabla w_j^h - \omega^2 u^h w_j^h)\, d\Omega = 0, \quad j = 1, 2, ..., \infty. \tag{2.5}$$

Concrete choice of the mesh $(R^2 \setminus \bar{\Omega})_h$, approximation of condition (2.3) and reducing of (2.5) to a finite-dimensional system of equations will be considered in the next Sections.

3 Fictitious domain (fictitious components) approach

The main idea of fictitious domain approach (in the treatment of [9, 10, 11, 14, 15] followed by the author) consists in using for solving the arising algebraic system iterative methods with separable preconditioner which corresponds to an approximation of the same differential operator in some enlarged domain. To introduce separable preconditioner we have to use rectangular mesh for rectangular domain. This method is most effective when the last mesh coincides almost everywhere with the mesh used for approximation except for the near-boundary region when the rectangular mesh is locally fitted to the boundary for its better approximation.

Implementation of the fictitious domain method for our problem includes the following stages.

3.1 Mesh constructing

Let us fix some polar system of coordinates (r, θ) and construct in R^2 a corresponding rectangular mesh $R_h^2 = \{(r_j, \theta_k),\ j = 0, 1, ..., \infty,\ k = 0, 1, ..., n_\theta\}$, where

$$0 = r_0 < r_1 < r_2 < ..., \quad 0 = \theta_0 < \theta_1 < ... < \theta_{n_\theta} = 2\pi.$$

The couples $[(r_j, \theta_k), (r_{j+1}, \theta_k)]$ and $[(r_j, \theta_k), (r_j, \theta_{k+1})]$ will be refered as "grid edges" and the quadrangles with vertices (r_j, θ_k), (r_{j+1}, θ_k), (r_{j+1}, θ_{k+1}), (r_j, θ_{k+1}) as "grid cells".

Now let us shift on $\partial\Omega$ some mesh nodes neighboring to it and triangulate all the modified grid cells in such a way that edges of the new mesh should approximate this boundary. Description of algorithms of this local modifying is beyond this paper (you can find one of them in [5]).

It should be emphasized that all lines of this mesh satisfy the equation

$$C_1 r + C_2 \theta + C_3 = 0 \tag{3.6}$$

with some constants $C_1,\ C_2,\ C_3$.

Thus, the mesh $(R^2 \setminus \bar{\Omega})_h$ is built as a union of non-modified cells and triangles belonging to $R^2 \setminus \bar{\Omega}$ (see examples in Fig. 1,6).

3.2 Choice of H_h^1 and construction of the preconditioner

For approximation (2.5) we will use the subspace $H_h^1((R^2 \setminus \bar{\Omega})_h)$ of continious functions u^h bilinear in each non-modified cell:

$$u^h = (\alpha_1 r + \beta_1)(\alpha_2 \theta + \beta_2) \tag{3.7}$$

and linear in each triangle of $(R^2 \setminus \bar{\Omega})_h$:

$$u^h = \alpha r + \beta\theta + \gamma, \tag{3.8}$$

with the usual finite element basis $w_j^h,\ j = 1, 2, ..., \infty$.

Together with it we consider the subspace $H_h^1(R_h^2)$ of piecewise-bilinear functions v^h defined on the non-modified mesh R_h^2 and build the following auxiliary system of equations "without obstacle":

$$\int_{R_h^2} (\nabla v^h \cdot \nabla \tilde{w}_j^h - \omega^2 v^h \tilde{w}_j^h)\, d\Omega = 0, \quad j = 1, 2, ..., \infty, \tag{3.9}$$

where $\tilde{w}_j^h$ are the corresponding basic functions.

It is easy to see that the last system is separable. Its solving will be used as the preconditioning procedure for solution of (2.5) (see Section 3.4).

3.3 Approximation of the Sommerfeld condition and its transformation

Hereafter we will suppose that the mesh R_h^2 is uniform in the both directions:

$$r_j = j h_r,\ j = 0, 1, ..., \infty, \quad \theta_k = k h_\theta,\ k = 0, 1, ..., n_\theta - 1, \text{ where } h_\theta = \frac{2\pi}{n_\theta}. \tag{3.10}$$

Let us denote n_r the maximum index j of nodes neighboring to $\partial\Omega_h$. For $j \geq n_r$ we can look for the solution u_{jk} of (2.5) in the form:

$$u_{jk} = \sum_{l=0}^{n_\theta - 1} u_j^l \varphi_k^l, \tag{3.11}$$

where (denoting $m_\theta = [\frac{n_\theta - 1}{2}]$ - the integer part of the number $\frac{n_\theta - 1}{2}$):

$$\varphi_k^l = \begin{cases} \sqrt{\frac{1}{n_\theta}} & \text{for } l = 0; \\ \sqrt{\frac{2}{n_\theta}} \sin \frac{2\pi l k}{n_\theta} & \text{for } l = 1, 2, ..., m_\theta; \\ \sqrt{\frac{2}{n_\theta}} \cos \frac{2\pi (l - m_\theta) k}{n_\theta} & \text{for } l = m_\theta + 1, m_\theta + 2, ..., 2m_\theta; \\ \sqrt{\frac{1}{n_\theta}} (-1)^k & \text{for } l = n_\theta - 1. \end{cases} \tag{3.12}$$

The function $\varphi_k^{n_\theta - 1}$ of the last type exists iff n_θ is an even number.

Having substituted (3.11)-(3.12) into (2.5) we obtain the set of n_θ independent tridiagonal systems of equations for u_j^l, $l = 0, 1, ..., n_\theta - 1$:

$$\begin{gathered} -\beta_{j-1} u_{j-1}^l + \alpha_j u_j^l - \beta_j u_{j+1}^l = 0, \ j = n_r, n_r + 1, ..., \infty, \\ \alpha_j = (1 - \tfrac{1}{6}\mu_l)(2 - \tfrac{2}{3} h_r^2 \omega^2) j h_\theta + \mu_l ((j+1)^2 \ln \tfrac{j+1}{j} + (j-1)^2 \ln \tfrac{j}{j-1} - 2j) h_\theta^{-1}, \\ \beta_j = (1 - \tfrac{1}{6}\mu_l)(1 + \tfrac{1}{6} h_r^2 \omega^2)(j + \tfrac{1}{2}) h_\theta - \mu_l (j + \tfrac{1}{2} - j(j+1) \ln \tfrac{j+1}{j}) h_\theta^{-1}, \\ \mu_l = \begin{cases} 0 & \text{for } l = 0; \\ 4 \sin^2 \frac{\pi l}{n_\theta} & \text{for } l = 1, 2, ..., m_\theta; \\ 4 \sin^2 \frac{\pi (l - m_\theta)}{n_\theta} & \text{for } l = m_\theta + 1, m_\theta + 2, ..., 2m_\theta; \\ 4 & \text{for } l = n_\theta - 1 \text{ if } n_\theta \text{ is an even number.} \end{cases} \end{gathered} \tag{3.13}$$

Now let us choose some number $n_\infty \gg n_r$ and suppose that for $r \geq R_\infty = n_\infty h_r$ the exact solution of (1.1) takes its asymptotic form (2.3) with a sufficient precision. Thus, for $j > n_\infty$ we may present the solutions of (3.13) in the form:

$$u_j^l = G_h^l \frac{(q_l)^j}{(j h_\theta)^{1/2}}. \tag{3.14}$$

We can take the value of q_l directly from (2.3):

$$q_l = e^{i\omega h_r}. \tag{3.15}$$

But it seems to the author to be more natural and correct to deduce this one from equations (3.13) themselves. Substituting (3.14) into (3.13) and neglecting terms of the order of j^{-2} we get the equation for q_l:

$$q_l^2 - \gamma q_l + 1 = 0, \text{where } \ \gamma = \frac{2 - \frac{2}{3} h_r^2 \omega^2}{1 + \frac{1}{6} h_r^2 \omega^2}, \tag{3.16}$$

from which we have

$$q_l = \frac{\gamma}{2} + i\sqrt{1 - \frac{\gamma^2}{4}}. \tag{3.17}$$

This number is complex iff the condition

$$h_r < \sqrt{6}\omega^{-1} = \frac{\sqrt{6}}{2\pi}\lambda \approx 0.4\lambda$$

is satisfied (hereafter $\lambda = \frac{2\pi}{\omega}$ is the wavelength). Of course, it must be actually so for the sake of approximation.

Some remarks:

1) the value of q_l does'nt depend on l and $|q_l| = 1$ in both expressions (3.15) and (3.17);

2) the conjugate value $\bar{q}_l$ corresponds to the wave moving from infinity towards the centre;

3) the error of representation (3.14) with q_l taken from (3.17) is of the order of $(n_r/j)^2$.

Thus, we have obtained the following boundary condition:

$$u^l_{n_\infty+1} = q_l\sqrt{\frac{j}{j+1}}u^l_{n_\infty}. \tag{3.18}$$

Now let us transform it to a boundary condition on the mesh line $j = n_r$ by means of the usual Gauss elimination in (3.13) for each l, i.e. we substitute (3.18) into (3.13) and then move towards the centre:

$$\begin{gathered} z^l_{n_\infty} = q_l\sqrt{\frac{j}{j+1}}, \\ z^l_j = \frac{\beta_j}{\alpha_{j+1} - \beta_{j+1}z^l_{j+1}}, \quad j = n_\infty - 1, n_\infty - 2, ..., n_r. \end{gathered} \tag{3.19}$$

Thus, we get the needed boundary condition:

$$u^l_{n_r+1} = z^l_{n_r}u^l_{n_r}. \tag{3.20}$$

It has been proved that process (3.19) is numerically stable. The proof is not complicated but rather cumbersome.

Let us order equations of (2.5) and corresponding nodes with $j \le n_r$ and represent this problem in the algebraic form (using (3.20) for excluding $u^l_{n_r+1}$):

$$Au = f. \tag{3.21}$$

Then let us prolong the node ordering onto R^2_h and introduce the matrix B corresponding to system of equations (3.9). Of course, the dimension N of the matrix A is less than the dimension N_0 of the matrix B. Hereafter we will identify the N_0-dimensional vectors and the grid functions on R^2_h using the introduced node ordering.

3.4 Enlargement of the problem and iterative process

As it has been mentioned in Section 3, fictitious domain method consists in using the matrix B^{-1} as the preconditioner in an iterative process for solving problem (3.21). But before that we have to enlarge (3.21) to obtain an equivalent N_0-dimensional algebraic problem:

$$\overset{\circ}{A}\overset{\circ}{u}=\overset{\circ}{f}\,. \tag{3.22}$$

Let us represent the matrix B in the block form:

$$B=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \tag{3.23}$$

where B_{11} is an $N \times N$ matrix.

The following ways of enlargement (3.22) are known [11, 15]:

$$\overset{\circ}{A}=\begin{bmatrix} A & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} \end{bmatrix},\quad \overset{\circ}{f}=\begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{3.24}$$

with some submatrices $\tilde{B}_{21}$, $\tilde{B}_{22}$ ("lower enlargement") and

$$\overset{\circ}{A}=\begin{bmatrix} A & \tilde{B}_{12} \\ 0 & \tilde{B}_{22} \end{bmatrix},\quad \overset{\circ}{f}=\begin{bmatrix} f \\ 0 \end{bmatrix}, \tag{3.25}$$

with some submatrices $\tilde{B}_{12}$, $\tilde{B}_{22}$ ("upper enlargement").

Let us represent the vector $\overset{\circ}{u}$ from (3.22) in the same form:

$$\overset{\circ}{u}=\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{3.26}$$

Obviously, the vector u_1 and the solution u of (3.21) always coincide in the first case and they coincide in the second case if the matrix $\tilde{B}_{22}$ is not degenerate.

Now we can apply to solving (3.22) an iterative method with the preconditioner B^{-1}, e.g. the generalized minimum residual method (GMRES):

$$\begin{gathered} \overset{\circ}{u}{}^{0}= B^{-1}\,\overset{\circ}{f}, \\ B(\overset{\circ}{u}{}^{n}-\overset{\circ}{u}{}^{n-1})=-\sum_{m=1}^{p}\gamma_{n,m}\xi^{n-m},\ n\geq 1, \end{gathered} \tag{3.27}$$

where $\xi^n=\overset{\circ}{A}\overset{\circ}{u}{}^{n}-\overset{\circ}{f}$ is the nth residual, p is some number and the parameters $\gamma_{n,m}$ are chosen from the condition

$$\|\xi^n\|_2=\min\,.$$

To implement this method it is necessary to solve system of equations with the matrix B at each step. It can be done by means of the well-known fast direct method that uses separation of variables (see Section 5).

It is well known [1, 9, 11, 15] that the convergence of the iterative method strongly depends on the chosen way of enlargement (3.22) and on the choice of $\tilde{B}_{21}$, $\tilde{B}_{12}$, $\tilde{B}_{22}$. The first way (3.24) is usually used with $\tilde{B}_{21}=0$, $\tilde{B}_{22}=0$ because it is the best choice of them for self-adjoint problems [9, 11, 15]. Application of this variant to solving the Helmholtz wave equation has been considered in [12, 13].

The second variant (3.25) of the method had not been applied before to the problem under consideration. The author has implemented it and carried out numerical experiments for the both variants with comparison of results (see Section 7).

Remark. If for some number j_0 all the nodes (r_j, θ_k), $j \le j_0$, $k = 0, ..., n_\theta - 1$ belong to $\bar{\Omega}_h$ then we can decrease the dimension of B prescribing $v_{j_0,k} = 0$, $k = 0, ..., n_\theta - 1$ in (3.9).

4 Implementation within the subspace

Let us suppose that all the rows of submatrices $\tilde{B}_{22}$ and B_{22} coincide except for (maybe) the rows corresponding to nodes of $\partial\Omega_h$ and neighboring ones. Let us introduce the matrix

$$C = B - \overset{\circ}{A} . \tag{4.28}$$

It is easy to see that all the rows of B and $\overset{\circ}{A}$ coincide except for the rows corresponding to nodes of $\partial\Omega_h$ and neighboring nodes of R_h^2. We denote by S_h the union of these sets of nodes. Thus, all the rows of C are equal to zero except for the rows corresponding to nodes of S_h. The total number of such nodes is of the order of $N^{1/2}$.

It follows from (3.27) that

$$\xi^0 = -CB^{-1} \overset{\circ}{f} . \tag{4.29}$$

Hence, $\xi^0 \in \operatorname{im} C$, i.e. this residual vector is equal to zero everywhere except on the nodes of S_h.

Method (3.27) can be rewritten as follows:

$$\begin{aligned} \xi^0 &= -CB^{-1} \overset{\circ}{f}, \quad C \overset{\circ}{u}{}^0 = CB^{-1} \overset{\circ}{f}, \\ \xi^n &= \xi^{n-1} - \sum_{m=1}^{p} \gamma_{n,m} \overset{\circ}{A} B^{-1} \xi^{n-m}, \; n \ge 1, \\ C \overset{\circ}{u}{}^n &= C \overset{\circ}{u}{}^{n-1} - \sum_{m=1}^{p} \gamma_{n,m} C B^{-1} \xi^{n-m}, \; n \ge 1, \end{aligned} \tag{4.30}$$

and

$$\overset{\circ}{A} B^{-1} \xi = (I - CB^{-1})\xi \tag{4.31}$$

for any vector ξ (using the definition of C). It is easy to see from (4.30)-(4.31) that all the vectors ξ^n, $\overset{\circ}{A} B^{-1} \xi^n$ belong to the image of C, i.e. are equal to zero everywhere except on the nodes of S_h.

Thus, we can implement this iterative process by storing only $O(N^{1/2})$ components of vectors ξ^n, $\overset{\circ}{A} B^{-1} \xi^n$ (because the other components are equal to zero). Calculation of linear combinations of vectors in (4.30) and scalar products in GMRES requires $pO(N^{1/2})$ arithmetic operations. The last property of the method is very important especially when p is a big number.

It is easy to see from (4.31) that to calculate vector $\overset{\circ}{A} B^{-1} \xi$ we need values of $B^{-1}\xi$ only at nodes of S_h (because a value of $CB^{-1}\xi$ depends only on them). While a vector $B^{-1}\xi$ is known the calculation of $\overset{\circ}{A} B^{-1} \xi$ requires $O(N^{1/2})$ arithmetic operations.

It follows from above that the implementation of the method under consideration requires $O(N^{1/2})$ computer memory locations and $p\,O(N^{1/2})$ arithmetic operations per step except for calculation of vector $v = B^{-1}\xi$. The last problem is considered in the following Section.

Remark. We don't know exact values of some coefficients of the matrices A and B due to the used approximation of the Sommerfeld condition. But it is not necessary for us to know them. It is sufficient that they coincide in the both matrices.

5 Partial solution algorithm

To implement the considered method the linear system of equations

$$Bv = \xi, \tag{5.32}$$

should be solved at each step, where a right-hand side ξ belongs to the subspace $\operatorname{im} C$, i.e. it can be distinct from zero only at nodes of S_h. It is necessary to calculate a solution v only at the same nodes. This problem is called "the partial solution problem".

Now we will describe an algorithm for solution of this problem which was proposed in [4, 9]. It is a modification of the well-known fast direct method that uses separation of variables.

We will try the solution to (5.32) in the form (3.11) for $j \leq n_r$:

$$v_{jk} = \sum_{l=0}^{n_\theta - 1} v_j^l \varphi_k^l. \tag{5.33}$$

Let us represent a right-hand side of (5.32) in the same form:

$$\xi_{jk} = \sum_{l=0}^{n_\theta - 1} \xi_j^l \varphi_k^l, \tag{5.34}$$

It is well known that

$$\xi_j^l = (\varphi^l, \xi_j) = \sum_{k=0}^{n_\theta - 1} \xi_{jk} \varphi_k^l, \; j = 0, ..., n_r, \; l = 0, 1, ..., n_\theta. \tag{5.35}$$

Substituting (5.33) and (5.34) into (5.32) we obtain n_θ independent linear systems of equations with the tridiagonal matrices:

$$\begin{gathered}
\alpha_0 v_0^0 - \beta_0 v_1^0 = \xi_0^0, \\
-\beta_{j-1} v_{j-1}^l + \alpha_j v_j^l - \beta_j v_{j+1}^l = \xi_j^l, \quad j = 1, ..., n_r, \\
l = 0, 1, ..., n_\theta - 1, \\
v_{n_r+1}^l = z_{n_r}^l v_{n_r}^l,
\end{gathered} \tag{5.36}$$

where

$$\alpha_0 = 0.5\,(1 - \tfrac{1}{3} h_r^2 \omega^2) h_\theta, \quad \beta_0 = 0.5\,(1 + \tfrac{1}{6} h_r^2 \omega^2) h_\theta,$$

$v_0^l = 0$ for $l \neq 0$ and the coefficients α_j, β_j for $j > 0$ have been presented in (3.13).

Thus, we have the following algorithm for solving (5.32):

Algorithm. For $l = 0, 1, ..., n_\theta - 1$:

1. Calculate ξ^l_j, $j = 0, 1, ..., n_r$, for a given l by means of formula (5.35).
2. Solve system (5.36) for a given l by means of the Gauss elimination method for tridiagonal matrices, i.e. we calculate the coefficients v^l_j.
3. Add the lth items in (5.33) only for vertices of S_h.

Let us estimate the number of arithmetic operations in this Algorithm. The first step requires $O(N^{1/2})$ operations for each l because ξ_{jk} can be nonzero only at vertices $(r_j, \theta_k) \in S_h$ and $\dim S_h = O(N^{1/2})$. The third step requires $O(N^{1/2})$ operations for each l as well because we calculate v_{jk} only at the same nodes. The second step obviously requires $O(n_r)$ operations for each l. Hence, partial solution of problem (5.32) requires $O(N^{1/2} + n_r) \cdot n_\theta = O(N_0)$ arithmetic operations.

It is easy to see that it requires $O(N^{1/2})$ computer memory locations.

It follows from the present and previous Sections that the method under consideration requires $O(N^{1/2})$ computer memory locations and each step requires $O(N_0 + pN^{1/2})$ arithmetic operations.

Remark. To be numerically stable the first stage of the Gauss elimination method should be implemented in the direction "from the boundary $j = n_r$ to the centre". But it is attractive to implement it in the opposite direction because then we would get real matrix coefficients (except for the last one) and hence less computing time. Maybe, the latter variant with non-monotonic pivoting can be used.

6 Calculation of an approximate solution and post-processing

After completion of the iterative method (4.30)-(4.31) we don't yet have the approximate iterative solution $\tilde{u}$ itself. We only have the vector $C \overset{\circ}{u}^{n_{\max}}$. To calculate $\tilde{u}$ we use the following obvious relation for the precise solution $\overset{\circ}{u}$:

$$B \overset{\circ}{u} = C \overset{\circ}{u} + \overset{\circ}{f} . \tag{6.37}$$

Substituting $C \overset{\circ}{u}^{n_{\max}}$ by $C \overset{\circ}{u}$ in (6.37) we obtain

$$\tilde{u} = B^{-1}(C \overset{\circ}{u}^{n_{\max}} + \overset{\circ}{f}) = B^{-1} C \overset{\circ}{u}^{n_{\max}} + \overset{\circ}{u}^{0} . \tag{6.38}$$

Only some components of $\tilde{u}$ are usually needed in practice for postprocessing. To calculate them we can solve the corresponding partial solution problem (5.32) with the right-hand side $C \overset{\circ}{u}^{n_{\max}} + \overset{\circ}{f}$.

In practice it is often necessary to calculate the values $G_{k,h}$, $k = 0, 1, ..., n_\theta - 1$ in the asymptotic expression (2.3). Obviously, the corresponding Fourier coefficients G^l_h satisfy the relations (see (3.19)):

$$G^l_h = (z^l_{n_r} z^l_{n_r+1} ... z^l_{n_\infty} (n_\infty + 1)^{1/2} / q^{n_\infty + 1}) \tilde{u}^l_{n_r}, \; l = 0, ..., n_\theta - 1. \tag{6.39}$$

Thus, we can calculate the multiplications of z_j^l for each l in advance (during process (3.19)), use them in (6.39) and then obtain $G_{k,h}$ by means of the reverse Fourier transformation:

$$G_{k,h} = \sum_{l=0}^{n_\theta - 1} G_h^l \varphi_k^l. \tag{6.40}$$

7 Numerical experiments

The author has carried out a lot of numerical experiments where the considered problem was solved by means of the methods described above. Variant (3.24) was considered with $\tilde{B}_{21} = 0$, $\tilde{B}_{22} = 0$, both without and with the strengthening from [13]. Variant (3.25) was considered with $\tilde{B}_{12} = B_{12}$, $\tilde{B}_{22} = B_{22}$.

Here the main conclusions from these experiments are:

1. Iterative method (4.30) with $p = 100$ and upper enlargement (3.25) converged for all the considered obstacles Ω and values of ω.

2. It always required less computing time than variant with lower enlargement (3.24). In many cases the latter one didn't converge at all (even if the strengthening from [13] was used).

3. If an obstacle was fixed then the number of iteration in the variant with upper enlargement weakly depended on the meshsize for a fixed value of ω but was approximately proportional to the last one for a fixed value of λ/h.

4. This variant of fictitious domain method works sufficiently faster than other known methods for solving the problem.

5. It is able to solve huge problems with big values of N and number of wavelengthes per obstacle.

Results of the experiments with this variant for several concrete obstacles Ω are presented below. All of the experiments were carried out with $p = 100$ (see (4.30)) and $n_\infty = 50\, n_r$ (see Section 3.3).

7.1 Circle

This obstacle was considered for testing the approximation and the computer program because the exact solution of (1.1) was known. In the experiment we took $\lambda = 2R$, $h_r = R/10$ and $n_\theta = 80$ where R was the radius of the sphere. The iterative method was stopped when the following criterium was satisfied:

$$\varepsilon = \frac{\|\xi\|_2}{\|f\|_2} \leq 10^{-6}. \tag{7.41}$$

The comparison of the calculated solution and the exact one was carried out within the ring $R \leq r \leq 2R$ in the C^0-norm (courtesy of F.Baron). We got the precision equaled 1.2 %. The same comparison for the piecewise-linear approximation of the problem gave the precision equaled 2.6 %.

7.2 The Π-shaped open resonator

This obstacle is presented in Fig. 1 together with the locally fitted mesh used for the calculations. In the experiment we took λ equaled to the inner width of the resonator, $h_r \approx \lambda/20$ and

$n_\theta \approx 2\pi n_r$ (i.e. the grid cells near the artificial boundary $r = n_r h_r$ were approximately equilateral). The mesh in Fig. 1 has the following parameters: $n_r = 47$, $n_\theta = 280$, $N_0 = 13161$. The experiments were carried out for various values of the incidence angle α (see Section 1), from $\alpha = 0°$ to $\alpha = 90°$, with computer Apollo-DN5500.

The curve "$\log_{10} \frac{\|\xi\|_2}{\|f\|_2}$ versus CPU time" for $\alpha = 30°$ is presented in Fig. 2. It doesn't start from zero due to an overhead cost ≈ 20 sec (including mesh generating and calculation (3.19) of the coefficients $z^l_{n_r}$, $l = 0, 1, ..., n_\theta - 1$). To obtain the precision $\varepsilon = 10^{-6}$ (see (7.41)) computing time was about 5 min and number of iterations was from 55 to 60 for all values of α.

The calculated solution for $\alpha = 30°$ is presented in Fig. 3 (20 isolines of the real part) and in Fig. 4 (20 isolines of the imaginary part).

Besides that the same problem was solved on the finer mesh with $n_r = 93$, $n_\theta = 560$, $N_0 = 52081$. Computing time was about 25-30 min and number of iterations was about 60-70. The calculated solution for $\alpha = 30°$ is presented in Fig. 5 (20 isolines of the real part).

7.3 The airplane-like obstacle

This obstacle is presented in Fig. 6 together with the locally fitted mesh. The experiments were carried out for various values of the incidence angle α and of the coefficient ω, with computer Apollo-DN10000.

Parameters and results of three experiments for this obstacle are presented in the following Table:

ω	Wavelengthes per obstacle	$\frac{\lambda}{h_r}$	α	ε	n_r	n_θ	N_0	Number of iterations	Computing time
1.2	3.6	26	90°	10^{-6}	53	320	14400	33	41 sec
8	24	10	0°	10^{-6}	130	750	83250	213	23 min
15	45	10	0°	10^{-6}	259	1500	330000	406	3 h
30	90	10	0°	10^{-4}	518	3200	1414400	401	10 h

Table 1. Results of calculations for the airplane-like obstacle

The corresponding solutions are presented in Fig. 7-10 (isolines of the real part)

8 Conclusion

Thus, the considered variant presented itself in the numerical experiments in a very good light. But there are still several open questions concerning to it:

1. **Correctness of the method.** As it has been mentioned in Section 3.4 the method with upper enlargement is correct if the matrix $\tilde{B}_{22}$ is non-degenerate. But, generally speaking, it is not always so. We can only say that the degeneracy of $\tilde{B}_{22}$ (or existing of its eigenvalues very close to zero) has a little probability in real practical problems.

2. **Convergence of the method.** It has not been yet theoretically proved saying nothing about theoretical estimations of its rate.

3. **Optimal choice of the matrices $\tilde{B}_{12}$, $\tilde{B}_{22}$ for better convergence.**

All of them require subsequent investigations.

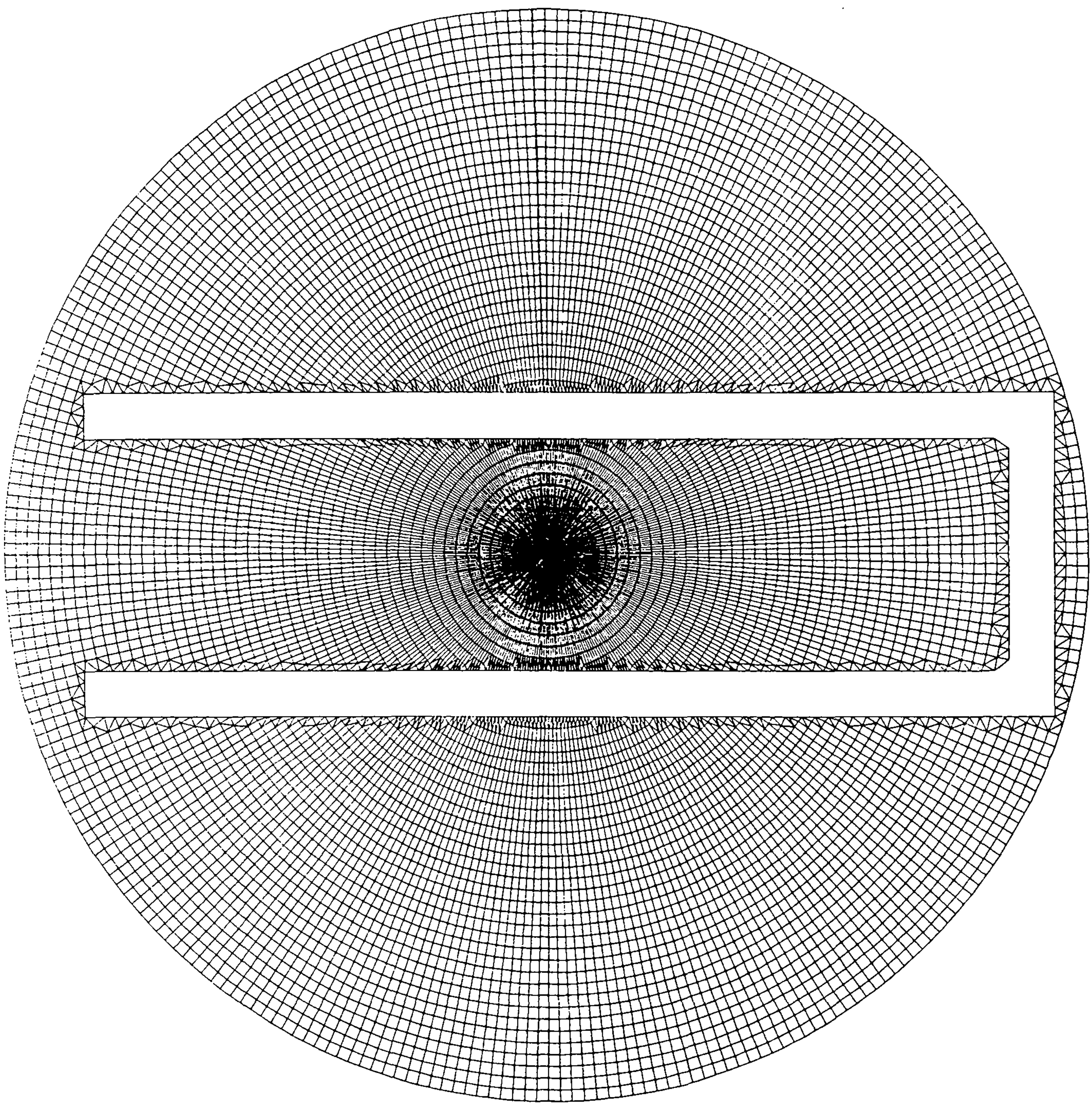

Figure 1. The locally fitted mesh for the Π-shaped open resonator

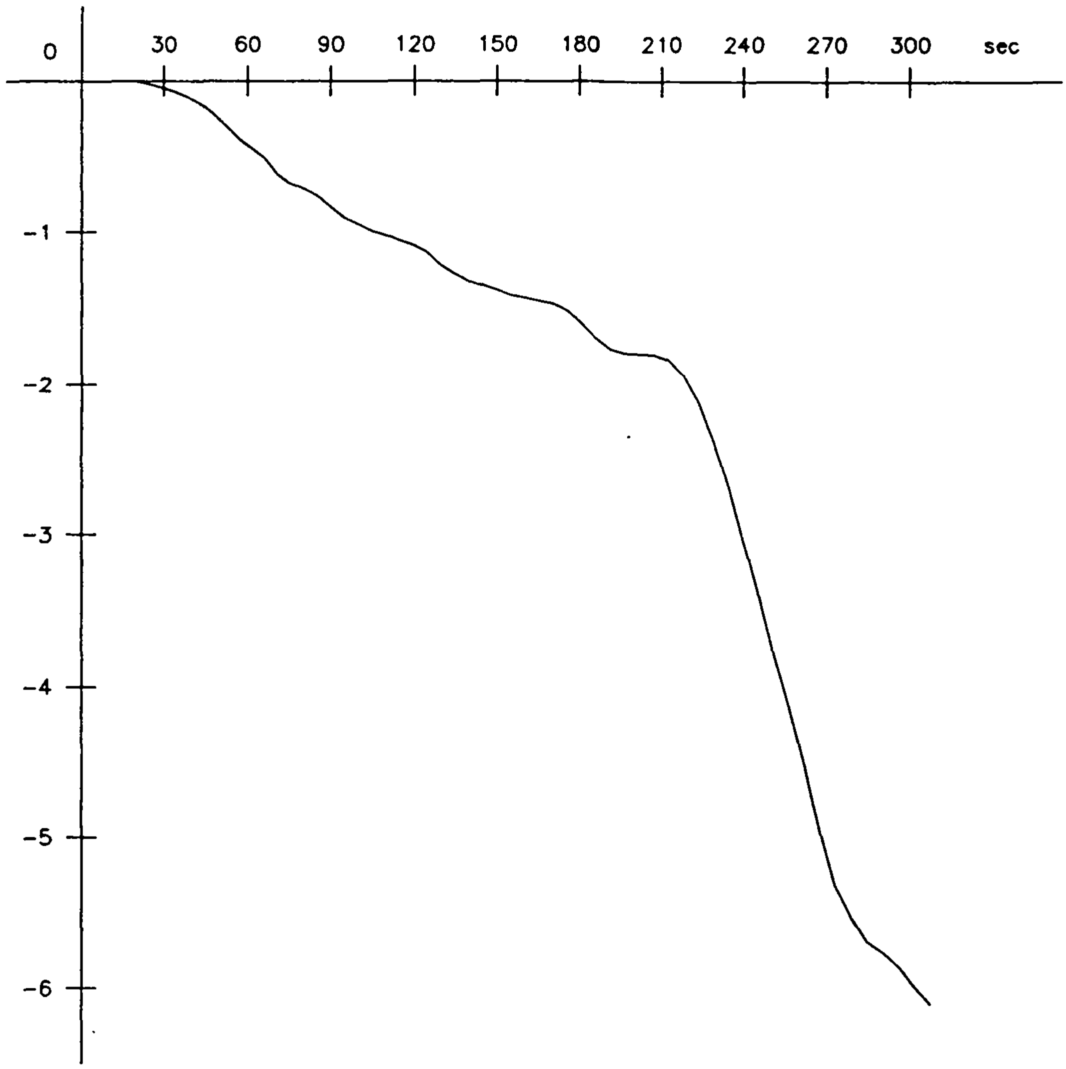

Figure 2. $\log_{10} \frac{\|\xi\|_2}{\|f\|_2}$ versus CPU time for the Π-shaped open resonator

$n_r = 47,\ n_\theta = 280,\ N_0 = 13161,\ \alpha = 30°$

The computer - Apollo DN5500

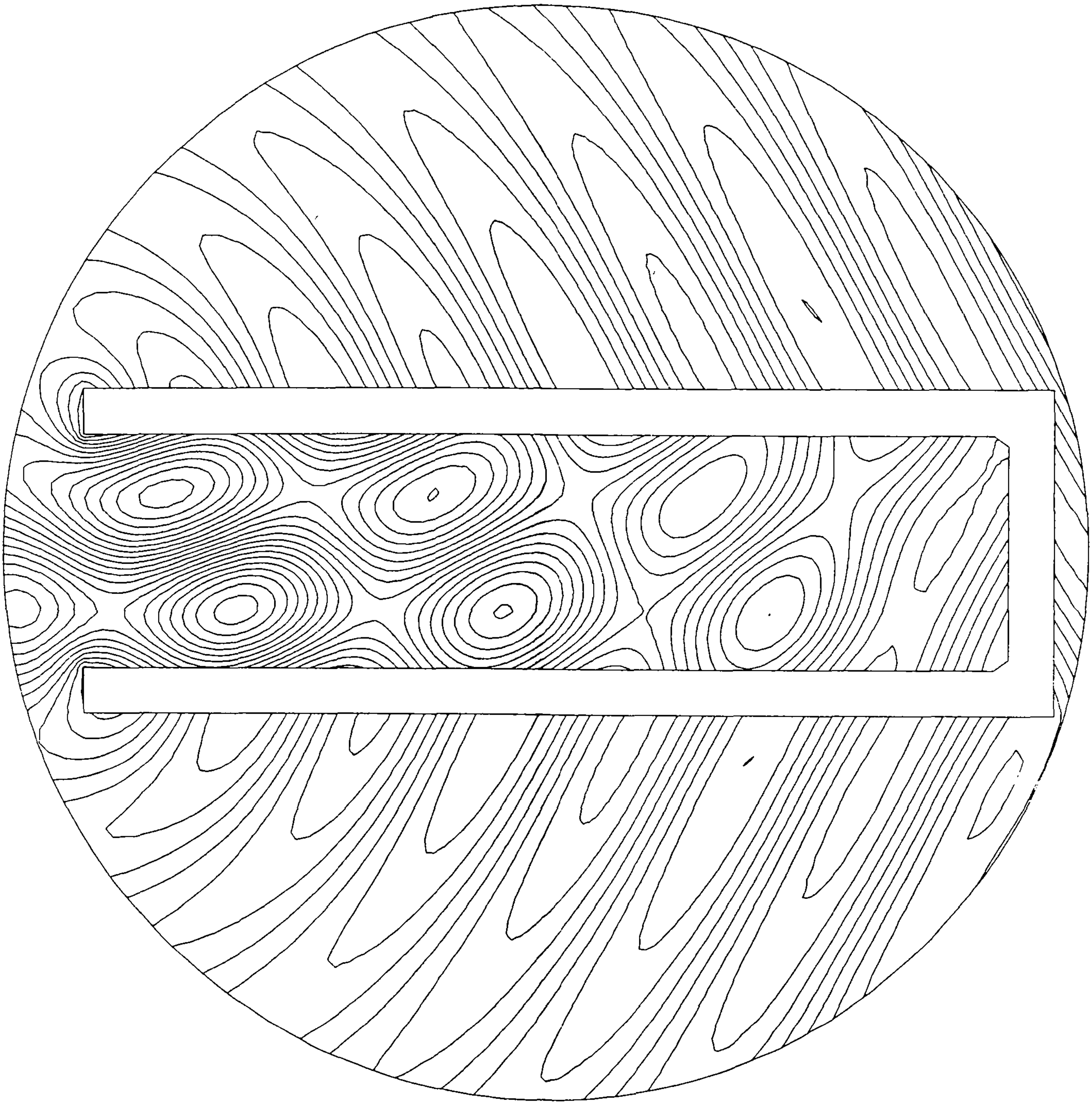

Figure 3. Solution for the Π-shaped open resonator
$n_r = 47,\ n_\theta = 280,\ N_0 = 13161,\ \alpha = 30^\circ$
20 isolines of the real part
$\min = -2.817, \quad \max = 2.721$

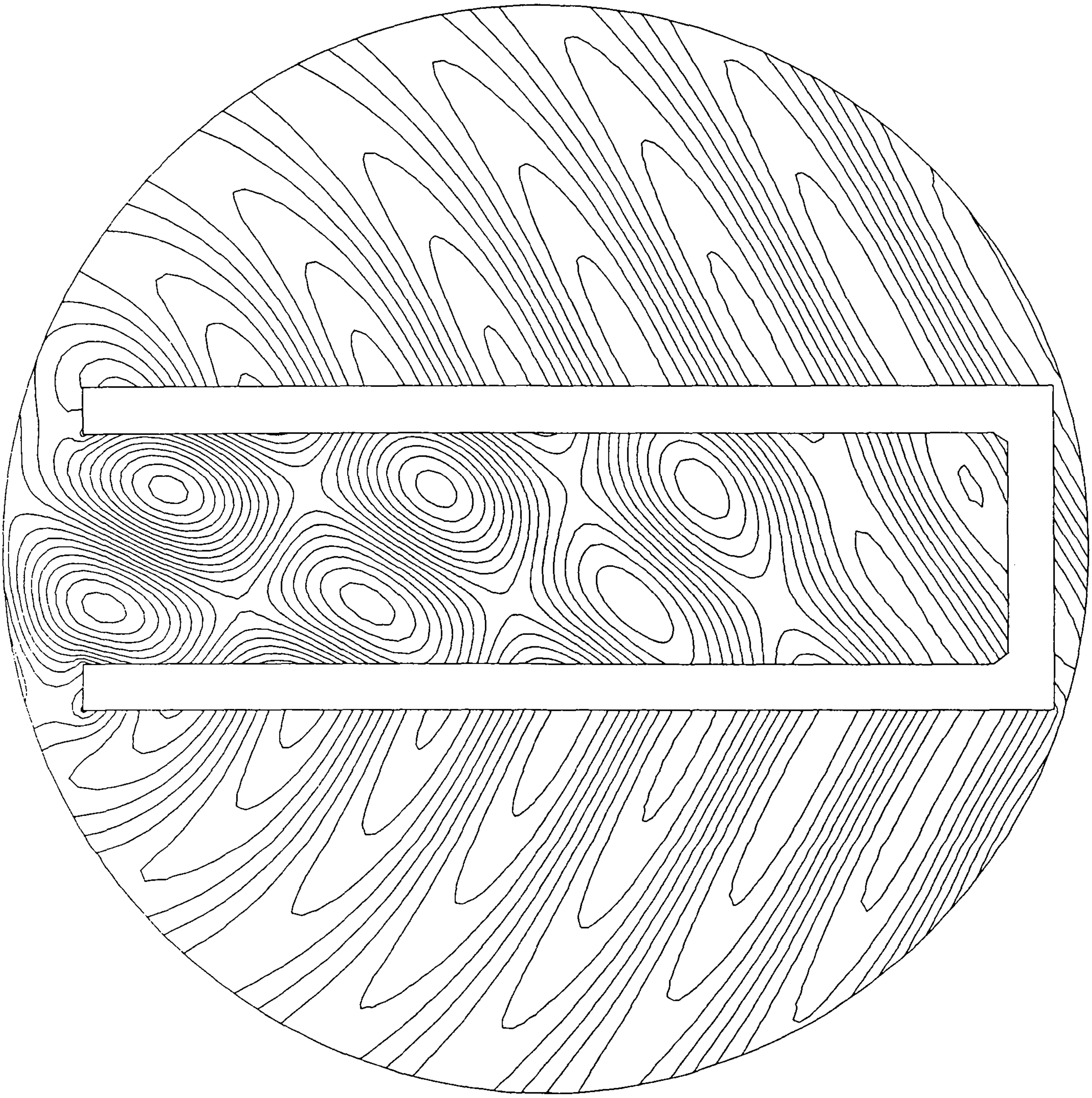

Figure 4. Solution for the Π-shaped open resonator
$n_r = 47,\ n_\theta = 280,\ N_0 = 13161,\ \alpha = 30°$
20 isolines of the imaginary part
min $= -2.382$, max $= 2.218$

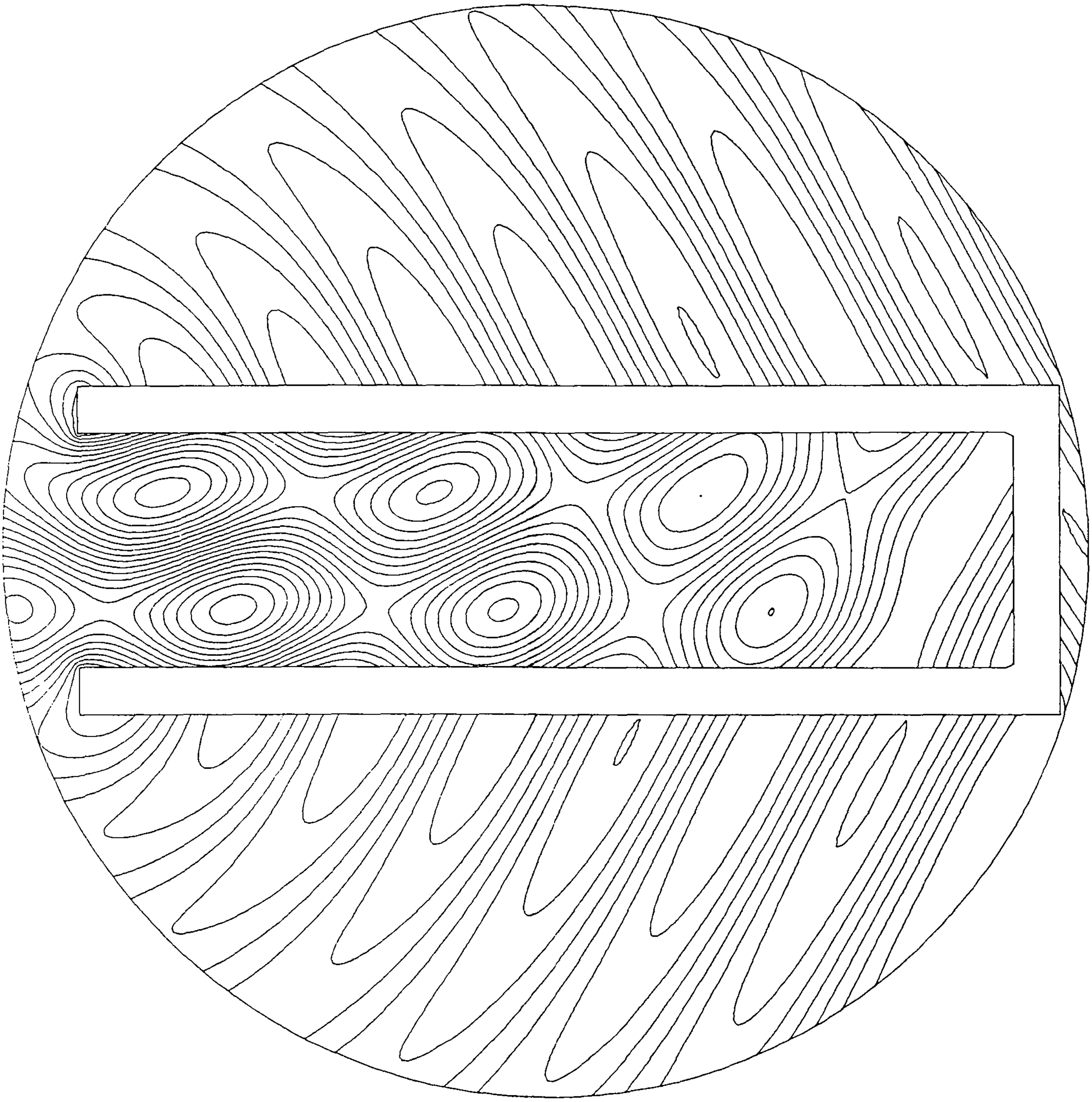

Figure 5. Solution for the Π-shaped open resonator
$n_r = 93,\ n_\theta = 560,\ N_0 = 52081,\ \alpha = 30°$
20 isolines of the real part
$\min = -2.990, \quad \max = 2.924$

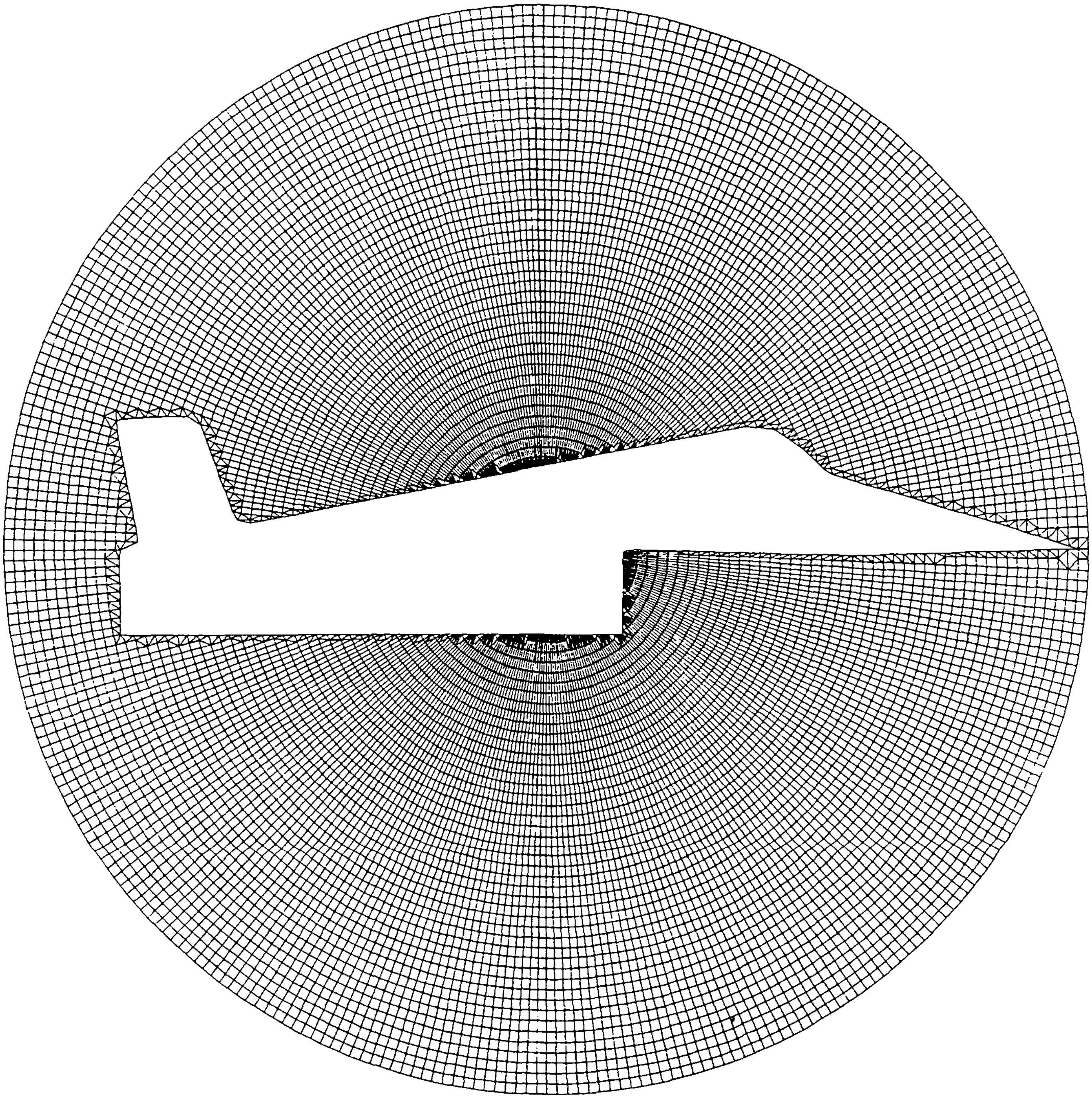

Figure 6. The locally fitted mesh for the airplane-like obstacle

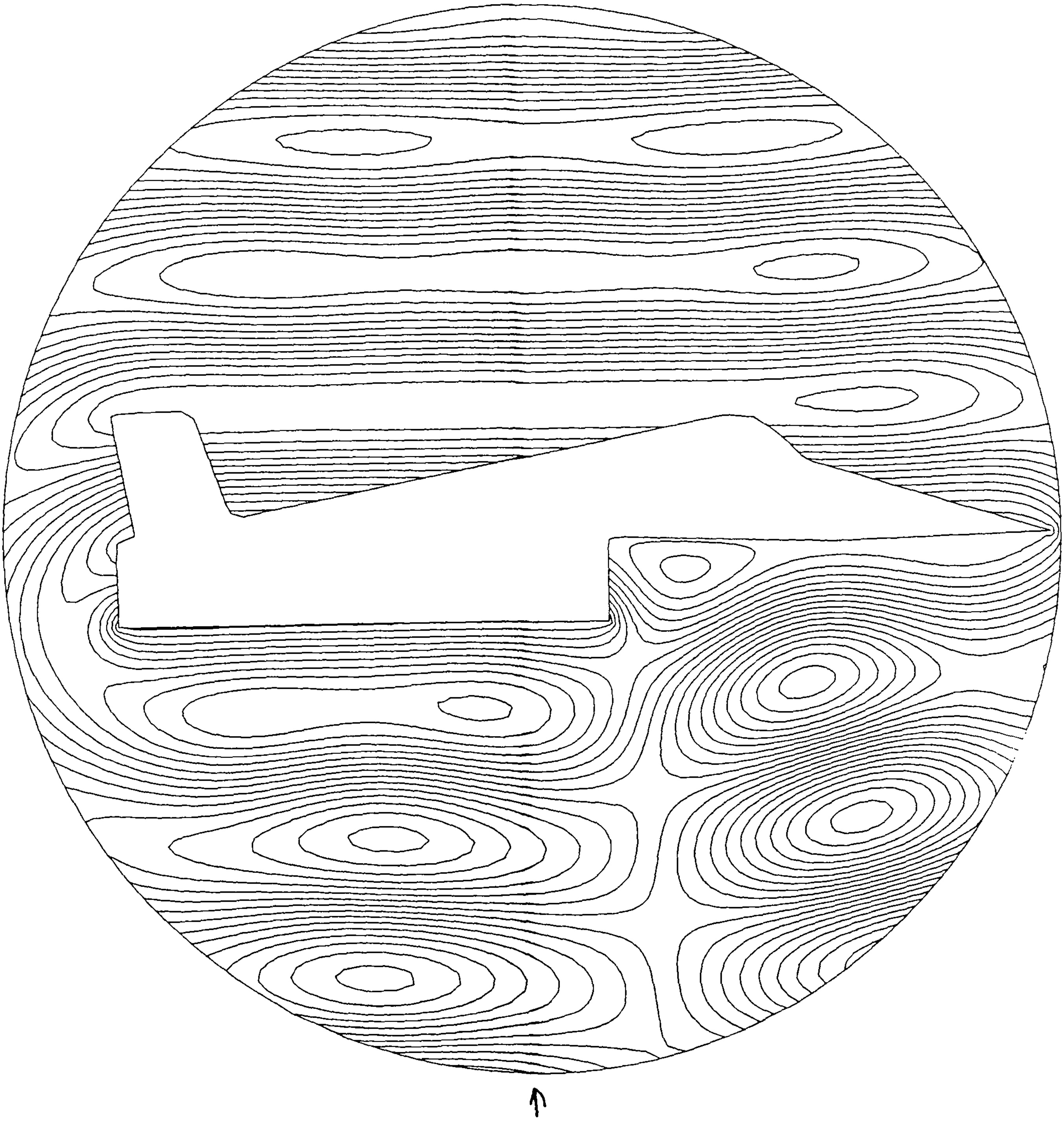

Figure 7. Solution for the airplane-like obstacle
$\omega = 1.2,\ \alpha = 90^\circ,\ n_r = 53,\ n_\theta = 320,\ N_0 = 14400$
20 isolines of the real part

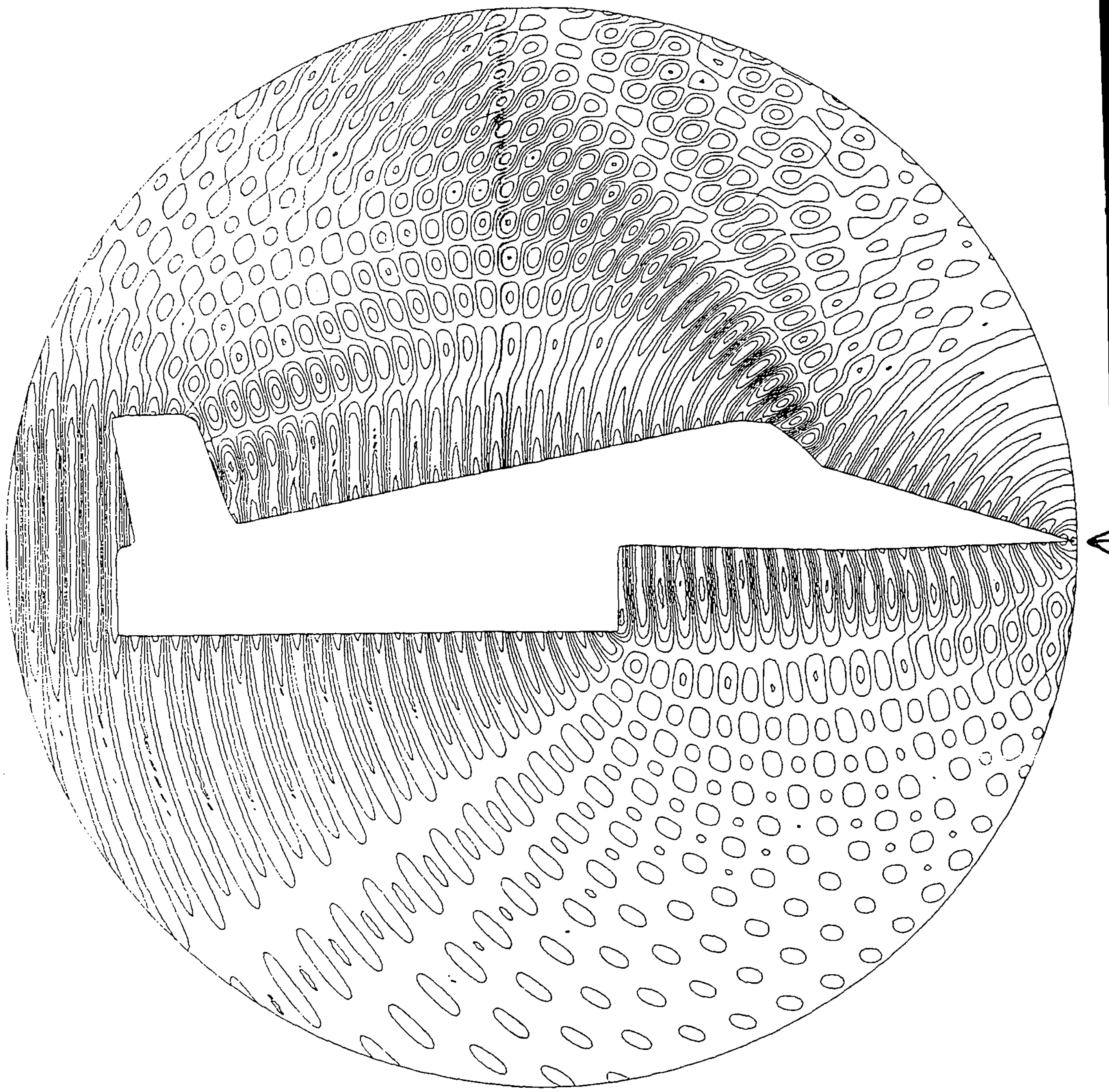

Figure 8. Solution for the airplane-like obstacle
$\omega = 8,\ \alpha = 0°,\ n_r = 130,\ n_\theta = 750,\ N_0 = 83250$
10 isolines of the real part

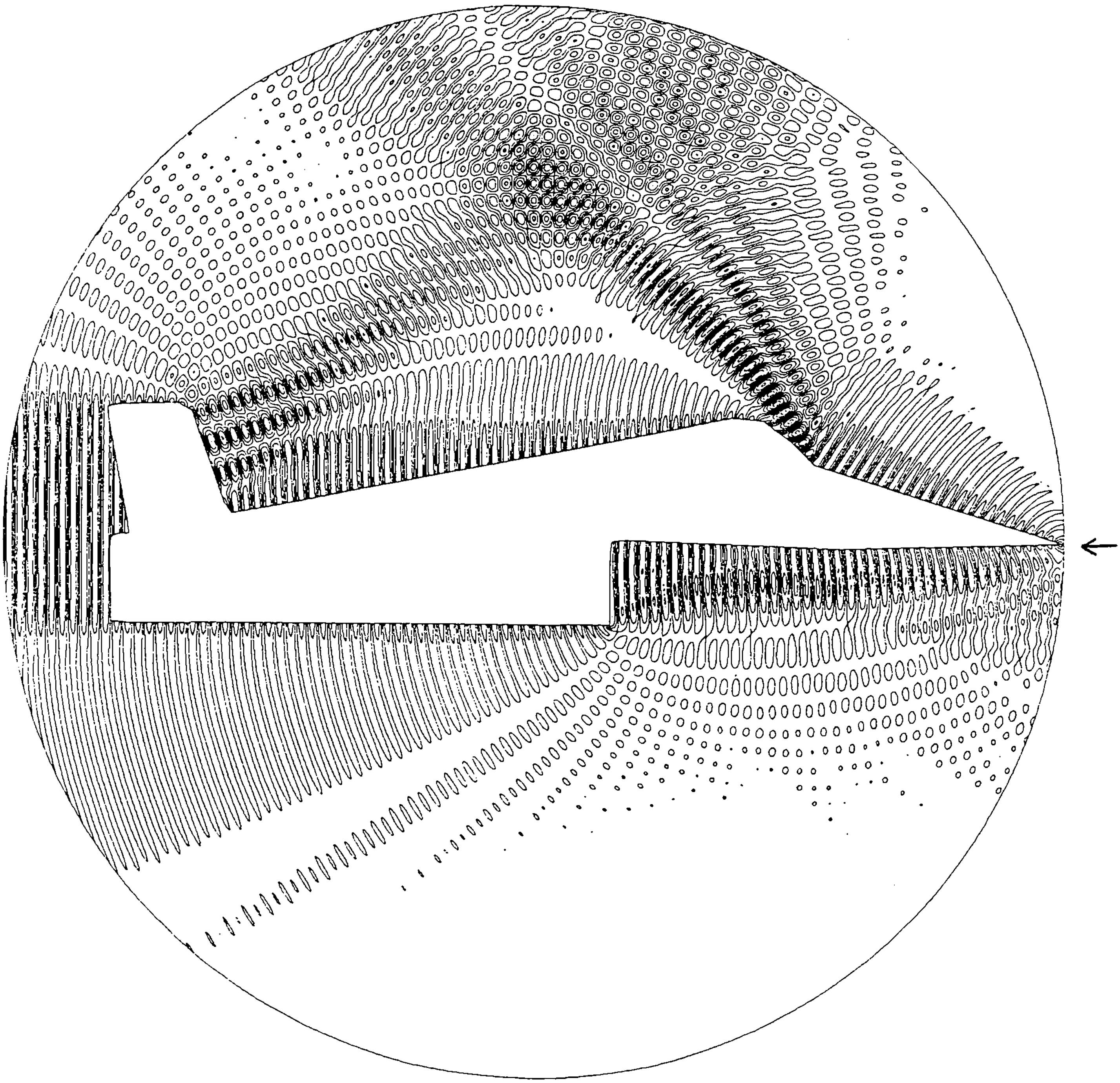

Figure 9. Solution for the airplane-like obstacle
$\omega = 15,\ \alpha = 0°,\ n_r = 259,\ n_\theta = 1500,\ N_0 = 330000$
10 isolines of the real part

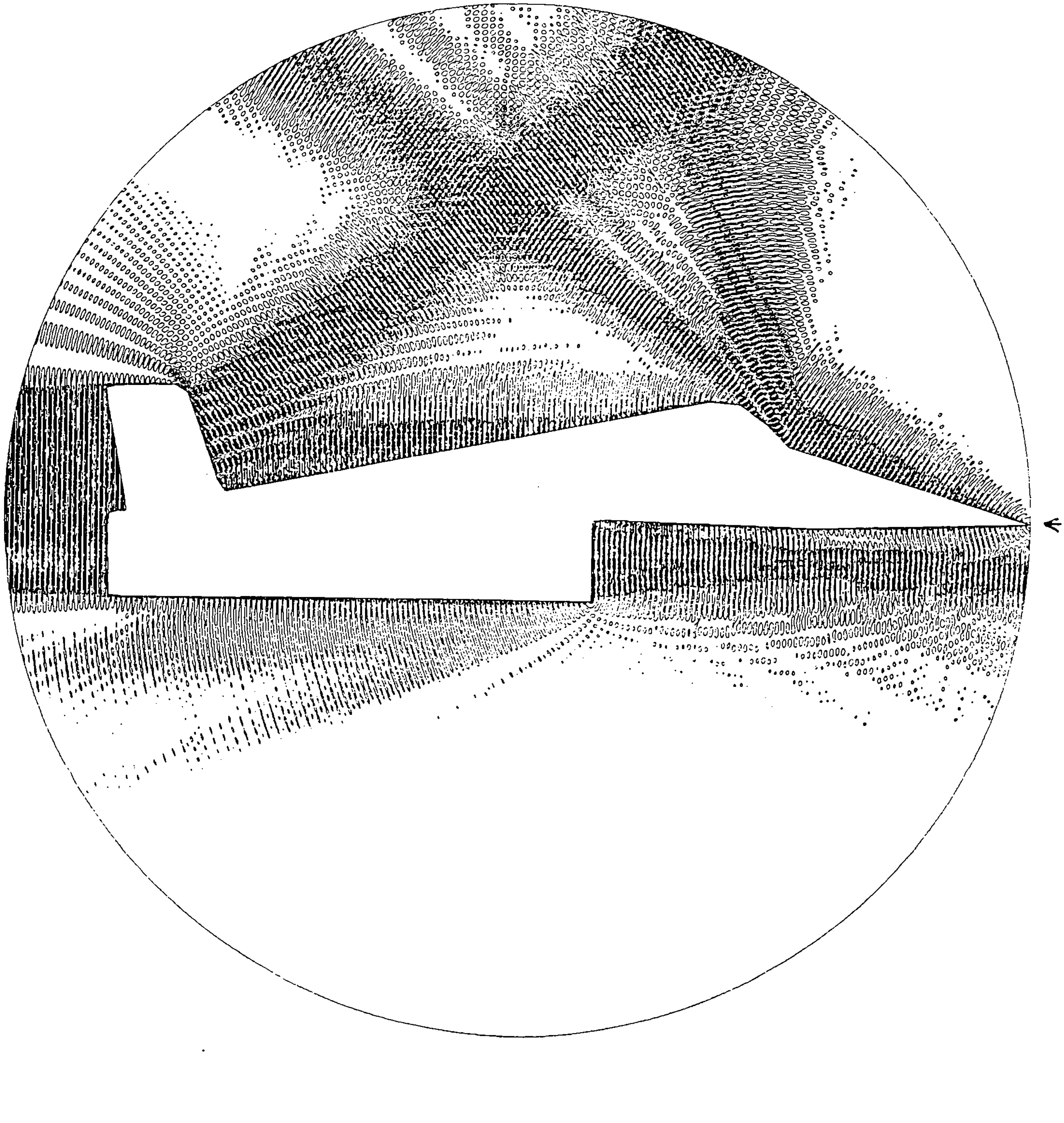

Figure 10. Solution for the airplane-like obstacle
$\omega = 30,\ \alpha = 0°,\ n_r = 518,\ n_\theta = 3200,\ N_0 = 1414400$
10 isolines of the real part

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ISSN 0249-6399