Automated mathematical induction

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Automated Mathematical Induction

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Abstract

Proofs by induction are important in many computer science and artificial intelligence applications, in particular, in program verification and specification systems. We present a new method to prove (and disprove) automatically inductive properties. Given a set of axioms, a well-suited induction scheme is constructed automatically. We call such an induction scheme a test set. Then, for proving a property, we just instantiate it with terms from the test set and apply pure algebraic simplification to the result. This method needs no completion and explicit induction. However it retains their positive features, namely, the completeness of the former and the robustness of the latter. It has been implemented in the theorem-prover SPIKE [1].

Contents

[1] Preliminary versions of the results in this paper have been presented at the *First International Symposium on Artificial Intelligence and Mathematics, Fort Lauderdale, Florida, January 1990* and in the *Bulletin of European Association for Theoretical Computer Science, 41:216-226, June 1990.*

1 Introduction

1.1 Motivation

Inductive reasoning is simply a method of performing inferences in domains where there exists a well-founded relation on the objects. It is fundamental when proving properties of numbers, data-structures, or programs axiomatized by a set of equations and or conditional axioms. The use of conditional axioms as definitions is wellknown in Computer Science, since programs in logical or functional style and data-structures can be expressed in this framework. As opposed to deductive theorems, inductive theorems are usually valid only in some particular models of the axioms. For instance, Herbrand models or initial models fit nicely with the semantics of data-type specifications, logic and functional programming. As a classical example, consider the data-structure of nonnegative integers built up using the 0 and S function symbols. Every element of this structure can be represented by a variable-free (ground) term that involves 0 and S only. Suppose we define the addition operation $+$ by the following axioms:

$$\begin{aligned} x + 0 &= x \\ x + S(y) &= S(x + y) \end{aligned}$$

Clearly, given two integers, using the above equations for adding them yields a non-negative integer. For instance, $S(S(S(0))) + S(S(0))$ equates to $S(S(S(S(S(0)))))$ by deductive reasoning: apply just twice the second axiom and once the first. Consider now the property of associativity of $+$:

$$(x + y) + z = x + (y + z)$$

This is a typical example of an identity whose proof requires some kind of induction. As everybody knows from his experience, it might be difficult, not only to find an

appropriate well-founded relation to support inductive inferences, but also to guess suitable induction hypothesis. Two main approaches have been proposed to overcome these difficulties. The first applies explicit induction arguments on the structure of terms [Aub79, Bur69, BM79, BHHW86, GG89, ZKK88]. The second one involves a proof by consistency: this is the *inductionless induction* method [Mus80, HH82, KM87, JK86, Fri86, KNZ86, Bac88, Küc89, BK89, Gra89]. To prove the associativity of + by explicit induction we can use the following scheme:

Basic Case: $(x + y) + 0 = x + (y + 0)$

Induction Step: $(x + y) + Su = x + (y + Su)$

The proof of Basic Case is trivial, and the proof of Induction Step follows from the defining axioms and the induction hypothesis $(x + y) + u = x + (y + u)$. On the other hand, to prove the same property by inductionless induction we first try to compile the axioms into a convergent (i.e., terminating and confluent) set of rewrite rules. In the above example it is sufficient to orient the axioms from left to right. Then the associativity is added as a rule oriented from right to left and a completion-like procedure is started in order to transform the whole set of rules into a convergent one by superposing left-hand sides. Here, all superpositions lead to trivial identities and therefore the associativity of + is proved.

However, both methods have many limitations, either on the theorems that could be proved or on the underlying theory. For instance, explicit induction techniques are unable to provide us automatically with induction schemes, and cannot help to disprove false conjectures. Note also that guiding a proof by explicit induction requires some skill for finding the right axioms or hypothesis to apply. On the other side, the inductionless induction technique often fails where explicit induction succeeds. If the associativity of + were oriented from left to right in the example this method fails to prove it, since the completion procedure loops in this case. Moreover, there does not exist any realistic inductionless induction procedure for conditional theories. However the main advantage of inductionless induction is that it does not require to build an hierarchy of lemmas to be proved. All the conjectures and intermediate results are considered at the same level (and proved in a same round) without any notion of hierarchy.

In this paper we present an alternative proof system for automatizing inductive reasoning in theories defined by conditional axioms. We show how to prove (and disprove) equations and, more generally, clauses in Herbrand models. This proof system combines the power of explicit induction and inductionless induction.

1.2 The aims of the paper

The presented method relies on the notion of *test sets* (which, in essence, is a finite description of the initial model) and uses only pure algebraic simplification. The key-idea of the simplification strategy is to use axioms, previously proved conjectures, and instances of the conjecture itself as soon as they are smaller than the currently considered proposition with respect to a well-founded relation. This last point captures

the notion of Induction Hypothesis in the proof by induction paradigm. The main observation is that when the axioms are oriented as a terminating set of rules, they provide a natural well-founded ordering which can be used to support induction.

The main arguments in favour of our method are:

- it works even when some functions are incompletely defined and when there are relations between constructors which is not the case for Boyer-Moore system;
- for its correctness, it does not require the given theory to be turned into a confluent set of axioms;
- it provides automatically induction schemes through algorithms for computing test sets;
- it allows to refute false conjectures when the axioms are presented by a ground convergent conditional rewrite system; it is also refutationally complete in the following sense : any equation that is not valid in the initial model will be disproved, provided that the axioms can be turned into a ground confluent unconditional rewrite system;
- it does not require any hierarchy for handling the induction hypothesis;
- it is not restricted to equational theories but also applies to conditional theories;
- it is not restricted to the proof (and disproof) of equations but also applies to non-orientable equations and general clauses.

Hofbauer and Kutsche [HK88] were the first to notice that inductionless induction techniques still work when the ground confluence property of the input set of equations is relaxed. However their procedure is still influenced by the completion framework since it is based on computing critical pairs and testing conjectures for ground reducibility. In the procedure we propose here, and whose first version appeared in january 1990 in [KR90a], we rather combine these two steps through the use of test sets. Our approach is developed for general *conditional* axiomatizations and do not require either the ground convergence property for correctness. Moreover, due to the property of test sets, it can be easily proved to be refutationally complete when the theory is presented by a ground convergent set of equational rules. Reddy presented a related method for the *nonconditional* case in july 1990 [Red90]. He used a notion of *covering sets* developed by [ZKK88], analogous to test sets, for computing induction schemes. However, no procedure is known for deriving such covering sets. This is not surprising since for any equational theory, a set of variables is always a covering set (and such a set is particularly useless for induction). We do not understand how the adepts of *covering sets* assert to perform induction *automatically* without providing a constructive definition of induction schemata.

On the other hand, we give here, following [KR90b, KR90c], a procedure to obtain test sets even in the conditional case. Moreover, the notion of covering set cannot be used to disprove false conjectures. Reddy has to simulate his procedure by inductive completion in order to prove its refutational completeness. Our proof of refutational

completeness is self-contained and can be extended to many strategies. For instance, we allow simplification of conjectures by other conjectures, which is a fundamental feature for efficiency as shown by computer experiments.

1.3 Layout of the paper

The structure of the paper is as follows. In Section 2 we present our inference system on a simple example. In Section 3 we introduce the essential notion used throughout this paper. In particular, we define some useful rewriting relations. In Section 4 we provide the basic theorems on proving and disproving inductive conjectures. In Section 5 we give a general inference system to perform induction and show its correctness. We also prove its refutational completeness for convergent equational theories. In Section 6 we introduce some methods to get a completely operational proof system. In particular, we show how the convergence property (required for the refutational completeness, but not for correctness) can be obtained either by a Knuth-Bendix like procedure or by semantic techniques specific to hierarchical axiomatizations. Whereas the computation of test sets is generally undecidable, in this section we also propose a method to obtain test sets in conditional theories over a free set of constructors. The Section 7 is dedicated to computer experiments with our SPIKE software.

2 Overview of our approach on examples

Before discussing technical details of the method that we propose for mechanizing proofs of inductive theorems, we first describe our inference system on a simple example, namely, positive integers with cut-off and *gcd* functions and the *less* predicate. The arrow $\rightarrow$ just indicates how to apply a (conditional) equation for simplification:

$$x - 0 \rightarrow x \tag{1}$$
$$0 - x \rightarrow 0 \tag{2}$$
$$succ(x) - succ(y) \rightarrow x - y \tag{3}$$
$$(0 < succ(x)) \rightarrow true \tag{4}$$
$$(x < 0) \rightarrow false \tag{5}$$
$$x < y = true \Rightarrow succ(x) < succ(y) \rightarrow true \tag{6}$$
$$x < y = false \Rightarrow succ(x) < succ(y) \rightarrow false \tag{7}$$
$$x < y = true \Rightarrow gcd(succ(x), succ(y)) \rightarrow gcd(succ(x), y - x) \tag{8}$$
$$x < y = false \Rightarrow gcd(succ(x), succ(y)) \rightarrow gcd(x - y, succ(y)) \tag{9}$$
$$gcd(x, 0) \rightarrow x \tag{10}$$
$$gcd(0, x) \rightarrow x \tag{11}$$

Consider the conjectures:

$$x - x = 0 \tag{12}$$
$$x < x = false \tag{13}$$

$$x < succ(x) = true \tag{14}$$
$$x < y = true \ \vee \ x < y = false \tag{15}$$
$$x < y = false \ \vee \ y < z = false \ \vee \ x < z = true \tag{16}$$
$$gcd(x, x) = x \tag{17}$$
$$x < succ(x) = false \tag{18}$$
$$x < y = false \ \vee \ y < x = true \tag{19}$$

Except the two last ones all these propositions are valid in the standard arithmetic: note that 13,15 and 16 state that $<$ is a total ordering on integers. Let us prove them by induction. With our method the first step consists in computing a test set (see definition 4.1). Using techniques of section 6, we get the following test set: $\{0, succ(x), true, false\}$. The next step consists in replacing variables of the conjecture by the elements of the test set and checking these instances using pure simplification.

The simplification strategy may use axioms, previously proved conjectures, and instances of the conjecture itself as long as they are smaller (w.r.t. a noetherian relation that contains the rewriting relation) than the currently considered proposition. This last point captures the notion of Induction Hypothesis in the proof by induction paradigm (see theorem 4.1,4.2).

For the equation 12 two instances need to be checked: $0 - 0 = 0$ and $succ(x) - succ(x) = 0$. The first one reduces immediately to a trivial identity. For the second one consider the reduction (notice the use of 12 as an induction hypothesis):

$$succ(x) - succ(x) \mapsto_3 x - x \mapsto_{12} 0$$

For 13 the only non-trivial instance is $succ(x) < succ(x) = false$

$$succ(x) < succ(x) \mapsto_{7[13]} false$$

For the last derivation, we have used an induction hypothesis to satisfy the condition of 7 (cf., theorem 4.1). For 14 the same argument can be employed. For 15 there are four instances by test set. The only non-trivial case is

$$succ(x) < succ(y) = true \vee succ(x) < succ(y) = false$$

Using case rewriting (cf., definition 3.3), we can split this formula into the conjunction of 20 and 21:

$$\neg x < y = true \vee true = true \vee succ(x) < succ(y) = false \tag{20}$$
$$x < y = true \vee succ(x) < succ(y) = true \vee succ(x) < succ(y) = false \tag{21}$$

20 is trivial and 21 is split again in:

$$x < y = true \vee \neg x < y = false \vee false = false \vee succ(x) < succ(y) = true \tag{22}$$

$$x < y = true \ \vee \ x < y = false \vee$$
$$succ(x) < succ(y) = false \ \vee \ succ(x) < succ(y) = true \tag{23}$$

22 is trivial again and 23 is a subclause of the initial conjecture: this is the induction step (cf., theorem 4.2). The 16 and 17 are proved exactly in the same way. Consider now 18. To disprove it, we are going to use the convergence property of the initial system. Note that theorem 5.1 proves the ground convergence of the given set of axioms. The instances to be considered are : $0 < succ(0) = false$, $succ(x) < succ(succ(x)) = false$. The first one reduces to $true = false$ whose members are irreducible and different. By theorem 4.3 the conjecture is false. For 19 consider the following instance $0 < succ(x) = false \vee succ(x) < 0 = true$. It can be reduced to $true = false \vee false = true$ and therefore 19 is not valid.

3 Preliminaries

We assume that the reader is familiar with the basic notions of Horn theories and rewrite systems. We introduce the essential terminology below and refer to Huet-Open [HO80], and Padawitz [Pad88b] for more detailed presentations.

3.1 Conditional theories

3.1.1 Terms and substitutions

A signature Σ is a pair (S, F) where S is a set of *sorts* and F is a finite set of function symbols such that F is equiped with a mapping *type:* $F \rightarrow S^* \times S$. For any $g \in F$ the value *type(g)* is the *type* of g, $sort(\Sigma)$ will denote S, $fun(\Sigma)$ will denote F and $g : \alpha \rightarrow s$ will denote $g \in F$ with $type(g) \equiv \alpha, s$. Constants are represented by nullary function symbols. From now on we assume that a given signature Σ is *sensible*, i.e., it admits at least one ground term of each sort.

Every signature Σ defines a set of formulas that can be built from function symbols taken from Σ and free variables taken from a denumerable set X of variables. Let X be a family $\{X_s\}_{s \in S}$ of sets of free variables indexed by S that is disjoint with Σ. For every sort $s \in \Sigma$ let $T(\Sigma, X)_s$ be the set of Σ-terms of sort s constructed using function symbols in F of type α, s and variables in X. Σ-terms will be denoted by t, s, l, r, ... $Var(t)$ stands for the set of all variables appearing in t. If t is a Σ-term and $x \in Var(t)$ then $\sharp(x, t)$ denotes the number of occurrences of the variable x in t. If $Var(t)$ is the empty set then t is a *ground* term. By $T(\Sigma)_s$ we will denote the set of all ground terms constructed using function symbols in F of type α, s. A term t is *linear* iff $\sharp(x, t) = 1$ for all variables in $Var(t)$.

Furthermore, let N^* be the set of sequences of positive integers, ϵ the empty sequence in N^* and $\cdot$ the concatenation operation on sequences. We call the elements of N^* *positions* and denote them u, v, w, p, q ... We now define the *prefix ordering* $\leq$ in N^* by $u \leq v$ iff there exists w such that $v = u.w$; in this case we define $v/u = w$. We write $u < v$ iff $u \leq v$ and $u \neq v$. Positions u and v are said to be *disjoint*, denoted $u|v$, iff neither $u \leq v$ nor $v \leq u$.

For any term t $dom(t) \subseteq N^*$ denotes its set of positions and the expression t/u denotes the *subterm of t at a position u.* Also let $t(u)$ denote the symbol of t at position u. A position u in a term t is said to be *a strict position* if $t(u) = f \in F$, a *variable*

position if $t(u) = x \in X$ and $\sharp(x,t) = 1$, a *non-linear variable position* if $t(u) = x \in X$ and $\sharp(x,t) > 1$. We use $sdom(t)$ to denote the set of strict positions in t. We write $t[s]_u$ to indicate that s is a subterm of t at position u. We denote $s[t]$ the term s whose a subterm is t.

A $\Sigma - substitution$ $\eta = \{x_1 \leftarrow t_1,\ x_2 \leftarrow t_2,\ \ldots, x_k \leftarrow t_k\}$, where $x_i \neq t_i$, assigns $\Sigma - terms$ of appropriate sorts to variables. It is applied to a $\Sigma - terms$ t, by simultaneous replacement of all occurrences of each x_i in t by t_i. The $\Sigma - terms$ $t_1,\ t_2,\ \ldots,t_k$ are the *values* of η. Greek letters $\sigma,\ \eta,\ \ldots$ denote substitutions. Composition of substitution σ and η is denoted by $\sigma\eta$. The $\Sigma - term$ $t\eta$ obtained by applying a substitution η to t is called an *instance* of t. If η is a ground substitution, we say that $t\eta$ is a *ground instance* of t. A term t *unifies* with a term s if there exists a substitution σ such that $t\sigma \equiv s\sigma$.

3.1.2 Conditional equations and clauses

Let $\Sigma = (S, F)$ be a signature. A Σ-*equation* is a pair $e =_s e'$ where $e,\ e' \in T(\Sigma, X)_s$ are terms of the same sort sort s. A *conditional* Σ-equation is either a Σ-equation or an expression of one of the following forms: $e_1 \wedge \ldots \wedge e_n \Rightarrow e$, or $e_1 \wedge \ldots \wedge e_n \Rightarrow$, or $\Rightarrow$ where $e, e_1, \ldots e_n$ are Σ-equations. Given a conditional Σ-equation , $e_1, \ldots e_n$ are said to be the *conditions* and e is the *conclusion*. A Σ-formula (or a Σ-clause) is built from Σ-equations using the symbols $\neg$ and $\vee$. In other words, a Σ-formula is an expression of the form $\neg e_1 \vee \neg e_2 \vee \ldots \vee \neg e_n \vee e'_1 \vee \ldots \vee e'_m$. When Σ is clear from the context we omit the prefix Σ in $\Sigma - \{expression\}$.

In this paper axiomatizations are built from conditional equations and goals to be proved are clauses, i.e. disjunction of equational literals, since $=$ is the only predicate [2].

3.1.3 Inductive theory

Given a set of conditional equations Ax of the signature Σ, we remind that a Herbrand model of Ax is a model of Ax whose domain is the set of ground terms (axioms for equality are implicitly assumed to be valid, too). A formula $\mathcal{F}$ is a *deductive theorem of* Ax if it is valid in any model of Ax. This will be denoted by $Ax \models \mathcal{F}$. Deductive theorems can be proved by refutation, by deriving a contradiction from $\neg\mathcal{F} \wedge Ax$. Usually $\neg\mathcal{F}$ is transformed into a universal sentence $\mathcal{U}$ by introducing *skolem functions*. Hence the signature Σ has to be extended. Most theorem-proving techniques rely on Herbrand's theorem, which implies that $\mathcal{U} \wedge Ax$ is unsatisfiable iff it has no Herbrand model (w.r.t. the extended signature). The notion of inductive theory can be related to special kinds of models: Herbrand models, initial models, or constructor models [Pad88a, Pad88c, Pad88b, Zha88]. We study here the initial and the Herbrand model approaches:

[2] we identify a conditional equation and its corresponding representation as a clause

Definition 3.1 *Let Ax be a set of conditional equations of the signature Σ. A clause e is an* inductive theorem of Ax *iff for any ground substitution σ, $e\sigma$ is valid in Ax. This will be denoted by $Ax \models_{ind} e$.*

For clauses validity in all Herbrand models differs, in general, from validity in the initial model. However, these two notions of validity coincide for unconditional equations:

Proposition 3.1 ([Pad88b]) *A clause e is an* **inductive theorem** *iff it is valid in any Herbrand model of Ax. An unconditional equation e is an inductive theorem iff it is valid in the initial model of Ax.*

3.2 Rewrite Relations

Given a binary relation $\rightarrow$, $\rightarrow^*$ denotes its reflexive and transitive closure. Given two binary relations R, S, let RoS denote their composition. A relation R is noetherian if there is no infinite sequence $t_1\ R\ t_2\ R\ \ldots$. In the following we suppose given a *reduction ordering* $\succ$ on the set of terms, that is, a transitive irreflexive relation that is noetherian, monotonic ($s \succ t$ implies $w[s] \succ w[t]$) and stable ($s \succ t$ implies $s\sigma \succ t\sigma$). We write $s \lhd t$ if s is a strict subterm of t and $t \succ s$. If neither $t \prec s$ nor $t \succ s$ nor $t \equiv s$ then we say that s and t are incomparable and we denote this by $t \divideontimes s$.

A reduction ordering can be extended to literals by comparing the multisets of their members using the multiset extension of $\succ$ (see [DM79]). Formulas are compared using the multiset extension of this last ordering to the multiset of their atomic subformulas. Since there is no ambiguity, all these extensions will be denoted by $\succ$, too.

An equation $s = t$ will be written $s \rightarrow t$ provided that $s\sigma \succ t\sigma$ for all ground substitutions σ. In that case we say that the equation is *orientable* and that $s \rightarrow t$ is a rule. A conditional equation $a_1 = b_1 \wedge \ldots a_n = b_n \Rightarrow s = t$ will be written as $a_1 = b_1 \wedge \ldots a_n = b_n \Rightarrow s \rightarrow t$ if $\{s\sigma\} \succ \{t\sigma, a_1\sigma, b_1\sigma, \ldots, a_n\sigma, b_n\sigma\}$ for all ground substitutions σ; in that case we say that the conditional equation is *orientable* and that $a_1 = b_1 \wedge \ldots a_n = b_n \Rightarrow s \rightarrow t$ is a *conditional rule.* The term s is the *left-hand side* of the rule. A set of conditional rules is a *rewrite system.*

Conditional rewriting

The idea of rewriting is to impose a direction when using equations in proofs. This direction is indicated by an arrow when it is independent from the instantiation: $l \rightarrow r$ means that we can replace l by r in some context. When an instance of a conditional equation is orientable and has a valid conditional part it can be applied as a rule. The conditions are checked by a recursive call to the theorem-prover. Termination is ensured by requiring the conditions to be smaller (w.r.t. the reduction ordering $\succ$) than the conclusion. Various conditional rewrite relations have been studied in the literature ([KR89, DOS88, Vor89]).

Definition 3.2 *Let H be a set of conditional equations and let A be a term. We write:*

$$A[s\sigma] \mapsto_H A[t\sigma]$$

if there exists a substitution σ *and a conditional equation* $a_1 = b_1 \wedge \ldots \wedge a_n = b_n \Rightarrow s = t$ *in* H *such that :*

1. $s\sigma \succ t\sigma$;
2. $\forall i \in \{1 \ldots n\}$ $\exists c$ *such that* $a_i\sigma \mapsto_H^* c$ *and* $b_i\sigma \mapsto_H^* c$;
3. $\{A[s\sigma]\} \succ \{a_1\sigma,\ b_1\sigma,\ \ldots\ a_n\sigma,\ b_n\sigma\}$

A term A is reducible w.r.t. $\mapsto_H$ if there is a term B such that $A \mapsto_H B$. Otherwise we say that A is *R-irreducible* we denote that by $t \not\downarrow_H$.

The system H will be qualified as *convergent* if

$$\forall a, b \in T(F)\ \ H \models a = b \quad \Rightarrow \quad \exists c \text{ such that } a \mapsto_H^* c \text{ and } b \mapsto_H^* c.$$

Note that when H is a set of conditional rules then the relation $\mapsto_R$ is similar to the notion of decreasing rewriting of Dershowitz, Okada and Sivakumar [DOS88].

The relation $\mapsto_H$ will be extended to sets of clauses in a natural way: by definition, $S \cup \{c\} \mapsto_H S \cup \{d\}$ whenever $c \mapsto_H d$.

Case rewriting

Case reasoning is a technique, which is the basis of many theorem proving strategies. It is most important rule in the context of inductive theorem proving where case splitting arises naturally from an induction hypothesis. We propose here a notion of case rewriting, which is well-suited to inductive reasoning.

Definition 3.3 (Case rewriting) *Let* H *be a set of conditional equations and let* $c \Rightarrow s = t$ *be a conditional equation in* H *(where* c *is a positive literal). Let* $A[s\sigma]_n$ *be a clause (where* σ *is a substitution) and let* S *be a set of clauses. The case rewriting rule can be stated as follows*

$$S \cup \{A[s\sigma]_n\} \rightarrow_H S \cup \{(c\sigma \vee A[s\sigma]_n), (\neg c\sigma \vee A[t\sigma]_n)\}$$

if neither $c\sigma$ *nor* $\neg c\sigma$ *is a subclause of* $A[s\sigma]_n$, n *is a position of a maximal literal of* A, $s\sigma \succ t\sigma$, $(s = t)\sigma \succ c\sigma$ *and* $A[s\sigma]_n \succ c\sigma$

This definition can be generalized to the case where c is a conjunction of positive literals in a straightforward way. Let us denote $\rightarrow_H \cup \mapsto_H$ by $\hookrightarrow_H$. The following proposition is the basis for proving (or disproving) clausal theorems.

Proposition 3.2 *The case rewriting rule is sound (the derived set of clauses is logically equivalent to the initial set provided* S *contains* H*). The relation* $\hookrightarrow_H$ *is noetherian.*

The first part is trivial. To prove the second one we use the following complexity measure on clauses: For a clause C define $s(C) = (a, b, c)$ where a is the set of maximal atoms of C, b is the number of literals of C, and c is the set of atoms in C. Given C and C' we say that $s(C) = (a, b, c) < s(C') = (a', b', c')$ if $a \prec a'$ or $a = a'$ and $b > b'$, or $a = a'$ and $b = b'$ and $c \prec c'$. Clauses sets will be compared by the multiset extension of this ordering.

Supported rewriting

We introduce now a new rewrite relation, which will be useful for expressing that an inductive hypothesis can help the proof of the premises of a conditional rule.

Definition 3.4 (Supported rewriting) *Let H be a set of conditional equations, W be a set of equations and A be a term. We define the supported rewriting relation $\mapsto_{H[W]}$ by:*

$$A[s\sigma] \mapsto_{H[W]} A[t\sigma]$$

if there exists a substitution σ and a conditional equation $a_1 = b_1 \wedge \ldots \wedge a_n = b_n \Rightarrow s = t$ in H such that :

1. $s\sigma \succ t\sigma$
2. $\forall i \in \{1 \ldots n\}$ $\exists c$ *such that* $a_i\sigma \mapsto^*_{H \cup W} c$ *and* $b_i\sigma \mapsto^*_{H \cup W} c$
3. $\{A[s\sigma]\} \succ \{a_1\sigma,\ b_1\sigma,\ \ldots\ a_n\sigma,\ b_n\sigma\}$

If W is empty or H is a set of unconditional equations we write $\mapsto_H$ instead of $\mapsto_{R[W]}$.

Relaxed rewriting

The following rewrite relation will be useful to handle non-orientable equations.

Definition 3.5 (Relaxed rewriting) *Let H be a set of unconditional equations. We define the relaxed rewriting relation by: $g[u\theta] \rightsquigarrow_H g[v\theta]$ if there is $(u = v) \in H$ and there is a substitution θ such that $u\theta \not\equiv v\theta$.*

4 How to prove and disprove inductive theorems

In this section, we propose general methods to prove (and disprove) automatically that sets of clauses are inductive consequences of theories axiomatized by sets of conditional rules. This technique allows us to replace inductive reasoning by pure (algebraic) simplification. This mechanization of inductive proofs is based on the notion of *test set*, which, in essence, provides a finite description of the initial model of a given conditional theory.

4.1 Test sets

A rewrite rule $c \Rightarrow s \rightarrow r$ is *left-linear* if s is linear. A rewrite system R is *left-linear* if every rule in R is left-linear, otherwise R is said to be *non-left-linear*. A term is *strong R-irreducible* if none of its non-variable subterms matches a left-hand side of R.

Further, if u is a position, then $|u|$ (the *length* of the corresponding string) gives us its *depth*. If t is a term, then the *depth* of t is the maximum of the depths of the positions in t and denoted *depth(t)*. The *strict depth* of t, written as *sdepth(t)*, is the maximum of the depths of the strict positions in t. The *depth of a rewrite system* R, denoted *depth(R)*, is defined as the maximum of the depths of the left-hand sides of

R. Similarly, the *strict depth* of R, written a *sdepth(R)*, is the maximum of the depths of the strict positions in the left-hand sides of R.

If R is a rewrite system, then R can be partitioned into left-linear rules R_{ll} and non-left-linear rules R_{nl}, i.e, $R = R_{ll} \cup R_{nl}$. $depth(R_{ll})$ denotes the maximum of the depths of the left-linear rules R_{ll} of R and $depth(R_{nl})$ denotes the maximum of the depths of the non-left-linear rules R_{nl}, then the number *D(R)* is equal to $depth(R)$ if $depth(R_{nl}) < depth(R_{ll})$ and $sdepth(R) < depth(R_{ll})$, otherwise *D(R)* is equal to depth(R)+1. $D(R)$ is said to be a *bound* for R.

Definition 4.1 *If R is a set of conditional rules, then a* test set $S(R)$ for R *is a finite set of R-irreducible terms that has the following properties:*

completeness: *For any R-irreducible ground term s there exists a term t in $S(R)$ and a ground substitution σ such that $t\sigma = s$;*

transnormality: *For any non-ground term t in $S(R)$ and for any position u in t for which t/u is a non-ground term and $|u| = depth(R)$, there exist infinitely many strong R-irreducible ground instances t_0, t_1, ... of t such that $t_0/u \neq t_1/u$, ... ;*

coveredness: *Any non-ground term in $S(R)$ has variables* only *at depth greater or equal than $D(R)$.*

Let us show that this notion of test set is what is really needed for automating induction. First, the *completeness* property allows us to prove theorems by induction on the domain of irreducible terms rather than on the whole set of terms (see theorem 4.1 and 4.2). Second, the *transnormality* and *coveredness* properties are crucial for the refutation of inductive conjectures (see theorem 4.3). In order to derive useful lemmas we must be able to apply the rewrite rules to some instances of the conjecture. For this reason, these instances have to be deep enough: this is also ensured by the *coveredness* property.

For left-linear rules the *transnormality* property may be weakened: it is sufficient to consider non-ground terms that admit at least one strong irreducible instance. Moreover, $D(R)$ can be taken equal to $depth(R) - 1$ in left-linear theories defined over a free set of constructors. This is important for efficiency purpose.

It is possible to compute test sets for equational theories in a relatively efficient way (see [Kou90, Hub91]). Unfortunately no algorithm exists for the general case of conditional theories. However, in the last section, we will give a method to compute test sets in conditional theories defined over a free set of constructors.

Example 4.1
a) If $F = \{a, f, g\}$ and $R = \{f(x,x) \rightarrow x,\ f(x,g(x)) \rightarrow g(x),\ f(g(x),x) \rightarrow x,\ g(g(a)) \rightarrow a\}$, then the set $S(R)$ containing the terms $\{a,\ g(a)\}$ is a test set for R.
b) Let us come back to the introductory example. Let H be the set of axioms $1, 2, \ldots, 11$. As we pointed out, $S(H) = \{0, succ(x), true, false\}$ may be considered a test set. Note that the set of ground instances of the members of the test set contains all ground irreducible terms w.r.t. the relation $\rightarrow_H$. Moreover, the three properties of the definition are verified.

Definition 4.2 *Let R be a set of conditional rules and let C be a clause. A variable $x \in C$ is an* induction variable *if it occurs in a subterm s of C such that i) s is unifiable with the left-hand side l of a rule $c \Rightarrow l \rightarrow r$ of R, ii) the position of x in s is a strict or a non-linear variable position of l or the position of a variable of l that also occurs in c.*

Terms in test sets are used to build *test-substitutions.*

Definition 4.3 *If $S(R)$ is a test set for a set R of conditional rules and C is a clause, then a* test instance *of C with respect to $S(R)$ is an instance of C obtained by substituting renamed terms of $S(R)$ for the induction variables of C. A test-substitution is a substitution that maps any variable from its domain to a renaming of an element of $S(R)$.*

Example 4.2 *Let R be $\{x + 0 = x, x + succ(y) = succ(x + y)\}$. Consider for C, the associativity property $(x + y) + z = x + (y + z)$. The induction variables are y and z. The test instances of C with respect to $\{0, succ(x')\}$ are $(x + 0) + 0 = x + (0 + 0), (0 + y) + succ(z') = x + (0 + succ(z')), (x + succ(y')) + 0 = x + (succ(y') + 0), (x + succ(y')) + succ(z') = x + (succ(y') + succ(z'))$*

4.2 Inductive proofs by simplification

Our notion of induction uses a noetherian ordering on ground terms that contains the conditional rewriting relation. We can use as an inductive hypothesis any instance of the theorem we want to prove, as soon as this instance is smaller (w.r.t. $\succ$) than the one that is currently considered. We propose here a rewriting relation which is sound, with regard to the use of induction hypotheses. For instance, an inductive hypothesis can help to satisfy the conditions of a conditional rule. The following theorem shows how to prove equations in initial models of conditional theories.

Theorem 4.1 *Let R be a set of conditional rules, $S(R)$ be a test set, and $u = v$ be an equation. In this case we define $\mapsto\!\!\!\!\rightarrow$ as the reflexive closure of the relation:*

$$(\mapsto_{R[u=v]}) o (\mapsto_{R \cup \{u=v\}})^*$$

If for all test-substitutions ν there is a term α such that $u\nu \mapsto\!\!\!\!\rightarrow \alpha$ and $v\nu \mapsto\!\!\!\!\rightarrow \alpha$ then $u = v$ is an inductive theorem for R.

It is straightforward to generalize the previous method to prove clauses. In the following, we call a *tautology* a clause that contains either two complementary literals or an instance of $x = x$.

Theorem 4.2 *Let R be a set of conditional rules, $S(R)$ be a test set, and C be a clause. If for all test-substitutions ν we have $\{C\nu\} \hookrightarrow_R^* \{p_1, p_2 \cdots, p_n\}$, and every clause p_j is either* a tautology *or* is subsumed by an axiom *or* contains an instance of C that is strictly smaller w.r.t. $\succ$ than $C\nu$, *then C is an inductive theorem of R.*

Example 4.3 *Let us prove the transitivity of* $<$ *(see axioms of the introductory example).*

$$x < y = false \vee y < z = false \vee x < z = true$$

The only non-trivial test instance among eight of them is C:

$$succ(x) < succ(y) = false \vee succ(y) < succ(z) = false \vee succ(x) < succ(z) = true$$

After three steps of case-rewriting we get only one non-tautological clause, namely:

$$x < y = false \vee y < z = false \vee x < z = true \vee$$

$$succ(x) < succ(y) = false \vee succ(y) < succ(z) = false \vee succ(x) < succ(z) = true$$

This clause contains a subclause that is a strictly smaller instance of the property C to be proved. Hence by theorem 5.2, the proof of transitivity is achieved.

4.3 Disproving inductive conjectures

The notion of a test set is particularly useful for refuting inductive properties. The next definition provides the criteria to reject conjectures that are not inductive theorems.

Definition 4.4 *Suppose that we are given a set of conditional rules R and a test set* $S(R)$*. Then a clause* $\neg e_1 \vee \ldots \vee \neg e_m \vee g_1 = d_1 \vee \ldots \vee g_n = d_n$ *is* **quasi-inconsistent** *with respect to R if there is a test instance* $C\sigma$ *such that for all* $i \leq m$ $e_i\sigma$ *is an inductive theorem and for all* $j \leq n$ *at least one of the following is satisfied:*

- $g_j\sigma \not\equiv d_j\sigma$*, and both* $g_j\sigma$ *and* $d_j\sigma$ *are strong irreducible by R.*
- $g_j\sigma \succ d_j\sigma$*, and* $g_j\sigma$ *is strong irreducible by R.*
- $g_j\sigma \prec d_j\sigma$*, and* $d_j\sigma$ *is strong irreducible by R.*

The next result shows that, when the set of axioms is convergent, a quasi-inconsistent clause cannot be inductively valid. This is proved by building a well-chosen ground instance of the clause, that is false in some Herbrand model of the axioms. In particular, if the clause is an equation then it is not valid in the initial model.

Theorem 4.3 *Let R be a convergent set of conditional rules and* $S(R)$ *be a test set for R. If C is quasi-inconsistent w.r.t. R then C is not an inductive theorem of R.*

The proof is given in the appendix.

Example 4.4 *Consider the following conditional axioms for integers with* $+$*, odd, and even.*

$$x + 0 \rightarrow x \tag{24}$$

$$x + s(y) \rightarrow s(x + y) \tag{25}$$

$$even(0) \rightarrow true \tag{26}$$

$$even(s(0)) \rightarrow false \tag{27}$$

$$even(s(s(x))) \rightarrow even(x) \tag{28}$$

$$even(x) = true \Rightarrow odd(x) \rightarrow false \tag{29}$$

$$even(x) = false \Rightarrow odd(x) \rightarrow true \tag{30}$$

Here the test set is $\{0, s(0), s(s(z)), true, false\}$. *Note that the axioms satisfy the convergence property. Consider the conjecture* $even(x) = true \vee odd(x) = false$. *It is quasi-inconsistent as shown by the following instance* $even(s(0)) = true \vee odd(s(0)) = false$.

4.4 A general procedure for proof by induction

4.4.1 Inference rules for inductive proofs

In the previous subsection we have given a technique for proving inductive theorems in one step. However, most of the time several rounds are needed before getting a proof. For instance, several successive instantiations by test sets may be necessary. The easiest way to present this process is to use the formalism of inference rules as in Bachmair [Bac91] and Reddy [Red90]. The main advantage of this approach is that there is no hierarchy between the intermediate lemmas to be proved, and therefore no difficulty for the management of inductive hypotheses: *every intermediate lemma to be proved will be put in the same set as the initial conjectures and no priority will be (a priori) attached to them.* In that sense our procedure can be viewed as being close to inductionless induction.

Let R be a system of conditional rules. Our proof by induction procedure (which is detailed in [Bou91]) modifies incrementally two sets of equations:

1. E, the set of equations to be proved.
2. H, the set of equations that have been *reduced* to equations of E, and therefore can be used as induction hypotheses.

Let us give now the set of inference rules I:

$$Generate \quad \frac{(E \cup \{e=e'\}, H)}{(E \cup (\cup_\sigma E_\sigma), H \cup \{e=e'\})} \quad if \quad \left\{ \begin{array}{l} For\ any\ test-instance\ e\sigma = e'\sigma,\ there\ is\ b\ such\ that: \\ either\ \ e \not\downarrow e'\ and\ (e\sigma \mapsto_{R[H \cup E \cup \{e=e'\}]} b)\ and\ E_\sigma = \{b = e'\sigma\} \\ or\ \ e \ast e'\ and\ (e'\sigma \mapsto_{R[H \cup E \cup \{e=e'\}]} b)\ and\ E_\sigma = \{e\sigma = b\} \\ or\ \ e\sigma \equiv e'\sigma\ and\ E_\sigma = \emptyset \end{array} \right.$$

$$Simplify_1 \quad \frac{(E \cup \{a=b\}, H)}{(E \cup \{a'=b\}, H)} \quad if \quad a \mapsto_{R[H \cup E \cup \{a=b\}]} a'$$

$$Simplify_2 \quad \frac{(E \cup \{a=b\}, H \cup \{g=h\})}{(E \cup \{a'=b\}, H \cup \{g=h\})} \quad if \quad \left\{ \begin{array}{l} a[g\sigma]_u \mapsto_H a' \equiv a[h\sigma]_u \\ g\sigma \lhd a \ \ or \ \ g \succ h \end{array} \right.$$

$$Simplify_3 \quad \frac{(E \cup \{c=d\} \cup \{a=b\}, H)}{(E \cup \{c=d\} \cup \{a'=b\}, H)} \quad if \quad \left\{ \begin{array}{l} a[c\sigma]_u \mapsto_E a' \equiv a[d\sigma]_u \\ c\sigma \lhd a \end{array} \right.$$

$$Simplify_4 \quad \frac{(E \cup \{a=b\}, H)}{(E \cup \{a'=b\}, H)} \quad if \quad \left\{ \begin{array}{l} a \rightsquigarrow_H a' \\ a \ast b \\ a' \preceq b \end{array} \right.$$

$$Delete \quad \frac{(E \cup \{a=a\}, H)}{(E, H)}$$

$$Fail \ \frac{(E \cup \{e=e'\}, H)}{\square} \ if \ \begin{cases} there\ is\ a\ test-instance \quad e\sigma = e'\sigma \ \ such\ that\ e\sigma \not\equiv e'\sigma \ and: \\ (e \succ e' \ \wedge \ e\sigma \not\downarrow_{R[H \cup E \cup \{e=e'\}]}) \ or \\ (e \not\asymp e' \ \wedge \ e\sigma \not\downarrow_{R[H \cup E \cup \{e=e'\}]} \ \wedge \ e'\sigma \not\downarrow_{R[H \cup E \cup \{e=e'\}]}) \end{cases}$$

The rule *Generate* allows to initialize inductive steps by replacing conjectures by *implicitly smaller conjectures.* The rule $Simplify_1$ simplifies a conjecture with conditional axioms from R, using conjectures from $H \cup E$ when checking the convergence of preconditions. The rules $Simplify_2$, $Simplify_3$ and $Simplify_4$ apply to a conjecture using either induction hypotheses or other conjectures. Note that $Simplify_4$ allows one to handle non orientables equations. The rule *Delete* helps getting rid of tautologies. The rule *Fail* applies when *Generate* cannot be applied.
An *I-derivation* is a sequence of states:

$$(E_0, H_0) \vdash_I (E_1, H_1) \vdash_I \ldots \vdash_I (E_n, H_n) \vdash_I \ldots$$

An I-derivation is *failed* if it ends with an application of the *Fail* rule.

Example 4.5 *Let* $R = \{p(0) = True, \ p(x) = True \Rightarrow p(s(x)) = True\}$. *The initial state is* $(\{p(x) = True\}, \emptyset)$. *We have:*

$$(\{p(x) = True\}, \emptyset) \vdash_{Generate} (\{True = True\}, \{p(x) = True\}) \vdash_{Delete} (\emptyset, \{p(x) = True\})$$

The Generate step is justified by noting that $p(s(x)) \mapsto_{R[\{p(x) = True\}]} True$ *since*

$$R \cup \{p(x) = True\} \models (p(x) = True)$$

Definition 4.5 *An I-derivation* $(E_0, \emptyset) \vdash_I (E_1, H_1) \vdash_I \ldots$ *is fair if either it is failed or the set of persisting equations* $(\cup_i \cap_{j \geq i} E_j)$ *is empty.*

Lemma 4.1 *Let* $(E_0, H_0) \vdash_I (E_1, H_1) \vdash_I \ldots$ *be a fair I-derivation such that* $R \not\models_{ind} E_0$. *Then there exists* k *such that the rule Fail is applied to* (E_k, H_k).

This lemma is proved by considering the smallest invalid ground equation that is an instance of an element of $\cup_i E_i$. See appendix.

Theorem 4.4 (Soundness) *Let* $(E_0, \emptyset) \vdash_I (E_1, H_1) \vdash_I \ldots$ *be a fair derivation. If this derivation is non failed then* $R \models_{ind} E_0$.

Note that E_0 is valid even when the derivation is infinite (and *Fail* never applies).

Corollary 4.1 *Given a set of equations* E *and a set of conditional equations* R, *if there is an I-derivation from the state* $(E, \emptyset)$ *to some state* $(\emptyset, H)$, *where* H *is any set of equations, then any element of* E *is an inductive theorem of* R.

Let us emphasize that we deal here with conditional theories, and that we allow simplification of conjectures by conjectures, unlike Reddy (see Simplify3).

The failure to prove some theorem with the above procedure may be due to the absence of some inference rule for case reasoning. We propose to enhance the procedure by allowing case reasoning through *case rewriting.* More generally, note also that at any step i, one of the current conjectures in E_i can be proved by any other induction proof technique. This is expressed by the generic rule:

Other $\frac{(E \cup \{C\}, H)}{(E, H \cup \{C\})}$ *if* $R \models_{ind} C$

We can easily prove that the system remains correct when we add this inference rule to the previous ones.

Example 4.6 *Consider exemple 4.4 and let us prove first the commutativity of* $+$. *We just consider the non-trivial instance by a test-substitution:*

$$s(s(x)) + s(s(y)) = s(s(y)) + s(s(x)) \tag{31}$$

After simplification, we have to consider the goal:

$$s(s(s(s(x)) + y)) = s(s(s(s(y)) + x)) \tag{32}$$

The rule $simplify_4$ *applies to* $s(s(s(s(x)) + y))$ *and yields* $s(s(y + s(s(x))))$ *since* $s(s(y + s(s(x)))) \prec s(s(s(s(y)) + x))$. *Then we get:*

$$s(s(y + s(s(x)))) = s(s(s(s(y)) + x)) \tag{33}$$

Applying Generate and $simplify_2$ *using the hypotheses* $s(s(y+s(s(x)))) = s(s(s(s(y))+x))$ *finishes the job.*

Having proved the commutativity of $+$, *we can prove :* $even(x + x) = true$. *To prove it, we proceeds as follow :* $even((s(s(x)) + (s(s(x)))))$ *reduces successively to* $even(s(s(s(s(x+x)))))$ *and then to* $even(x+x)$. *Now* $even(x+x)$ *is simplified to true by using the induction hypothesis.*

Let us prove now $odd(x+s(x)) = true$. *The non trivial case is* $odd(s(s(x))+s(s(s(x))))$ *which simplifies to true by supported rewriting.*
Therefore, $x + y = y + x$, $even(x + x) = true$ *and* $odd(x + s(x)) = true$ *are valid in the initial model of 24-30.*

4.4.2 Refutation of conjectures

Conditional systems

Let us consider the set of inference rules J obtained by extending I with the following new inference rule:

Disproof $\frac{(E \cup \{C\}, H)}{\square}$ *if* C *is quasi − inconsistent.*

If the given theory is a (ground) convergent set of conditional rules *Disproof* allows to detect many false conjectures. This is an easy consequence of the theorem 4.3 and this is formally expressed by the following corollary:

Corollary 4.2 *Let* R *be a convergent set of conditional equations and let* $(E_0, \emptyset) \vdash_J (E_1, H_1) \vdash_J \ldots$ *be a J-derivation. If there exists* k *such that Disproof is applied to* (E_k, H_k) *then* $R \not\models_{ind} E_k$.

When discovering an inconsistency at some step (E_k, H_k) we can conclude that the input set E_0 is not valid. This is stated in the next lemma:

Lemma 4.2 *Let $(E_0, \emptyset) \vdash_J (E_1, H_1) \vdash_J \ldots$ be a J-derivation where neither Fail nor Disproof are applied. If for all $j < k$ $R \models_{ind} E_j$ then $R \models_{ind} E_k$.*

Proof: If $(E_{k-1}, H_{k-1}) \vdash_J (E_k, H_k)$ by a simplification rule, then the equations which are used for simplification occur in some E_j $(j < k)$ and therefore are valid in R by induction (on k). Hence, E_k is valid too in R. If $(E_{k-1}, H_{k-1}) \vdash_J (E_k, H_k)$ by *Generate* on $e = e'$, every auxiliary equation which is used for rewriting an instance of $e = e'$ by a test-substitution is either in R or $E_{j'}$ $(j' < k)$ and hence E_k is valid in R. □

Theorem 4.5 *Assume that R is convergent and that $(E_0, \emptyset) \vdash_J (E_1, H_1) \vdash_J \ldots$ is a J-derivation. If there exists k such that Disproof is applied to (E_k, H_k), then $R \not\models_{ind} E_0$.*

Proof: Let $(E_0, \emptyset) \vdash_J (E_1, H_1) \vdash_I \ldots$ be a J-derivation. Assume that there exists k such that *Disproof* is applied to (E_k, H_k). It is clear that *Disproof* has not been applied before step k. By corollary 4.2 $R \not\models_{ind} E_k$ and by lemma 4.2. $R \not\models_{ind} E_0$. □

Note also that the soundness theorem remains valid for the inference system J if we replace, in the statement, "*Fail*" by "*Fail* or *Disproof*".

A refutationally complete system for equations

Let us consider the particular case of a convergent equational system R. We take the same inference rules as above, except that we replace $R[W]$ by R. We also notice that, in this situation, the rules *Fail* and *Disproof* are identical. Therefore we can prove that our inference system I $(= J)$ is refutationally complete. This means that the method allows to detect any false conjecture by the *Fail* rule. Note that we need not prove this fact by translating derivations to narrowing derivations. This would be hard work due to the numerous simplification rules.

Theorem 4.6 *Let $(E_0, \emptyset) \vdash_I (E_1, H_1) \vdash_I \ldots$ be a fair derivation. Then $R \not\models_{ind} E_0$ if and only if the derivation is failed.*

Proof: Let $(E_0, \emptyset) \vdash_I (E_1, H_1) \vdash_I \ldots$ be a fair derivation

- Assume that $R \not\models_{ind} E_0$. By theorem 4.4. then the derivation is failed.
- Assume that there exists k such that *Fail* is applied to (E_k, H_k). By corollary 4.3. $R \not\models_{ind} E_k$ and by lemma 4.2. $R \not\models_{ind} E_0$. □

5 How to get the convergence property

Convergent systems of equations have the property that two terms are equal if and only if they simplify to identical ones. In this section we provide several methods to obtain the convergence property, which is crucial in our framework for disproving conjectures.

5.1 The saturation technique

The Knuth and Bendix procedure [KB70] has been designed to derive convergent systems from equational presentations. The *saturation* technique is a natural extension of Knuth and Bendix algorithm to conditional theories and has first been introduced in [KR87, Rus87]. This technique is based on a set of inference rules which is refutationally complete for conditional equations.

We assume in this section that $\prec$ is total on the set of ground terms and that for any term t and for any proper subterm s of t we have $s \prec t$.

As the main deduction rule, we use *superposition*, which is a refinement of paramodulation (see also [RW69, Rus88, HR91]): the only inferences which are allowed are those obtained by paramodulating maximal members of conclusions into maximal members of conclusions (*maximal* refers here to the ordering $\prec$). Let E be a set of conditional equations. We formally define the superposition rule as follows:

$$\frac{c \Rightarrow l[u'] = r \quad q \Rightarrow u = v}{(c \wedge q \Rightarrow l[v] = r)\sigma}$$

where u' is not a variable and $u'\sigma \equiv u\sigma$ and $l\sigma \succ c\sigma, r\sigma$ and $u\sigma \succ q\sigma, v\sigma$

There is also a refinement of paramodulation to deal with conditions. This is the *conditional narrowing* rule: paramodulation is performed from maximal members of conclusions into maximal conditions.

$$\frac{c \wedge (l[u'] = r) \Rightarrow e \quad q \Rightarrow u = v}{(c \wedge q \wedge (l[v] = r) \Rightarrow e)\sigma}$$

where u' is not a variable and $u'\sigma \equiv u\sigma$ and $l\sigma \succ c\sigma, r\sigma$ and $u\sigma \succ q\sigma, v\sigma$

Finally, the last inference rule allows to solve conditions by pure unification. This rule is called *reflexion*:

$$\frac{c \wedge s = t \Rightarrow e}{(c \Rightarrow e)\sigma}$$

where $s\sigma \equiv t\sigma$ and $(s = t)\sigma \succeq c\sigma, e\sigma$

There are also a certain number of rules for simplification, tautology deletion and subsumption that we will not detail here since they are standard. When the application of superposition, conditional narrowing and reflexion to a set of clauses S generates only clauses that can be deleted using simplification, tautology deletion and subsumption, then we say that S is saturated. Applying the set of inference rules to conditional equations, we can generalize the theorem of Knuth and Bendix: if an application of the rules above allows us to derive from S' a saturated set S then $\mapsto_S$ is convergent. This technique is detailed in [KR87] and we will only show how it works on an example. Let us note also that a more general version of it does not fail in the presence of equations that cannot be turned into rewrite rules (in particular, we allow extra-variables in the conditions or non-orientable equations). Moreover, purely negative conditional equations are allowed, too. The following example is inspired by [Pla85], where its confluence is proved by semantic techniques. Here we can show that our saturation procedure immediately stops. This ensures that the system has the convergence property.

Example 5.1 *Consider the following set of conditional equations, which defines the* minimum *of a list of integers. We assume axioms 4,5,6,7 of the introductory example for $<$ and the lrpo [Der87] as reduction ordering with the following precedence on functions:*

$$min \succ < \succ cons \succ nil \succ succ \succ 0 \succ true \succ false$$

$$true = false \Rightarrow \tag{34}$$

$$succ(x) = 0 \Rightarrow \tag{35}$$

$$succ(x) = succ(y) \Rightarrow x = y \tag{36}$$

$$(0 < succ(x)) \rightarrow true \tag{37}$$

$$(x < 0) \rightarrow false \tag{38}$$

$$succ(x) < succ(y) \rightarrow x < y \tag{39}$$

$$min(cons(x, nil)) \rightarrow x \tag{40}$$

$$(x < min(l)) = true \Rightarrow min(cons(x, l)) \rightarrow x \tag{41}$$

$$(x < min(l)) = false \Rightarrow min(cons(x, l)) \rightarrow min(l) \tag{42}$$

$$min(l) = x \Rightarrow min(cons(x, l)) \rightarrow min(l) \tag{43}$$

Let us apply the saturation procedure: by superposition between 40 and 42 we get the following interesting consequence:

$$(x < min(nil)) = false \Rightarrow x = min(nil) \tag{44}$$

Superposing 40 and 41 we get:

$$(x < min(l)) = false \wedge (x < min(l)) = true \Rightarrow x = min(l) \tag{45}$$

By simplification of the conditions of 45, we obtain the following clause which is subsumed by 34.

$$(x < min(l)) = false \wedge false = true \Rightarrow x = min(l) \tag{46}$$

No new equation can be deduced. This implies that the system is convergent on ground terms.

The next example shows that extra-variables are allowed in the conditions:

Example 5.2 *We assume the following precedence $p \succ s \succ true$:*

$$x < y = true \Rightarrow x < s(x) = true \tag{47}$$

$$p(x) < x = true \tag{48}$$

$$s(p(x)) = x \tag{49}$$

To apply conditional narrowing we need to find a substitution such that:

$$\sigma(x < y) \succ \sigma(x < s(x)), \sigma(x' < y') \prec \sigma(x' < s(x')), \sigma(x < y) \equiv \sigma(x' < s(x'))$$

But these constraints have no solutions. No inference steps would add a non-trivial clause to the initial system. Hence, it is convergent.

5.2 Hierarchical axiomatization techniques

Hierarchical axiomatizations are natural tools for building structured specifications. They are obtained by incremental extensions of base theories with new function definitions.

Let $\Sigma B \equiv (S, FB)$ be a subsignature of $\Sigma \equiv (S, F)$. In other words, Σ differs from ΣB only in new function symbols. The elements of ΣB are often called *constructors* and those of $\Sigma - \Sigma B$ are called *defined* function symbols. Terms, substitutions, equations, conditional equations, and formulas over ΣB are called constructor Σ – *terms*, constructor Σ – *substitutions*, constructor Σ – *equations*, constructor Σ – *conditional equations*, and constructor Σ – *formulas*, respectively.

For hierarchical axiomatizations, ground confluence can be obtained by semantic methods. The next theorem is the one underlying D.Plaisted's work [Pla85]:

Theorem 5.1 *Let H' be a set of conditional rules over a signature $\Sigma B \cup (\Sigma - \Sigma B)$ and let $H \subset H'$ be a convergent set of rules over ΣB. Assume that the initial model of H' is a conservative extension of the initial model of H. If for every ground term $f(t_1, \ldots, t_n)$, where $f \in (\Sigma - \Sigma B)$, there exists a constructor Σ – term t' such that $f(t_1, \ldots, t_n) \mapsto_{H'}{}^* t'$ then H' is ground convergent.*

Proof: Let t be a term in $T(F)$. Suppose that $t \mapsto_{H'}{}^* t_1$ and $t \mapsto_{H'}{}^* t_2$. From the hypothesis, there exists two terms t'_1 and t'_2 in $T(\Sigma B)$, such that $t_1 \mapsto_{H'}{}^* t'_1$ and $t_2 \mapsto_{H'}{}^* t'_2$. Since H' and H share the same initial model, $t'_1 = t'_2$ is also a theorem in H. By the convergence property of H, we can find a term s such that $t'_1 \mapsto_H{}^* s$ and $t'_2 \mapsto_H{}^* s$. This achieves the proof. □

In the case of a hierarchical conditional theory H', our procedure for computing test sets (see next section) may be used to demonstrate convergence provided that the initial model of H' is a conservative extension of the initial model of H. Recall that the rewriting relation $\mapsto_{H'}$ was defined with respect to a reduction ordering on *all* terms and this ensures that it is noetherian. For instance, the introductory example is convergent since the axioms introduced to define successively $-, <, gcd$ do not modify the initial model (Peano arithmetic) and the rewrite rules eliminate these symbols from any ground term. This is the same as asking for the sufficient-completeness property w.r.t. a rewriting relation [GH78].

Further, verification of inductive properties often involves the proof of some lemmas. Adding these lemmas to the initial axiomatization does not destroy the convergence, as stated in the following result:

Theorem 5.2 *If H is ground convergent and C is a conditional equation that is an inductive theorem of H. Then $\mapsto_{H \cup \{C\}}$ is ground convergent.*

Proof: Let t, t_1, t_2 be ground terms such that:

$$t \mapsto_{H \cup \{C\}}{}^* t_1 \text{ and } t \mapsto_{H \cup \{C\}}{}^* t_2$$

Since C is an inductive theorem, all the ground instances of C that have been used in the proofs above are valid in H. Hence, $t_1 = t_2$ is valid in H, and the convergence property of H allows to conclude the proof.□

For instance, in the introductory example 1-11 and 12,13,14,17 are convergent (with an appropriately chosen reduction ordering).

6 How to get test sets

As we have already pointed out, the construction of test sets for conditional theories is undecidable. This lies in that such a computation requires some kind of induction. We propose here a method of computing test sets for conditional theories in which the set of function symbols of their signature can be partitionned into a set ΣB of constructors and a set $\Sigma - \Sigma B$ of defined functions. Therefore, we will assume that every left-hand side of a conditional rule has a symbol from $\Sigma - \Sigma B$. This corresponds to the well-known requirements for a *principle of definition* to hold in an equational theory (see [HH82]). In order to simplify our representation we shall assume that $\Sigma - \Sigma B$ contains only one function symbol. The key concept of the present method of computing test sets for conditional theories is the notion of *Pattern trees.*

Definition 6.1 *Given a linear term* $r = f(t_1, t_2, \ldots, t_k)$ *where* $f \in \Sigma - \Sigma B$ *and* $t_i \in T(\Sigma B, X)_{s_i}$ *for all* $i \leq k$, *the* sons *of* r *are all possible different terms (modulo variable renaming) obtained by replacing a fixed variable of sort* s *in* r *by all terms of sort* s *of the form* $g(x_1, \ldots, x_n)$, *where* g *is a function symbol of sort* s *of arity* n *in* ΣB *and* $x_1, \ldots, x_n$ *are fresh distinct variables of appropriate sorts.*

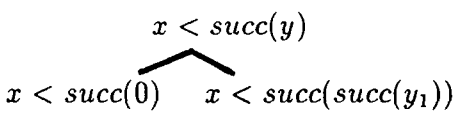
Example 6.1 *If* $F = \{a, f, g\}$ $FB = \{a, g\}$ *where* $a : \rightarrow s$, $f : s \times s \rightarrow s, g : s \rightarrow s$ *and* $r = f(g(x), y)$, *then the terms* $f(g(a), y)$ *and* $f(g(g(x_1)), y)$ *are the sons of* r: *they are obtained by replacing the variable* x *in* r *by the terms* a, $g(x_1)$.

Definition 6.1 is the basis for the concept of pattern trees:

Definition 6.2 *A* pattern tree **T** *of a linear term* t *is a tree, the root of which is labeled with* t *and the outgoing branches from each non-leaf node are labeled with the sons of the label at that node, w.r.t. some variable.*

Let us illustrate definition 6.2 on a simple example:

Example 6.2 *If* $FB = \{0, succ, true, false\}, F = FB \cup \{<\}$ *where* $0 : \rightarrow int$, $<: int \times int \rightarrow bool$, $true : \rightarrow bool$, $false : \rightarrow bool$, *and* $succ : int \rightarrow int$, *then the following tree is a pattern tree of the term* $x < succ(y)$:



Pattern trees enjoy some fundamental properties: Given a finite pattern tree of t,

- The set of ground instances of t is equal to the set of ground instances of all leaf labels.

- The sets of ground instances of two different leaf labels are disjoint.
- For any ground substitution η there exist a unique leaf r and a unique ground substitution σ such that $t\eta = r\sigma$.

The construction of a test set for a given conditional theory H consists in computing a suitable pattern tree of the term $f(x_1, x_2, \ldots, x_k)$ where $f \in \Sigma - \Sigma B$ and $x_i \in X_{s_i}$ for all $i \leq k$. In general, when we want to construct such a tree by expansion from its root, there are several questions that come naturally to mind: given a pattern tree **T**, then

a) What nodes have to be expanded?

b) What variables in them have to be replaced?

c) When the construction of **T** halts?

To answer these questions, we will define special kind of terms that we want to be the leaf labels of the pattern trees we are interested in. These terms possess a well-defined structure that, to a certain degree, mirrors the structure of the left-hand sides of the rules of the conditional theory under consideration.

Definition 6.3 *A term t is said to be* H-extensible at position u *if*

a) *u is a variable position in t of sort s, and*

b) *either the only functions symbols of sort s are nullary or u is a strict or a non-linear position of a left-hand side of a rule in H.*

A term t is said to H-extensible *if it is H-extensible at some position u. Otherwise, t is said to be* H-covering.

Definition 6.4 *Given a set H of conditional rules, a term t is said to be* pseudo-reducible *by H if there exists a sequence of conditional rules $C_1 \Rightarrow t_1 = r_1$, $C_2 \Rightarrow t_2 = r_2$, $\ldots$ $C_n \Rightarrow t_n = r_n$ in H and a sequence of positions $u_1, u_2, \ldots, u_n$ in t such that $t/u_1 = t_1\sigma_1$, $t/u_2 = t_2\sigma_2$, $\ldots$, $t/u_n = t_n\sigma_n$ and $C_1\sigma_1 \vee C_2\sigma_2 \vee \ldots \vee C_n\sigma_n$ is an inductive theorem of H. Otherwise, t is said to be* pseudo-irreducible *by H.*

Thus, if a term t is pseudo-reducible by H, then all its ground instances are reducible by H. But proving that a node label is pseudo-reducible by a given conditional theory H amounts to proving some inductive theorems. To avoid any vicious circle, either we can use a different method to prove these particular properties or we can use our method itself with a weaker notion of test set than the one we are currently computing. For instance, the set of all terms that are of depth not greater than $D(H)$ and with variables only at depth $D(H)$ suffices for these weaker test sets.

All of the necessary machinery is now at hand to resolve the questions **a, b** and **c** stated above. Let us introduce the different types of node labels we are dealing with by using the class of covering terms.

Definition 6.5 *Given a conditional theory H, let* **T** *be a pattern tree of* $f(x_1, \ldots, x_n)$. *A node label t in* **T** *is said to be of*

type 1 *if t is pseudo-reducible by H,*

type 2 *if t is pseudo-irreducible by H and is H-covering,*

type 3 *if t is an H-irreducible ground term.*

A node label t in **T** *is said to be* type-x-free *if t is neither of type 1, nor of type 2, nor of type 3. A pattern tree* **T** *of* $f(x_1, \ldots, x_n)$ *is said to be* complete *if each node label t in* **T** *of type 1, 2, 3 is a leaf label.*

In the following discussion, we denote node labels of type 1 using boldface letters.

Example 6.3 *If* $FB = \{0, succ, true, false\}, F = FB \cup \{<\}$ *where* $0 : \rightarrow int$, $<: int \times int \rightarrow bool$, $true : \rightarrow bool$, $false : \rightarrow bool$, *and* $succ : int \rightarrow int$, *and H is the following conditional theory.*

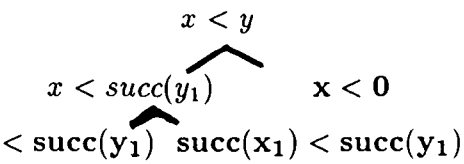
$$0 < succ(x) \rightarrow true \quad (50)$$

$$x < 0 \rightarrow false \quad (51)$$

$$x < y = true \Rightarrow succ(x) < succ(y) \rightarrow true \quad (52)$$

$$x < y = false \Rightarrow succ(x) < succ(y) \rightarrow false \quad (53)$$

then the following pattern tree of $x < y$ *is complete:*



In general, to compute a complete pattern tree T of the term $f(x_1, \ldots, x_n)$, $f \in F$, the following procedure may be used.

A procedure for computing complete pattern trees:

The procedure takes as input a conditional theory H over a signature $\Sigma = (S, F)$. Initially, T is reduced to a node labeled with the term $f(x_1, \ldots, x_n)$.
Repeat as long as type-x-free leaf labels are in T. If none remains terminate successfully: the pattern tree of $f(x_1, \ldots, x_n)$ is complete.

(1) Select a type-x-free leaf label r in T.

(2) Select a variable x in r at a position u where r is H-extensible.

(3) Expand r at x to cover all possibilities for x (i.e., compute the sons of r at x).

When the procedure terminates successfully, it returns as output a test set for H:

Theorem 6.1 *Given a set H of conditional rules, if there exists a complete pattern tree of $f(x_1, \ldots, x_n)$ all leaf labels of which are of type 1, then a test set $S(H)$ can be computed. This test set $S(H)$ contains:*
1. all arguments of leaf labels that are minimal with respect to the subsumption ordering;
2. all constructor terms with variables only at depth $D(H) - 1$ no instance of which is an argument of a leaf label.

For example, the leaf labels of the previous example are $x < 0$, $0 < succ(z_1)$, and $succ(x_1) < succ(y_1)$. The arguments of these leaf labels are: x, 0, 0, $succ(z_1)$, $succ(x_1)$, and $succ(y_1)$. Thus a test set for H is $\{0, succ(x_1), true, false\}$. Note that *true* and *false* are constructor terms and do not occur as argument of leaf labels.

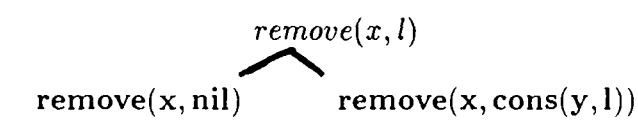
The method also gives a way to check that any ground term of a hierarchical axiomatization can reduce to a constructor term and, as a consequence, by theorem 5.1, to prove the convergence of the system.

Theorem 6.2 *Let H be a set of conditional rules over $\Sigma = \Sigma B \cup (\Sigma - \Sigma B)$ such that each left-hand side contains a symbol from $\Sigma - \Sigma B$. Assume also that T is a complete pattern tree w.r.t. H. Every term in $T(\Sigma)$ is reducible to a term in $T(\Sigma B)$ iff every leaf of T is of* **type 1**.

Example 6.4 *Consider the following theory H:*

$$dif(x,x) \rightarrow ff \quad (54)$$
$$dif(0,s(x)) \rightarrow tt \quad (55)$$
$$dif(s(x),0) \rightarrow tt \quad (56)$$
$$dif(s(x),s(y)) \rightarrow dif(x,y) \quad (57)$$
$$remove(x,nil) \rightarrow nil \quad (58)$$
$$dif(x,y) = tt \Rightarrow remove(x,cons(y,l)) \rightarrow cons(x,remove(y,l)) \quad (59)$$
$$dif(x,y) = ff \Rightarrow remove(x,cons(y,l)) \rightarrow remove(y,l) \quad (60)$$

then the following pattern tree of $x < y$ is complete and each leaf label is of type 1:



since $dif(x,y) = tt \vee dif(x,y) = ff$ is an inductive theorem. Therefore

a. the test set of H is $S(H) = \{tt, ff, 0, s(x), cons(x,l), nil\}$.

b. the system is ground convergent by theorem 5.1.

Note that the previous construction may be extended to the non-free constructors case provided that they are specified by a set of unconditional equations. However,

definition 6.1, 6.2 and 6.3 must be changed accordingly. In the following example the constructors s and p verify some relations.

$$0 < 0 \rightarrow ff \quad (61)$$
$$0 < s(0) \rightarrow tt \quad (62)$$
$$s(x) < y \rightarrow x < p(y) \quad (63)$$
$$p(x) < y \rightarrow x < s(y) \quad (64)$$
$$s(p(x)) \rightarrow x \quad (65)$$
$$p(s(x)) \rightarrow x \quad (66)$$
$$x < y = tt \Rightarrow x < s(y) \rightarrow tt \quad (67)$$
$$y < x = ff \Rightarrow y < p(x) \rightarrow ff \quad (68)$$

The test set here is $\{0, p(0), p(p(x)), s(0), s(s(x)), tt, ff\}$. To see that it is sufficient to verify that all the following terms are pseudo-reducible:
$0 < 0, 0 < p(0), 0 < p(p(x)), 0 < s(0), 0 < s(s(x)),\ p(0) < 0, p(0) < p(0), p(0) < p(p(x)), p(0) < s(0), p(0) < s(s(x)),\ p(p(x)) < 0, p(p(x)) < p(0), p(p(x)) < p(p(y)),\ p(p(x)) < s(0), p(p(x)) < s(s(y)),\ s(0) < 0, s(0) < p(0), s(0) < p(p(x)), s(0) < s(0), s(0) < s(s(x)),\ s(s(x)) < 0, s(s(x)) < p(0), s(s(x)) < p(p(y)), s(s(x)) < s(0),\ s(s(x)) < s(s(y))$.

7 Implementation and experimental Results

Our implementation of test set induction uses four main data structures:

- R, a rewrite system for a conditional theory, built with the constructor discipline. Note that this restriction is not required for purely equational theories.
- C, a set of conjectures to be proved.
- H, a set of inductive hypotheses.
- S, a test set of R

We can describe the procedure Prove_by_induction in this way:

Prove_by_induction *(C,H,R)*

1. *Compute a test set S of R.*

2. – *Simplify each conjecture in C:*
   - ⋆ *by the axioms of R.*
   - ⋆ *by the other conjectures.*
   - ⋆ *by the inductive hypotheses H*

   – *Elimination of trivial identities.*

3. **if** $C = \emptyset$

then *all the initial conjectures are inductive consequences of* R.
else
a. *Select a conjecture* c *in* C.
b. *Apply* Generate *or* Case rewriting *to derive new inductive hypotheses by instantiation of* c *by test-substitutions at inductive positions.*
c. **if** *step* **b.** *did not succeed*
then
if R *is convergent and* c *is a quasi-inconsistent equation.*
then C *is not an inductive consequence of* R.
else *failure.*
else *go to 2.*

Instead of being hierarchical, our induction proof handles all the induction hypotheses at the same level. This provides us with a fully automated procedure, where much more simplifications are permitted than with explicit hierarchical induction. Moreover, it permits to prove several properties in the same round, each of them being helpful to prove the others.

Below, we show partial transcripts of sessions with SPIKE on the example 4.6, to give more intuition about the abilities of the system. It illustrates the efficiency of our system thanks to the mutual simplification of the conjectures. A more detailed account is given in [Bou91].

```
R = { plus(x1,0) = x1 ;
      S(plus(x1,x2)) = plus(x1,S(x2)) ;
      even(0) = True ;
      even(S(0)) = False ;
      even(S(S(x1))) = even(x1) ;
      (even(x1)=True) => odd(x1) = False ;
      (even(x1)=False) => odd(x1) = True }

C0 = {even(S(plus(x1,x1))) = False ,
      even(plus(x1,x1)) = True ,
      odd(plus(x1,S(x1))) = True ,
      odd(plus(x1,x1)) = False ,
      plus(x1,x2) = plus(x2,x1) ,
      even(x1) = True v even(x1) = False ,
      odd(x1) = True v odd(x1) = False}

H0 = {}

Simplification of (odd(plus(x1,S(x1))) = True ) by
                R[H0 U C0]:
                  (True = True)

Simplification of (odd(plus(x1,x1)) = False ) by
                R[H0 U C0]:
                  (False = False)

Delete (True = True)

Delete (False = False)

E1 = {even(S(plus(x1,x1))) = False ,
      ...}

H1 = {}
```

```
Application of generate on (even(S(plus(x1,x1))) = False)

C2 = {False = False ,
      False = False ,
      even(S(plus(S(S(x1)),x1))) = False ,
      ...}

H2 = {even(S(plus(x1,x1))) = False}

Delete (False = False)

Simplification of (even(S(plus(S(S(x1)),x1))) = False ) by
                C2-{even(S(plus(S(S(x1)),x1))) = False}:
                  (even(S(plus(x1,S(S(x1))))) = False )

C3 = {even(S(plus(x1,S(S(x1))))) = False ,
      ...}

H3 = {even(S(plus(x1,x1))) = False}

Simplification of (even(S(plus(x1,S(S(x1))))) = False ) by
                R[H3 U C3]:
                  (even(S(plus(x1,x1))) = False )


E4 = {even(S(plus(x1,x1))) = False ,
      ...}

H4 = {even(S(plus(x1,x1))) = False}

Simplification of (even(S(plus(x1,x1))) = False ) by H4:
                  (False = False )

Delete (False = False)

C5 = {plus(x1,x2) = plus(x2,x1) ,
      ...}

H5 = {even(S(plus(x1,x1))) = False}

...

Application of generate on (plus(x1,x2) = plus(x2,x1))
   => The induction variables: x2

C10 = {x1 = plus(0,x1) ,
       S(x1) = plus(S(0),x1) ,
       S(S(plus(x1,x2))) = plus(S(S(x2)),x1) ,
       ...}

H10 = {even(S(plus(x1,x1))) = False ,
       even(plus(x1,x1)) = True ,
       plus(x1,x2) = plus(x2,x1)}

Simplification of (S(S(plus(x1,x2))) = plus(S(S(x2)),x1) ) by H10:
                  (S(S(plus(x2,x1))) = plus(S(S(x2)),x1) )

C11 = {S(S(plus(x2,x1))) = plus(S(S(x2)),x1) ,
       odd(x1) = True or odd(x1) = False ,
       even(x1) = True or even(x1) = False}

H11 = {even(S(plus(x1,x1))) = False ,
       even(plus(x1,x1)) = True ,
       plus(x1,x2) = plus(x2,x1)}

...
```

```
Application of case rewriting on:
          odd(x1) = True or odd(x1) = False:

 1) odd(x1) = False v True = True v  ~ (even(x1) = False).

 2) odd(x1) = True v odd(x1) = False v even(x1) = False.

C12 = {even(x1) = True or even(x1) = False ,
       odd(x1) = False v True = True v  ~ (even(x1) = False) ,
       odd(x1) = True v odd(x1) = False v even(x1) = False}

H12 = {even(S(plus(x1,x1))) = False ,
       even(plus(x1,x1)) = True ,
       plus(x1,x2) = plus(x2,x1) ,
       x1 = plus(0,x1) ,
       S(x1) = plus(S(0),x1) ,
       S(S(plus(x1,x2))) = plus(S(S(x1)),x2)}

Delete  odd(x1) = False v True = True v ~ (even(x1) = False)

...

Application of generate on:
        even(x1) = True or even(x1) = False

C14 = {True = True v True = False ,
       False = True v False = False ,
       even(S(S(x1))) = True v even(S(S(x1))) = False ,
       ...}

H14 = {even(S(plus(x1,x1))) = False ,
       even(plus(x1,x1)) = True ,
       plus(x1,x2) = plus(x2,x1) ,
       x1 = plus(0,x1) ,
       S(x1) = plus(S(0),x1) ,
       S(S(plus(x1,x2))) = plus(S(S(x1)),x2) ,
       even(x1) = True v even(x1) = False}

Delete True = True v True = False

Delete False = True v False = False

Simplification of even(S(S(x1))) = True v even(S(S(x1))) = False by  R:
  even(x1) = True v even(x1) = False

Delete even(x1) = True v even(x1) = False.
 which contains an instance of:
       even(x1) = True v even(x1) = False.
 smaller than:
       even(S(S(x1))) = True v even(S(S(x1))) = False.

C15 = {}

H15 = {even(S(plus(x1,x1))) = False ,
       even(plus(x1,x1)) = True ,
       plus(x1,x2) = plus(x2,x1) ,
       x1 = plus(0,x1) ,
       S(x1) = plus(S(0),x1) ,
       S(S(plus(x1,x2))) = plus(S(S(x1)),x2) ,
       even(x1) = True v even(x1) = False}

>>> All the initial conjectures are inductive consequences of R. <<<

Have a nice day !

(): unit

/tmp_mnt/users/eureca/bouhoula/SPIKE/example.ml loaded
```

8 Conclusion

We have presented new methods for inductive reasoning. These methods try to exploit as much as possible the power of rewriting. Rewriting systems are a natural framework for inductive reasoning, since they provide well-suited noetherian relations. Proofs in the initial model usually require checking infinite sets of ground equations. The concept of test set allows to reduce this set to a finite one. Moreover, when the axioms are convergent, test sets give a complete strategy to disprove theorems by producing counterexamples. We feel that the method presented here could challenge successfully the alternative method proposed by Boyer and Moore. Moreover, this method generalizes to the case where there are axioms in the theory with negation of equations in the conditions. The theorem-prover SPIKE based on our technique has solved a number of interesting problems. It is being currently extended to incorporate generalization mechanisms and tactics (see [Bun83, BvHSI89, Hut89]), which are necessary to solve many usual problems. Moreover, we are working on better algorithms for the computation of test sets.

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Appendix

proof of theorem 4.1 and 4.2

Theorem 4.1 is a particular case of theorem 4.2. However, we prefer to give the proof separately since it gives the motivations for theorem 4.2. Suppose that $u = v$ is not an inductive theorem of R. Let τ be a ground substitution such that $u\tau = v\tau$ is the smallest equation w.r.t. $\prec$ such that

$$R \not\models u\tau = v\tau$$

We can suppose that τ is irreducible, otherwise we could exhibit a smaller counter-example. Hence, there is a test substitution ϕ and a ground substitution θ such that $\tau = \phi\theta$. From the hypothesis, there is a term α such that

$$u\phi \rightarrow\!\!\!\!\!\!* \; \alpha \text{ and } v\phi \rightarrow\!\!\!\!\!\!* \; \alpha$$

By instantiation:

$$u\tau \rightarrow\!\!\!\!\!\!* \; \alpha\theta \text{ and } v\tau \rightarrow\!\!\!\!\!\!* \; \alpha\theta$$

It remains to show that every term replacement occurring in the proof above is valid (in the initial model) either because it is a logical consequence of R or by induction hypothesis. This will give a contradiction with the choice of τ.

1. suppose the instance $u\rho = v\rho$ of $u = v$ is used in some unconditional step. Since it is not the first step, the replacement is applied to a term which is smaller than the initial term $u\phi\theta$ (or $v\phi\theta$) since this last one has been rewritten at least once. We have again $\{u\phi\theta, v\phi\theta\} \succ \{u\rho, v\rho\}$.

2. if $u\rho = v\rho$ is used in some conditional step. Since conditions are evaluated only when they are smaller than the term to rewrite, every instance of an equation which is used in such a proof is smaller than the original goal $\{u\phi\theta, v\phi\theta\}$.

proof of theorem 4.3

Let L be the left-hand sides of the conclusions of the conditional rules in R. Let C be of the form $\neg e_1 \vee \neg e_2 \vee \ldots \vee \neg e_m \vee g_1 = d_1 \vee \ldots \vee g_n = d_n$, a clause which is quasi-inconsistent with respect to R. Then there is a test instance $C\sigma$ of C, such that $e_{l''}\sigma$ is an inductive theorem for all $l'' \leq m'$ and for all $l' \leq n$ at least one of the following fact holds:

A) $g_{l'}\sigma \not\equiv d_{l'}\sigma$ and no subterm of $g_{l'}\sigma$, $d_{l'}\sigma$ is an instance of L.

B) $g_{l'}\sigma \succ d_{l'}\sigma$ and no subterm of $g_{l'}\sigma$ is an instance of L.

C) $g_{l'}\sigma \prec d_{l'}\sigma$ and no subterm of $d_{l'}\sigma$ is an instance of L.

In order to show that C is not an inductive theorem of R, it is sufficient to show that $(g_1 = d_1 \vee \ldots \vee g_n = d_n)\ \sigma$ is not an inductive theorem of R. Indeed, all ground instances of $(\neg e_1 \vee \neg e_2 \vee \ldots \vee \neg e_m)\ \sigma$ are false in R, by hypothesis. Let us show that in the case where for all $l' \leq n\ \ g_{l'}\sigma \not\equiv d_{l'}\sigma$ and no subterm of $g_{l'}\sigma$, $d_{l'}\sigma$ is an instance of L, then C is not in inductive theorem of R.

Let $Var\ (g_1 = d_1 \vee \ldots \vee g_n = d_n) = \{x_1, \ldots, x_k\}$ and let $A\sigma$ be an instance of $A = g_1 = d_1 \vee \ldots \vee g_n = d_n$ such that $\sigma = \{x_1 \leftarrow t_1, \ldots, x_k \leftarrow t_k\}$, where $t_i \in S(H)$ for all $i \leq k$. We shall show that there exists at least a ground instance $A\sigma\eta$ of $A\sigma$ no subterm of which is an instance of L. This will be enough to ensure that $(g_1 = d_1 \vee \ldots \vee g_n = d_n)\ \sigma$ is not an inductive theorem of R. There are two cases depending on the groundness of $t_{k'}$:

Case 1: Suppose t_i is ground for all $i \leq k$. Then $A\sigma$ is a ground clause. Thus, there is no i such that $g_i\sigma$ and $d_i\sigma$ rewrite to the same term using R. Hence $(g_1 = d_1 \vee \ldots \vee g_n = d_n)\ \sigma$ is false using the ground convergent of R.

Case 2: Suppose there exists $i \leq k$ such t_i is non-ground. Suppose that the variables $x_1, \ldots, x_k$ occur at positions $v_1, \ldots, v_k$, and let m be the maximal number of non-ground subterms of $t_{k'}$, which are rooted at positions of depth equal to $depth(H)$. Let $\sigma\eta$ be a strong H-irreducible ground substitution instance of σ such that $|depth(A\sigma\eta/v_i w_{ij}) - (A\sigma\eta/v_{i'} w_{i'j'})| > depth(A\sigma)$ for all $l \leq i,\ i' \leq k$, and all $l \leq j' \neq j \leq m$. Note that such a substitution instance exists by using clause 2 of the definition of Test Sets. Let us show that $A\sigma\eta$ contains no subterm which is an instance of L. Were a subterm of $A\sigma\eta$ an instance of L, there would exist a strict position u in $A\sigma\eta$ (because $\sigma\eta$ is strong R-irreducible) and a term $s \in L$ such that $A\sigma\eta/u$ is an instance of s. There are two cases depending on the linearity of s:

Case 21: Suppose s is linear. Let first v be a non-variable position in s. Necessarily uv is a non variable position in $A\sigma$ by using clause 3 of definition 1. Since $A\sigma\eta/u$ is an instance of s, we have $A\sigma(uv) = A\sigma\eta\ (uv) = s(v)$. Now suppose that $VP(s)$ is the set of variable positions of s, and σ'' the substitution such that for any $s(w) = x \in X$, then $x\sigma'' = A\sigma\eta/uw$. Note that σ'' is well defined, because s is linear, so x cannot have several occurences. It follows that $A\sigma/u = s\sigma''$, thus $A\sigma$ contains a subterm which is an instance of $s \in L$, contradiction.

Case 22: Suppose s is non-linear. Let v a non-variable position in s. Necessarily uv is a non variable position in $t\sigma$ by using clause 3 of definition of Test Sets. Since $A\sigma\eta/u$ is an instance of s, we have $A\sigma(uv) = A\sigma\eta(uv) = s(v)$. Let $(u_1, u_2) \in VP(s)$, such that $s(u_1) = s(u_2) = x \in X$. Then $A\sigma\eta/uu_1 = A\sigma\eta/uu_2$. Since $A\sigma$ contains no subterm which is an instance of L, there must exist such a paire uu_1, uu_2 of position in $A\sigma$ such that $A\sigma/uu_1 \neq A\sigma/uu_2$. The remainder of the proof of this case is by subcases depending on the groundness of subterms $A\sigma/uu_1$ and $A\sigma/uu_2$:

Case 22a: Suppose $A\sigma/uu_1$ and $A\sigma/uu_2$ are ground. Then $A\sigma/uu_1 = A\sigma\eta/uu_1$ and $A\sigma/uu_2 = A\sigma\eta/uu_2$. Since $A\sigma\eta/uu_1 = A\sigma\eta/uu_2$, we have $A\sigma/uu_1 = A\sigma/uu_2$, contradiction.

Case 22b: Suppose $A\sigma/uu_1$ is ground and $A\sigma/uu_2$ is non-ground. Then $A\sigma/uu_1 = A\sigma\eta/uu_1$. On the other hand, since $u \in sdom(A\sigma)$ and $|u_2| \leq depth(H)$, there must exist a position v_iw_{ij} in $A\sigma$ such $A\sigma/v_iw_{ij}$ is non-ground, $uu_2 < v_iw_{ij}$, and $|w_{ij}| = depth(H)$, for some $i \leq k$ and $j \leq m$. Thus, $depth(A\sigma\eta/uu_2) > depth(A\sigma\eta/v_iw_{ij})$. Further, $depth(A\sigma\eta/v_iw_{ij}) > depth(A\sigma)$ by using definition of substitution $\sigma\eta$. Now, $depth(A\sigma) > depth(A\sigma/uu_1) = depth(A\sigma\eta/uu_1)$. Thus, $depth(A\sigma\eta/uu_2) > depth(A\sigma\eta$ $/uu_1)$, contradiction.

Case 22c: Suppose $A\sigma/uu_1$ and $A\sigma/uu_2$ are non-ground. Since $A\sigma/uu_1 \neq A\sigma/uu_2$, there must exist a position w such that $A\sigma/uu_1w \neq A\sigma/uu_2w$. There are again three subcases depending on the groundness of subterms $A\sigma/uu_1w$ and $A\sigma/uu_2w$:

(i) Suppose $A\sigma/uu_1w$ and $A\sigma/uu_2w$ are ground: the proof is identical to case 22a.

(ii) Suppose $A\sigma/uu_1w$ is ground and $A\sigma/uu_2w$ is non-ground: the proof is identical to case 22b.

(ii) Suppose $A\sigma/uu_1w$ and $A\sigma/uu_2w$ are non-ground: Since $A\sigma/uu_1w \neq A\sigma/uu_2w$, there must exist $l \leq i$, $i' \leq k$, such that one of the following cases hold:

(1) $uu_1w = v_iw_{ij}$ and $uu_2w = v_{i'}w_{i'j'}$,

(2) $uu_1w = v_iw_{ij}$ and $uu_2w > v_{i'}w_{i'j'}$,

(3) $uu_1w = v_iw_{ij}$ and $uu_2w < v_{i'}w_{i'j'}$,

(4) $uu_1w > v_iw_{ij}$ and $uu_2w = v_{i'}w_{i'j'}$,

(5) $uu_1w > v_iw_{ij}$ and $uu_2w < v_{i'}w_{i'j'}$,

(6) $uu_1w > v_iw_{ij}$ and $uu_2w > v_{i'}w_{i'j'}$,

(7) $uu_1w < v_iw_{ij}$ and $uu_2w = v_{i'}w_{i'j'}$,

(8) $uu_1w < v_iw_{ij}$ and $uu_2w < v_{i'}w_{i'j'}$,

(9) $uu_1w < v_iw_{ij}$ and $uu_2w > v_{i'}w_{i'j'}$,

(1) Suppose $uu_1w = v_iw_{ij}$ and $uu_2w = v_{i'}w_{i'j'}$. Then $depth(A\sigma\eta/uu_1w) > depth(A\sigma\eta/\ v_iw_{ij}) > depth(A\sigma\eta/v_{i'}W_{i'j'})$ bu using definition of substitution $\sigma\eta$. However, $depth(A\sigma\eta/v_{i'}W_{i'j'} = depth(A\sigma\eta/uu_2w)$ by case hypothesis, and thus $depth(A\sigma\eta$ $/uu_1w) > depth(A\sigma\eta/uu_2w)$, contradiction.
The proof of subcases **(2)**, **(3)**, **(4)**, **(5)**, **(7)** and **(9)** is identical to subcase **(1)**.

(6) Suppose $uu_1w > v_iw_{ij}$ and $uu_2w > v_{i'}w_{i'j'}$. Since $u \in sdom(t\sigma)$ and $|u_1| \leq depth(R)$, there must be $uu_1 < v_iw_{ij}$. Let $w = pq$, where p is such that $uu_1p = v_iw_{ij}$. Since $A\sigma/uu_1w \neq A\sigma/uu_2w$, we have $A\sigma/uu_1p \neq A\sigma/uu_2p$, and $uu_1p = v_iw_{ij}$ and

either $uu_2p = v_{i'}w_{i'j'}$, or $uu_2p < v_{i'}w_{i'j'}$, or $uu_2p > v_{i'}w_{i'j'}$. The proof of those of subcases is identical to **(1)**, **(2)**, **(3)**.

(8) Now Suppose $uu_1w < v_iw_{ij}$ and $uu_2w < v_{i'}w_{i'j'}$. Assume also that $v_{i'}w_{i'j'}$ is along a maximal path of $A\sigma\eta/uu_2$. Then $depth(A\sigma\eta/uu_1w) > depth(A\sigma\eta/v_iw_{ij}) > depth(A\sigma) + depth(A\sigma\eta/v_{i'}W_{i'j'})$ by using the definition of substitution η. However, $depth(A\sigma) + depth(\ A\sigma\eta\ /v_{i'}W_{i'j'}) > depth(A\sigma\eta/uu_2w)$, contradiction.

Therefore, no subterm of $A\sigma\eta$ is an instance of L and $g_{n'}\sigma\eta \not\equiv d_{n'}\sigma\eta$ for all $n' \leq n$. Therefore $R \not\models g_{n'}\sigma\eta = d_{n'}\sigma\eta$ since R is convergent. Thus, C is not an inductive theorem of R. This complete the proof of the theorem. □

proof of lemma 4.1

To prove this lemma, we introduce a notion of complexity of a proof step:

Definition 8.1 *Let $P \equiv s\sigma \leftrightarrow_{s=t} t\sigma$ be an equational proof step. We define the complexity of P by:*

$$C(P) = \begin{cases} (\{s\sigma\},\{t\sigma\}) & if \quad s \succ t \\ (\{t\sigma\},\{s\sigma\}) & if \quad s \prec t \\ (\{s\sigma,t\sigma\},\_) & otherwise \end{cases}$$

Let $P \equiv s\sigma \leftrightarrow_{s=t} t\sigma$ be an equational proof step, we say that $s = t$ justifies P. We say that P is valid in R if $R \models s\sigma = t\sigma$. We consider the set K of proof-steps which are justified by equations in $(\cup_i E_i)$ and we show that the rule $Fail$ applies to an equation $s = t$ of $(\cup_i E_i)$ which justifies a minimal proof-step of K which is not valid in R. It is sufficient to prove that $s = t$ cannot be simplified, and that *generate* cannot apply to $s = t$. Therefore $Fail$ is applied since the equation cannot persists in the derivation due to the fairness hypotheses.

Let $P \equiv s\sigma \leftrightarrow_{s=t} t\sigma$ be this proof-step and assume that $s \not\prec t$. We can assume that σ is irreducible by R. By hypothesis we have $R \not\models s\sigma = t\sigma$. Assume that $s = t \in E_j$ and $(E_j, H_j) \vdash_I (E_{j+1}, H_{j+1})$ by application of simplify or generate on $s = t$. We discuss now the situation according to the rule which is applied and we derive a contradiction in every case. In this proof, in order to simplify notations, we write E for E_j and H for H_j.

$Simplify_1$

We distinguish two cases:

Case 1: $s\sigma \mapsto_{R[H\cup E]} s'\sigma$ and $s \succ s'$. Assume that $e = f \in (H \cup E)$ has been used for proving the preconditions (with substitution θ). Consider $G \equiv e\theta \leftrightarrow_{e=f} f\theta$. By definition 3.4 $\{e\theta, f\theta\} \prec \{s\sigma\}$, hence $C(G) < (\{s\sigma\},\_)$ and therefore $C(G) < C(P)$. On the other hand $(H \cup E) \subset \cup_i E_i$, therefore G is valid in R and finally $R \not\models s'\sigma = t\sigma$. Let $Q \equiv s'\sigma \leftrightarrow_{s'=t} t\sigma$. We verify that $C(Q) < C(P)$:

- **If** $s \succ t$ **then** $C(P) = (\{s\sigma\},\{t\sigma\}) < C(Q)$ since $\{s'\sigma, t\sigma\} \prec \{s\sigma\}$.
- **Otherwise:** $C(P) = (\{s\sigma, t\sigma\},\_)$
  - **If** $s' \succ t$ **then** $C(Q) = (\{s'\sigma\},\{t\sigma\}) < C(P)$ since $s'\sigma \prec s\sigma$.
  - **If** $s' \prec t$ **then** $C(Q) = (\{t\sigma\},\{s'\sigma\}) < C(P)$ since $\{t\sigma\} \prec \{s\sigma, t\sigma\}$.
  - **Otherwise:** $C(Q) = (\{s'\sigma, t\sigma\},\_) < C(P)$ since $\{s'\sigma, t\sigma\} \prec \{s\sigma, t\sigma\}$.

Hence, Q is not valid in R and strictly smaller than $s\sigma \leftrightarrow_{s=t} t\sigma$. This contradicts the choice of $s = t$ (since $s' = t \in \cup_i E_i$).

Case 2: $t\sigma \mapsto_{R[H\cup E]} t'\sigma$ and $t \succ t'$. For the same reason as before we have: $R \not\models s\sigma = t'\sigma$. Let $Q \equiv s\sigma \leftrightarrow_{s=t'} t'\sigma$. We verify that $C(Q) < C(P)$:

- **If** $s \succ t$ **then** $C(P) = (\{s\sigma\}, \{t\sigma\})$.
  In this case we have $s \succ t \succ t'$, hence $C(Q) = (\{s\sigma\}, \{t'\sigma\}) < C(P)$ since $t'\sigma \prec t\sigma$.
- **Otherwise:** $C(P) = (\{s\sigma, t\sigma\}, \_)$
  - **If** $s \succ t'$ **then** $C(Q) = (\{s\sigma\}, \{t'\sigma\}) < C(P)$ since $\{s\sigma\} \prec \{s\sigma, t\sigma\}$.
  - **If** $s \prec t'$ **then** $C(Q) = (\{t'\sigma\}, \{s\sigma\}) < C(P)$ since $t'\sigma \prec t\sigma$.
  - **Otherwise:** $C(Q) = (\{s\sigma, t'\sigma\}, \_) < C(P)$ since $\{s\sigma, t'\sigma\} \prec \{s\sigma, t\sigma\}$.

Hence, Q is not valid in R and strictly smaller than $s\sigma \leftrightarrow_{s=t} t\sigma$, contradiction.

$Simplify_2$

We distinguish two cases:

Case 1: $s \mapsto_H s'$ and $s \succ s'$. Let $g = h$ be the equation of H which has been used to simplify s. Hence there exists τ such that $s \equiv s[g\tau]$, $s' \equiv s[h\tau]$ and $g \succ h$.

- **If** $g = h$ is valid in R **then** $R \not\models s'\sigma = t\sigma$ and on the other hand $C(s'\sigma \leftrightarrow_{s'=t} t\sigma) < C(P)$. (same proof as $Simplify_1$). **which is absurd.**
- **Otherwise:** $R \not\models (g\tau)\sigma = (h\tau)\sigma$.
  - **If** $g\tau \lhd s$ **then** $C((g\tau)\sigma \leftrightarrow_{g=h} (h\tau)\sigma) < P$ since $\{(g\tau)\sigma,\ (h\tau)\sigma\} \prec \{s\sigma\}$, contradiction.
  - **Otherwise:** $g \succ h$. Hence there exists k such that $\{g_i = h\sigma_i\}_{i=1\ldots n} \subset E_k$ with $\{\sigma_i\}_{i=1\ldots n}$ test-substitution for $(g = h)$ and $g\sigma_i \mapsto_{R[H \cup E]} g_i$. On the other hand let $\sigma' = (\tau\sigma)$, where σ' is a ground substitution. Assume that it is irreducible. Then there exists a test-substitution σ_{i_0} and a ground substitution θ such that $\sigma' = \sigma_{i_0}\theta$. Then we have:

    $$\begin{array}{ccc} (g\sigma_{i_0})\theta & \leftrightarrow_{g=h} & (h\sigma_{i_0})\theta \\ \downarrow_{R[H \cup E]} & \nearrow_{E_k} & \\ g_{i_0}\theta & & \end{array}$$

    $C(g_{i_0}\theta \leftrightarrow_{g_{i_0}=h\sigma_{i_0}} (h\sigma_{i_0})\theta) < C(P)$ since $\{g_{i_0}\theta, (h\sigma_{i_0})\theta\} \prec \{s\sigma\}$ since $g_{i_0}\theta \prec (g\sigma_{i_0})\theta \preceq s\sigma$ and $(h\sigma_{i_0})\theta \prec (g\tau)\sigma \preceq s\sigma$. On the other hand, $R \not\models g_{i_0}\theta = (h\sigma_{i_0})\theta$ since all the equations used in the proof of preconditions are valid (since they justify smaller proof-steps), contradiction.

Case 2: $t \mapsto_H t'$. Same reasoning as Case 1.

$Simplify_3$

We distinguish two cases:

Case 1: $s \mapsto_E s'$ and $s \succ s'$. Let $g = h$ be the equation of E which is used to simplify s. Hence there exists τ such that $s \equiv s[g\tau]$, $g\tau \lhd s$, $s' \equiv s[h\tau]$ and $g\tau \succ h\tau$.

- **If** $g = h$ is valid in R **then** $R \not\models s'\sigma = t\sigma$. On the other hand $C(s'\sigma \leftrightarrow_{s'=t} t\sigma) < C(P)$. (same proof as for $Simplify_1$), contradiction.
- **Otherwise:** $R \not\models (g\tau)\sigma = (h\tau)\sigma$. Let $Q \equiv (g\tau)\sigma \leftrightarrow_{g=h} (h\tau)\sigma$, we verify that $C(Q) < C(P)$:
  - **If** $g \succ h$ **then** $C(Q) = (\{(g\tau)\sigma\}, \{(h\tau)\sigma\}) < C(P)$ since $(g\tau)\sigma \prec s\sigma$.
  - **If** $g \prec h$ **then** absurd since $g\tau \succ h\tau$.
  - **Otherwise:** $C(Q) = (\{(g\tau)\sigma, (h\tau)\sigma\}, \_) < C(P)$ since $\{(g\tau)\sigma, (h\tau)\sigma\} \prec \{s\sigma\}$ due to the fact that $(g\tau)\sigma \prec s\sigma$ and $(h\tau)\sigma \prec (g\tau)\sigma \prec s\sigma$.

  contradiction.

Case 2. $t \mapsto_E t'$. Same reasoning as Case 1.

Simplify$_4$

We distinguish two cases:

Case 1. $s \rightsquigarrow_H s'$, $s \not\equiv s'$, $s' \preceq t$ and $s \not\equiv t$. $C(P) = (\{s\sigma, t\sigma\}, \_)$ since $s \not\equiv t$. On the other hand let $g = h$ be the equation of H which is used to simplify s. Hence there exists τ such that $s \equiv s[g\tau]$, $s' \equiv s[h\tau]$ and $g\tau \not\equiv h\tau$:

(a) **If** $s' \prec t$ **then** we have $C((g\tau)\sigma \leftrightarrow_{g=h} (h\tau)\sigma) < C(P)$ since $\{(g\tau)\sigma, (h\tau)\sigma\} \prec \{s\sigma, t\sigma\}$ hence $R \models (g\tau)\sigma = (h\tau)\sigma$ and so: $R \not\models s'\sigma = t\sigma$) with $s' = t \in (\cup_i E_i$. Let $Q \equiv s'\sigma \leftrightarrow_{s'=t} t\sigma$. We have $s' \prec t$ hence $C(Q) = (\{t\sigma\}, \{s'\sigma\}) < C(P)$, contradiction.
(b) **Otherwise:** $s' = t$, in this case we have:

$$\begin{array}{ccc} s\sigma & \leftrightarrow_{s=t} & t\sigma \\ \downarrow_H & & \\ t\sigma & & \end{array}$$

Hence: $R \not\models (g\tau)\sigma = (h\tau)\sigma$. Let $\sigma' = (\tau\sigma)$, where σ' is a ground substitution. Assume that it is irreducible, hence there exists σ_{i_0} a test-substitution and a ground substitution θ such that $\sigma' = \sigma_{i_0}\theta$. We distinguish two cases:

i.

$$\begin{array}{ccc} (g\sigma_{i_0})\theta & \leftrightarrow_{g=h} & (h\sigma_{i_0})\theta \\ \downarrow_{R[H \cup E]} & \nearrow_{g_{i_0}=h\sigma_{i_0}} & \\ g_{i_0}\theta & & (g_{i_0} = h\sigma_{i_0}) \in (\cup_i E_i) \end{array}$$

Let $Q \equiv g_{i_0}\theta \leftrightarrow_{g_{i_0}=h\sigma_{i_0}} (h\sigma_{i_0})\theta$, we verify that $C(Q) < C(P)$:

* **If** $g_{i_0} \succ h\sigma_{i_0}$ **then** $C(Q) = (\{g_{i_0}\theta\}, \{(h\sigma_{i_0})\theta\}) < C(P)$ since $g_{i_0}\theta \prec (g\sigma_{i_0})\theta \preceq s\sigma$.
* **If** $g_{i_0} \prec h\sigma_{i_0}$ **then** $C(Q) = (\{(h\sigma_{i_0})\theta\}, \{g_{i_0}\theta\}) < C(P)$ since $(h\sigma_{i_0})\theta \preceq s'\sigma = t\sigma$.

* **Otherwise:** $C(Q) = (\{g_{i_0}\theta, (h\sigma_{i_0})\theta\}, \_) < C(P)$ since $\{g_{i_0}\theta, (h\sigma_{i_0})\theta\} \prec \{s\sigma, t\sigma\}$.

Contradiction.

ii.

$$(g\sigma_{i_0})\theta \leftrightarrow_{g=h} (h\sigma_{i_0})\theta$$
$$\downarrow R[H \cup E]$$
$$g\sigma_{i_0} = h_{i_0} \quad h_{i_0}\theta \qquad (g\sigma_{i_0} = h_{i_0}) \in (\cup_i E_i)$$

Let $Q \equiv (g\sigma_{i_0})\theta \leftrightarrow_{g\sigma_{i_0}=h_{i_0}} h_{i_0}\theta$, we verify that $C(Q) < C(P) = (\{s\sigma, t\sigma\}, \_)$:

* **If** $g\sigma_{i_0} \succ h_{i_0}$ **then** $C(Q) = (\{(g\sigma_{i_0})\theta\}, \{h_{i_0}\theta\}) < C(P)$ since $(g\sigma_{i_0})\theta \preceq s\sigma$.
* **If** $g\sigma_{i_0} \prec h_{i_0}$ **then** $C(Q) = (\{h_{i_0}\theta\}, \{(g\sigma_{i_0})\theta\}) < C(P)$ since $h_{i_0}\theta \prec (h\sigma_{i_0})\theta \preceq s'\sigma = t\sigma$.
* **Otherwise:** $C(Q) = (\{(g\sigma_{i_0})\theta, h_{i_0}\theta\}, \_) < C(P)$ since $\{(g\sigma_{i_0})\theta, h_{i_0}\theta\} \prec \{s\sigma, t\sigma\}$.

Contradiction.

Case 2. $t \rightsquigarrow_H t'$. Same reasoning as 1.

Generate

Since σ is a ground irreducible substitution hence there exists a test-substitution σ_{i_0} and a ground substitution θ such that: $\sigma = \sigma_{i_0}\theta$. Assume that the rule *Generate* applies to $s = t$, then we distinguish two cases:

Case 1. $s \succ t$

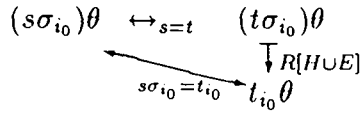
$$(s\sigma_{i_0})\theta \leftrightarrow_{s=t} (t\sigma_{i_0})\theta$$
$$\downarrow R[H \cup E]$$
$$s_{i_0}\theta \quad s_{i_0} = t\sigma_{i_0} \qquad (s_{i_0} = t\sigma_{i_0}) \in (\cup_i E_i)$$

Let $Q \equiv s_{i_0}\theta \leftrightarrow_{s_{i_0}=t\sigma_{i_0}} (t\sigma_{i_0})\theta$, we verify that $C(Q) < C(P)$:

- **If** $s \succ t$ **then** $C(Q) < C(P)$ since $\{s_{i_0}\theta, (t\sigma_{i_0})\theta\} \prec \{s\sigma\}$.
- **Otherwise:** $C(P) = (\{s\sigma, t\sigma\}, \_)$
  * **If** $s_{i_0} \succ t\sigma_{i_0}$ **then** $C(Q) = (\{s_{i_0}\theta\}, \{(t\sigma_{i_0})\theta\}) < C(P)$ since $s_{i_0}\theta \prec s\sigma$.
  * **If** $s_{i_0} \prec t\sigma_{i_0}$ **then** $C(Q) = (\{(t\sigma_{i_0})\theta\}, \{s_{i_0}\theta\}) < C(P)$ since $(t\sigma_{i_0})\theta = t\sigma$.
  * **Otherwise:** $C(Q) = (\{s_{i_0}\theta, (t\sigma_{i_0})\theta\}, \_) < C(P)$ since $\{s_{i_0}\theta, (t\sigma_{i_0})\theta\} \prec \{s\sigma, t\sigma\}$.

$R \not\models s_{i_0}\theta = (t\sigma_{i_0})\theta$ since all the equations used in the proof of preconditions are valid (since they justify smaller proof-steps). On the other hand, $C(Q) < C(P)$, contradiction.

Case 2. $s \nsucc t$, the situation is either as in A. or we have:



$(s\sigma_{i_0} = t_{i_0}) \in (\cup_i E_i)$

Let $Q \equiv (s\sigma_{i_0})\theta \leftrightarrow_{s\sigma_{i_0}=t_{i_0}} t_{i_0}\theta$, $C(P) = (\{s\sigma, t\sigma\}, \_)$ since $s \not\ast t$, we verify that $C(Q) < C(P)$:

- **If** $s\sigma_{i_0} \succ t_{i_0}$ **then** $C(Q) = (\{(s\sigma_{i_0})\theta\}, \{t_{i_0}\theta\}) < C(P)$ since $(s\sigma_{i_0})\theta = s\sigma$.
- **If** $s\sigma_{i_0} \prec t_{i_0}$ **then** $C(Q) = (\{t_{i_0}\theta\}, \{(s\sigma_{i_0})\theta\}) < C(P)$ since $t_{i_0}\theta \prec t\sigma$.
- **Otherwise:** $C(Q) = (\{(s\sigma_{i_0})\theta, t_{i_0}\theta\}, \_) < C(P)$ since $\{(s\sigma_{i_0})\theta, t_{i_0}\theta\} \prec \{s\sigma, t\sigma\}$.

contradiction. □

proof of theorem 6.1

Clearly, $S(H)$ is finite. Let us show that $S(H)$ satisfies the properties of $Definition 4.1$:

- Since all leafs labels are of type 1, every ground term of the form $f(t_1, \ldots, t_n)$, where $f \in \Sigma - \Sigma B$ and $t_i \in T(\Sigma B)_{s_i}$ for all i, is H-reducible.

  Therefore every ground irreducible term is a constructor Σ-term. By the properties of pattern trees, the completeness follows.

- Coveredness is obtained by construction.

- By construction of test sets the only terms in $S(H)$ are constructor terms. By hypothesis any left-hand sides in H contains a non constructor symbol. This means that all ground instances of terms in $S(H)$ are strong irreducible. By coveredness each term in $S(H)$ has variables occurring only at depth $Depth(H)$. Therefore all variables may be substituted by infinitely many different constructor terms (we assume that there exists at least one constructor symbol which is not a constant). Hence $S(H)$ has the transnormality property.

proof of theorem 6.2

Assume that every leaf of T is of type 1. Let $s = f(t_1, ..., t_n)$ be a ground term such that $f \in \Sigma - \Sigma B$ and for all i, $t_i \in T(\Sigma B)$. Necessarily, there exists a leaf t of T and a ground substitution σ such that $t\sigma = s$. Since t is of type 1, s is reducible. Let $s!$ be an irreducible term wich is equivalent to s w.r.t. H. If $s!$ admits a function symbol from $\Sigma - \Sigma B$, let $f(s_1, ..., s_n)$ be a subterm of $s!$ such that for all i, $s_i \in T(\Sigma B)$. Since $f(s_1, ..., s_n)$ is an instance of a leaf of T, it is reducible. This fact contradicts the irreducibility of $s!$. As a consequence, $s! \in T(\Sigma B)$.

Assume that every term in $T(\Sigma)$ is reducible to a term in $T(\Sigma B)$. If there is a leaf of T which is not of type 1, then t is either of type 2 or of type 3.

If t is of type 2, then t is ground and irreducible. Since t contains a non constructor symbol, this contradicts the assumption.

If t is of type 3 and is not an instance of a left-hand side of a rule in H, we reason as in Appendix B, case B. Assume that $V(t) = \{x_1, ..., x_k\}$. Let us consider a ground irreducible substitution ϕ such that for any $j \in \{1, \ldots, k\}$, $\mid x_i\phi \mid - \mid x_j\phi \mid \geq |t| + |H| + 1$. Note that ϕ is well defined thanks to the definition 6.1 and thanks to the fact that no constructor occurs as a top symbol. Then we can prove that $t\phi$ is ground and irreducible. Again, since t contains a non constructor symbol, this contradicts the assumption.

If t is of type 3 and $t = l\sigma$, where $c \Rightarrow l \rightarrow r$ is a rule in H. Let $\{c_1 \Rightarrow l_1 \rightarrow r_1, \ldots, c_n \Rightarrow l_n \rightarrow r_n\}$ be the set of conditional equations of H whose left-hand sides match l. Hence, $t/u_1 = l_1\sigma_1, \ldots, t/u_n = l_n\sigma_n$ and, since t is not pseudo-reducible, $C = C_1\sigma_1 \vee \ldots \vee C_n\sigma_n$ is not an inductive theorem of H. There is a ground irreducible substitution θ such that $C\theta$ is not valid. Consider now $t\theta$: $t\theta$ cannot be reducible at the root, because $C\theta$ is not true; $t\theta$ cannot be reducible at another position, since no proper subterm of $t\theta$ contains a defined function symbol. This leads again to a contradiction.

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