The Hinfini problem with control constraints

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LE PROBLEME H_∞ AVEC RESTRICTIONS SUR LE CONTROLE

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Abstract

Necessary and sufficient conditions for existence of a solution to the suboptimal H_∞-problem for input-output linear systems with control constraints are established.

Résumé

On établit des conditions nécessaires et suffisantes pour l'existence d'une solution sous optimale pour le problème H_∞ avec restrictions sur le contrôle.

Key words

Stabilizable feedback, Hamilton-Jaccobi equation, Differential game, Convex function, H_∞-problem.

Mots-clé

Feedback stabilisable, équation Hamilton-Jacobi, jeux différentiel, fonction convexe, problème H_∞.

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1 Problem formulation

Consider the input-out system

$$\begin{array}{ll} x'(t) = Ax(t) + B_2u(t) + B_1w(t), & t \in \mathbb{R}^+ = [0,\infty) \\ x(0) = x_0 & \\ z(t) = C_1x(t) + D_{12}u(t), & \text{a.e. } t \in \mathbb{R}^+ \\ u(t) \in U_0 & \text{a.e. } t > 0 \end{array} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup e^{At} on X, $B_2 \in L(U,X)$, $B_1 \in L(W,X)$, $C_1 \in L(X,Z)$, $D_{12} \in L(U,Z)$ and $x' = \dfrac{dx}{dt}$.

Here X, Z, U, W are separable real Hilbert spaces and U_0 is a closed convex subset of U such that $0 \in U_0$.

The system (1.1) will be studied under the following standard hypotheses

$$D_{12}^*D_{12} = I, \quad D_{12}^*C_1 = 0 \tag{1.2}$$

$$\textit{The pair } (A, C_1) \textit{ is exponentially detectable.} \tag{1.3}$$

Here and throughout in the sequel we shall use the asterisk symbol to denote the dual operators. Also we shall denote by $|.|$, $|.|_Z$, $|.|_U$, $|.|_W$ the norms in X, Z, U, W, respectively and by $(.,.)$, $(.,.)_Z$, $(.,.)_U$, $(.,.)_W$ the corresponding scalar products.

In system (1.1) $x \in X$, $u \in U$, $w \in W$ and $z \in Z$ are the state, the control, the exogeneous variables (disturbance) and the controlled input, respectively. By definition, an *admissible feedback control* is a mapping $F : X \to U_0$ such that for every measurable function $x = x(t)$, $t \to F(x(t))$ is measurable on R^+.

An admissible feedback control F is said to be *stabilizable* if for every $x_0 \in X$ and $f \in L^2(R^+;X)$ the Cauchy problem

$$x' = Ax + B_2Fx + f \text{ in } R^+ \; ; \; x(0) = x_0 \tag{1.4}$$

has at least one mild solution $x \in C(R^+;X) \cap L^2(R^+;X)$ with $u = Fx \in L^2(R^+;U)$.

We shall denote by $\mathcal{F}$ the set of all stabilizable feedback controls F. For every $F \in \mathcal{F}$ and $w \in L^2(R^+;W)$, $x_0 \in X$ we set

$$S_F(x_0, w) = z = C_1x + D_{12}u \tag{1.5}$$

where x is any mild solution to equation

$$x' = Ax + B_2Fx + B_1w \text{ in } R^+ \; ; \; x(0) = x_0 \tag{1.6}$$

i.e. (see e.g. [6])

$$(1.6)' \qquad x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)}(B_2Fx(s) + B_1w(s))ds, \quad \forall t \geq 0,$$

such that $C_1x \in L^2(R^+;X)$.

The operator $S_F : X \times L^2(R^+;W) \to L^2(R^+;Z)$ is in general multivalued but everywhere defined on $X \times L^2(R^+;W)$.

According to the theory of standard H_∞-problem ([3], [5]) we shall define the H_∞-suboptimal control problem for system (1.1) as follows : *Given* $\gamma > 0$ *find* $F \in \mathcal{F}$ *such that*

$$|S_F(x_0, w)|^2 \le \rho^2 \|w\|^2_{L^2(R^+;W)} + C|x_0|^2 \quad \forall (x_0, w) \in X \times L^2(R^+; W) \tag{1.7}$$

where $0 < \rho < \gamma$ *and* $C \in R$.

Here

$$|S_F(x_0, w)| = \sup\{\|\theta\|_{L^2(R^+;Z)} \; ; \; \theta \in S_F(x_0, w)\}.$$

The main result of this work, Theorem 1 below solves the above problem in terms of a stationary Hamilton-Jacobi equation and it is a generalization of known results ([3], [8]) from the unconstrained case. This results seems to be new even in the finite dimensional framework. Although the approach borrows an idea already used in the study of standard H_∞-problem, namely to reduce the problem to a differential game associated with system (1.1), the proof is quite different and there are significant differences between our treatment and the standard one.

2 The main result

Throughout in the sequel we shall assume that system (1.1) satisfies hypotheses (1.2) and (1.3).

Theorem 1 *Let* $\gamma > 0$. *If the* H_∞*-suboptimal control problem has a solution* $F \in \mathcal{F}$ *then there is a continuous, convex and Gâteaux differentiable function* $\varphi : X \to R$ *such that*

$$0 \le \varphi(x) \le C|x|^2 \quad \forall x \in X \tag{2.1}$$

$$\begin{aligned} &(Ax, \nabla\varphi(x)) + 2^{-1}|P_{U_0}(-B_2^*\nabla\varphi(x))|^2_U + (2\gamma^2)^{-1}|B_1^*\nabla\varphi(x)|^2 \\ &\quad + (B_2^*\nabla\varphi(x), P_{U_0}(-B_2^*\nabla\varphi(x)))_U + 2^{-1}|C_1 x|^2 = 0 \quad \forall x \in D(A). \end{aligned} \tag{2.2}$$

Moreover, the Cauchy problem

$$\begin{aligned} &x' = Ax + B_2 P_{U_0}(-B_2^*\nabla\varphi(x)) + \gamma^{-2} B_1 B_1^* \nabla\varphi(x) \\ &x(0) = x_0 \end{aligned} \tag{2.3}$$

has for every $x_0 \in X$ *at least one mild solution*

$$x^* \in C(R^+; X) \cap L^2(R^+; X) \; ; \; \lim_{t\to\infty} x^*(t) = 0.$$

Conversely, let us assume that either B_2 *or* e^{At} *are compact for all* $t > 0$.

If eq. (2.2) has a solution φ *with the above properties then the feedback* $F = P_{U_0}(-B_2^*\nabla\varphi)$ *is stabilizable and guarantees inequality (1.7) with* $\rho = \gamma$.

Here $P_{U_0} : U \to U_0$ is the projection operator on the set U_0 and $\nabla\varphi$ is the gradient of φ.

We note that in the case of unconstrained H_∞-control problem, i.e., $U_0 = U$, eq. (2.2) reduces to the Riccati equation corresponding to the regular H_∞- problem ([3], [4], [5]) whilist the closed loop inequality (1.7) becomes

$$\|S_F(0, w)\|_{L^2(R^+;Z)} \le \rho \|w\|_{L^2(R^+;W)} \quad \forall w \in L^2(R^+; W)$$

However, in our case a gap arises between the necessary and sufficient conditions for existence of solution to H_∞-problem. Perhaps in most significant cases the existence of a solution φ to (2.2) is necessary and sufficient for existence of a solution to H_∞-suboptimal control problem.

3 Proof of Theorem 1

We shall assume first that there is $F \in \mathcal{F}$ such that (1.7) is satisfied. Define on the space $L^2(R^+;U) \times L^2(R^+;W)$ the function

$$\begin{aligned} K(u,w) &= 2^{-1}\int_0^\infty (|z(t)|_Z^2 + h(u(t)) - \gamma^2 |w(t)|_W^2)dt \\ &= 2^{-1}\int_0^\infty (|C_1 x(t)|_Z^2 + h(u(t)) - \gamma^2 |w(t)|_W^2)dt \end{aligned} \tag{3.1}$$

where x is the mild solution to (1.1), $h(u) = |u|_U^2 + I_{U_0}(u)$ and $I_{U_0} : U \to (-\infty, +\infty]$ is the indicator function of U_0, i.e., $I_{U_0}(u) = 0$ for $u \in U_0$, $I_{U_0}(u) = +\infty$ for $u \bar{\in} U_0$.

Denote $\mathcal{U} = L^2(R^+;U)$, $\mathcal{W} = L^2(R^+;W)$ and consider the problem

$$\sup_{w\in\mathcal{W}} \inf_{u\in\mathcal{U}} K(u,w). \tag{3.2}$$

Lemma 1 *Problem (3.2) has a unique solution $(u^*, w^*) \in \mathcal{U} \times \mathcal{W}$.*

Proof. We will consider first the minimization problem

$$\inf\{K(u,w)\ ;\ u \in \mathcal{U}\} \tag{3.3}$$

where w is arbitrary but fixed in $\mathcal{W}$. Since the function $K(.,w)$ is strictly convex, lower semicontinuous coercive and $\not\equiv +\infty$, problem (3.3) has a unique solution $\tilde{u} = \Gamma w$, i.e.,

$$\Gamma w = \arg\inf\{K(u,w)\ ;\ u \in \mathcal{U}\}.$$

Denote by x_w the corresponding solution to system (1.1) and note that along with the operator Γ the function

$$\Phi(w) = 2^{-1}\int_0^\infty (|C_1 x_w|_Z^2 + h(\Gamma w))dt$$

is continuous and linear on segments, i.e.,

$$\Phi(\lambda w_1 + (1-\lambda)w_2) = \lambda\Phi(w_1) + (1-\lambda)\Phi(w_2) \tag{3.4}$$

$$\Gamma(\lambda w_1 + (1-\lambda)w_2) = \lambda\Gamma w_1 + (1-\lambda)\Gamma w_2, \quad \forall\lambda \in [0,1] \quad w_1, w_2 \in \mathcal{W}. \tag{3.5}$$

Indeed it is easily seen that $u = \lambda\Gamma w_1 + (1-\lambda)\Gamma w_2$ is a solution to problem (3.3) corresponding to $w = \lambda w_1 + (1-\lambda)w_2$. Since the function h is strictly convex this implies (3.5). Moreover, if $w_n \to w$ in $\mathcal{W}$ then $\{u_n = \Gamma w_n\}$ is bounded in $\mathcal{U}$ and so on a subsequence, $u_n \to u$ weakly in $\mathcal{U}$, $x_n(t) \to x_w(t)$ weakly in X for all $t > 0$ and $C_1 x_n \to C_1 x_w$ weakly in $L^2(R^+;Z)$. We have

$$\int_0^\infty (|C_1 x_n|_Z^2 + h(u_n))dt \le \int_0^\infty (|C_1 x_w|_Z^2 + h(\Gamma w)dt = 2\inf(3.3)$$

and hence $\lim \Phi(w_n) = \Phi(w)$. This clearly implies that $u_n \to \Gamma w$ strongly in $L^2(R^+;Z)$ as claimed.

Now if denote by $\Phi_0 : \mathcal{W} \to R$ the function

$$\begin{aligned} \Phi_0(w) &= 2^{-1}\int_0^\infty (\gamma^2|w(t)|_W^2 - |C_1 x_w(t)|_Z^2 - h(\Gamma w(t)))dt \\ &= 2^{-1}\gamma^2\|w\|_{\mathcal{W}}^2 - \Phi(w) \end{aligned}$$

we may equivalently write problem (3.2) as

$$\inf\{\Phi_0(w)\ ;\ w \in \mathcal{W}\}. \tag{3.6}$$

According to inequality (1.7) we have

$$\Phi_0(w) \geq \alpha\|w\|_{\mathcal{W}}^2 + \beta \quad \forall w \in \mathcal{W}$$

where $\alpha > 0$ and $\beta \in R$. On the other hand, as seen above Φ_0 is continuous and by (3.4) it is strictly convex. Hence Φ_0 attains its infimum on W in a unique point w^* and so problem (3.2) has a unique solution (u^*, w^*) as claimed. Throughout in the sequel we shall denote by $x^* = x_{w^*}$ the corresponding solution to system (1.1).

In order to obtain the Euler-Lagrange conditions of optimality corresponding to problem (3.2) we shall consider a family of approximating sup inf problems on the finite intervals $[0, n]$. Namely

$$\sup_{w\in\mathcal{W}_n} \inf_{u\in\mathcal{U}_n} K_n(u, w) \tag{3.7}$$

where

$$K_n(u, w) = 2^{-1}\int_0^n (|C_1x(t)|_Z^2 + h(u(t)) - \gamma^2|w(t)|_W^2)dt$$

x is the corresponding solution to (1.1) on $[0, n]$ and $\mathcal{U}_n = L^2(0, n; U)$, $\mathcal{W}_n = L^2(0, n; W)$.

Lemma 2 *Problem (2.8) has a unique solution (u_n, w_n) which is expressed as*

$$u_n(t) = P_{U_0}(B_2^*p_n(t))\ ;\ w_n(t) = -\gamma^{-2}B_1^*p_nt)\ a.e.\ t \in (0, n) \tag{3.8}$$

where

$$p_n' = -A^*p_n + C_1^*C_1x_n\ in\ [0, n]\ ;\ p_n(n) = 0. \tag{3.9}$$

Moreover, we have

$$\lim_{n\to\infty}\int_0^n (|C_1x_n(t) - C_1x^*(t)|_Z^2 + |u_n(t) - u^*(t)|_U^2 + |w_n(t) - w^*(t)|_W^2)dt = 0. \tag{3.10}$$

Proof It is readily seen that for every n there exists $F \in \mathcal{F}$ such that

$$\|S_F(x_0, w)\|_{L^2(0,n;Z)}^2 \leq \rho^2\|w\|_{L^2(0,n;W)}^2 + C|x_0|^2,$$

$$\forall(x_0, w) \in X \times L^2(0, n; W)$$

where $o < \rho < \gamma$ (it suffices to take in (1.7), $w = w_0$ on $(0, n)$ and $w = 0$ on (n, ∞)). Then arguing as in the proof of Lemma 1 it follows that problem (3.7) has a unique solution (u_n, w_n). Moreover, for every $w \in W_n$ the solution $\bar{u} = \Gamma_n w$ to optimal control problem

$$\inf\{K_n(u, w)\ ;\ u \in \mathcal{U}_n\}$$

satisfies the Euler-Lagrange system (see e.g. [2]), pp. 258)

$$p' = -A^*p + C_1^*C_1\bar{x}\ on\ (0, n)\ ;\ p(n) = 0. \tag{3.11}$$

$$B_2^*p(t) - \bar{u}(t) \in N_{U_0}(\bar{u}(t))\ \text{a.e. } t \in (0, n) \tag{3.12}$$

where $N_{U_0}(\bar{u})$ is the normal cone to U_0 at $\bar{u}$ and $\bar{x}$ is the corresponding solution to (1.1) with $u = \bar{u}$. Recall that (3.12) can be rewritten as

$$\bar{u}(t) = P_{U_0}(B_2^* p(t)) \text{ a.e. } t \in 0, n). \tag{3.13}$$

For $\bar{u} = u_n$, $\bar{x} = x_n$, Eqs. (3.11), (3.13) reduces to (3.9) and the first equation in (3.8) respectively.

In virtue of (3.4), which clearly remains valid for Γ_n, we have

$$\int_0^n (\gamma^2(w - w_n, w_n)_W - (C_1 x_n, C_1 y)_Z + (\Gamma_n w_n - \Gamma_n w, \Gamma_n w_n)_W) \geq 0$$

$$\forall w \in \mathcal{W}_n.$$

where

$$y' = Ay + B_2(\Gamma w - \Gamma w_n) + B_1(w - w_n) \text{ in } (0, n) \; ; \; y(0) = 0.$$

Multiplying the latter by p_n and integrating on $(0, n)$ we get by (3.11) that (without less of generality we may assume that y and p_n are differentiable)

$$\int_0^n (\gamma^2 w_n(t) + B_1^* p_n(t), w(t) - w_n(t))_W dt \geq 0 \quad \forall w \in \mathcal{W}_n.$$

Hence $\gamma^2 w_n(t) + B_1^* p_n(t) = 0$ a.e. $t \in (0, n)$ as claimed.

To prove (3.10) we note first that in virtue of inequality (1.7) we have

$$-\alpha^2 \int_0^n |w_n(t)|_W^2 dt \geq K_n(u_n, w_n) \geq \inf_{u \in \mathcal{U}_n} K_n(u, w) \geq -2^{-1} \int_0^n |w^*(t)|_W^2 dt.$$

Hence

$$\int_0^n |w_n(t)|_W^2 dt \leq C \quad \forall n.$$

Then extending w_n by 0 on $[n, +\infty)$, we may assume that as $n \to \infty$

$$w_n \to \tilde{w} \text{ weakly in } L^2(\mathbb{R}^+; W);$$

On the other hand, we have

$$\int_0^n (|C_1 x_n(t)|_Z^2 + |u_n(t)|_U^2) dt \leq \int_0^n (|C_1 x(t)|_Z^2 + |u(t)|_Z^2 + I_{U_0}(u(t)) dt \tag{3.14}$$

for all (x, u) satisfying (1.1) with $w = \tilde{w}$. In particular the latter implies that $\{u_n\}$ is bounded in $L^2(R^+; U)$ and therefore on a subsequence again denoted n, we have

$$\begin{array}{lll} u_n \to \tilde{u} & \text{weakly in} & L^2(R^+; U) \\ x_n(t) \to \tilde{x}(t) & \text{weakly in} & X \text{ on compacts intervals} \\ C_1 x_n \to C_1 \tilde{x} & \text{weakly in} & L^2(R^+; U). \end{array}$$

(We have extended u_n and x_n by zero on $[n, \infty)$). Then letting n tend to $+\infty$ in (3.14) we get

$$2^{-1} \int_0^\infty (|C_1 \tilde{x}(t)|_Z^2 + h(\tilde{u}(t)) dt \leq \inf\{K(u, \tilde{w}) \; ; \; u \in \mathcal{U}\}$$

because the function Φ is weakly lower semicontinuous. This implies that

$$\int_0^n (|C_1 x_n(t)|_Z^2 + |u_n(t)|_U^2)dt \to \int_0^\infty (|C_1\tilde{x}(t)|_Z^2 + |\tilde{u}(t)|_U^2)dt$$

and therefore

$$\int_0^n (|C_1(x_n(t) - \tilde{x}(t)|_Z^2 + |u_n(t) - \tilde{u}(t)|_U^2)dt \to 0 \tag{3.15}$$

as $n \to \infty$. On the other hand, we know that

$$-K_n(u_n, w_n) = -K_n(\Gamma w_n, w_n) \leq -K_n(\Gamma w^*, w^*)$$

and by (3.15) we infer that

$$-K(\Gamma\tilde{w}, \tilde{w}) \leq -K(\Gamma w^*, w^*).$$

Since as seen earlier, the functions $w \to -K(\Gamma w, w)$ is strictly convex and $w^* = \arg\inf K(\Gamma w, w)$ we conclude that $\tilde{w} = w^*$, $\tilde{u} = u^*$ and by (3.15) we have

$$\int_0^n |w_n(t) - w^*(t)|_W^2 dt \to 0 \text{ as } n \to \infty$$

thereby completing the proof of Lemma 2.

Define the functions, $\varphi : X \to R$, $\varphi_n : X \to R$

$$\varphi(x_0) = \sup_{w\in\mathcal{W}} \inf_{u\in\mathcal{U}} K(u,w) = K(u^*, w^*) \tag{3.16}$$

$$\varphi_n(x_0) = \sup_{w\in\mathcal{W}_n} \inf_{u\in\mathcal{U}_n} K_n(u,w) = K_n(u_n, w_n), \quad n = 1,2,\dots \tag{3.17}$$

As a supremum of the family of convex lower semicontinuous functions

$$x_0 \to \inf\{2^{-1}\int_0^\infty (|C_1 x(t)|_Z^2 + h(u(t)) - \gamma^2|w(t)|_W^2)dt \; ; \; u \in \mathcal{U}\}$$

the function φ is itself convex and lower semicontinuous. Since it is everywhere defined it is continuous on X. Similarly, the function φ_n are convex and continuous. Moreover, we have

Lemma 3 *The functions φ_n are Gâteaux differentiable and $\nabla\varphi_n(x_0) = -p_n(0)$ $\forall x_0 \in X$ where p_n is the solution to (3.9).*

Proof By (3.8), (3.9), and (1.1) it is readily seen that $-p_n(0) \in \partial\varphi_n(x_0)$, i.e.,

$$-(p_n(0), x_0 - y_0) \geq \varphi_n(x_0) - \varphi_n(y_0) \quad \forall y_0 \in X$$

(We have denoted by $\partial\varphi_n : X \to X$ the subdifferential of φ_n). Denote by $P_n : X \to X$ the operator : $P_n x_0 = -p_n(0)$. We shall prove that $P_n = \partial\varphi$. Since $P_n \subset \partial\varphi$ to this purpose it suffices to show that P_n is maximal monotone i.e., the range $R(\lambda I + T_n)$ is all of X for some $\lambda > 0$ (see e.g. [2]). To this end let $y_0 \in X$ be arbitrary but fixed. To solve the equation

$$\lambda x_0 + T_n x_0 = y_0 \tag{3.18}$$

consider the sup inf problem

$$\sup_{w\in\mathcal{W}_n} \inf_{u\in U_n, x(0)\in X} \{\int_0^n (|C_1 x(t)|_Z^2 + h(u(t)) - \gamma^2|w(t)|_W^2)dt$$
$$+\lambda|x(0)|^2 - 2(x(0), y_0) : \text{subject to } (1.1)\} \tag{3.19}$$

Clearly the inf problem has for every $w \in W_n$ a unique solution $\bar{u} = \hat{\Gamma}_n w$ given by (see [2], p. 258)

$$
\begin{aligned}
&\bar{u}(t) = P_{U_0}(B_2^*\bar{p}(t)) \text{ a.e. } t \in (0,n) \\
&\bar{p}' = -A^*\bar{p} + C_1^*C_1\bar{x} \; in \; (0,n) \\
&\bar{p}(0) = \lambda\bar{x}(0) - y_0 \; ; \; \bar{p}(n) = 0.
\end{aligned}
\tag{3.20}
$$

Now in virtue of inequality (1.7) we have for all λ sufficiently large

$$
\begin{aligned}
&\int_0^n (|C_1\bar{x}(t)|_Z^2 + h(\bar{x}(t)) - \gamma^2|w(t)|_W^2)dt + \lambda|\bar{x}(0)|^2 - 2(\bar{x}(0), y_0) \\
&\quad \leq -\alpha^2 \int_0^n |w(t)|_W^2 dt + \lambda|\bar{x}(0)|^2 - 2(\bar{x}(0), y_0) + C|\bar{x}(0)|^2 \\
&\quad \leq -\alpha_0^2 \int_0^n |w(t)|_W^2 dt \quad \forall w \in W_n,
\end{aligned}
\tag{3.21}
$$

because

$$
\begin{aligned}
&\int_0^n (|C_1\bar{x}(t)|_Z^2 + h(\bar{u}(t)))dt + \lambda|\bar{x}(0)|^2 - 2(\bar{x}(0), y_0) \\
&\quad \leq \int_0^n |C_1\tilde{x}(t)|_Z^2 dt \leq M \int_0^n |w(t)|_W^2 dt \quad \forall w \in W_n
\end{aligned}
\tag{3.22}
$$

where $\tilde{x}' = A\tilde{x} + B_1 w$; $\tilde{x}(0) = 0$. Then by (3.20) and (3.22) we see that $\lambda^2|\bar{x}(0)|^2 \leq M_1 \int_0^n |w(t)|_W^2 dt$ which implies (3.21) for λ sufficiently large.

This implies as in the proof of Lemma 1 that problem (3.19) has a unique solution $(\bar{u}_n, \bar{w}_n) \in \mathcal{U}_n \times \mathcal{W}_n$ which is given by

$$
\bar{u}_n(t) = P_{U_0}(B_2^*\bar{p}_n(t)), \quad \bar{w}_n(t) = -\gamma^{-2}B_1^*\bar{p}_n(t)
$$

where $\bar{p}_n$ is the solution to (2.20) with $\bar{x} = \bar{x}_n$.

On the other hand, it is readily seen that $(\bar{u}_n, \bar{w}_n)$ is also the solution to problem (3.17) where $x_0 = \bar{x}_n(0)$, i.e.,

$$
(\bar{u}_n, \bar{w}_n) = \arg \sup_w \inf_u \{K_n(u,w) \; ; \; x(0) = \bar{x}_n(0)\}
$$

Then by (3.20) we conclude that $x_0 = \bar{x}_n(0)$ is the solution to Eq (3.18). Hence $\partial\varphi_n = \mathbb{P}_n$ and therefore φ_n is Gâteaux differentiable because $\partial\varphi_n$ is single valued (see e.g. [2], pp. 107).

Lemma 4 *There is $C > 0$ independent of n such that*

$$
|p_n(t)| \leq C \quad \forall t \in [0,n]. \tag{3.23}
$$

Proof Consider the function $\psi_n : [0,n] \times X \to R$ defined by

$$
\psi_n(t, x_0) = \sup_w \inf_u \{2^{-1} \int_t^n (|C_1 x|_Z^2 + h(u) - \gamma^2 |w|_W^2)dt\} \tag{3.24}
$$

subject to $u \in L^2(t,n;U)$, $w \in L^2(t,n;W)$ and

$$
x' = Ax + B_2 u + B_1 w \text{ in } [t,n] \; ; \; x(t) = x_0.
$$

The function $\psi_n(t,.)$ is convex, continuous, Gâteaux differentiable and (see Lemma 3)

$$
\nabla_x \psi_n(t, x_0) = -p_n^t(t)
$$

where p_n^t is the solution to

$$(p_n^t)' = -A^* p_n^t + C_1^* C_1 x_n^t \text{ in } [t,n] \; ; \; p_n^t(n) = 0$$

and u_n^t, w_n^t, x_n^t are optimal in (3.29). Moreover, it is readily seen (the dynamic programming principle) that if $x_0 = x_n(t)$ then $u_n^t = u_n$, $w_n^t = w_n$, $x_n^t = x_n$ on $[t,n]$ where (u_n, w_n) is the solution to problem (3.7). We may therefore infer that in this case $p^n(s) = p_n(s) \quad \forall s \in [t,n]$ and

$$p_n(t) = -\nabla_x \psi_n(t, x_n(t)) \quad \forall t \in [0,n] \tag{3.25}$$

where p_n is the solution to eq (3.9).

On the other hand, since $\psi_n(t,.)$ is convex we have

$$\psi_n(t, x_n(t)) \leq \psi_n(t, x_n(t) + \beta\theta) - \beta(\nabla_x \psi_n(t, x_n(t)), \theta)$$

for all $\theta \in X$ and $\beta > 0$. This yields for $|\theta| = 1$

$$|\nabla \psi_n(t, x_n(t))| \leq \beta^{-1} \psi_n(t, x_n(t) + \beta\theta) + C \tag{3.26}$$

(We shall denote by C several positive constants independent of n). On the other hand, by Lemma 2 we know that

$$\lim_{n\to\infty} \psi_n(t,x) = \sup_w \inf_u \left\{ \frac{1}{2} \int_t^\infty (|C_1 x|_Z^2 + h(u) - \gamma^2 |w|_W^2) dt \right.$$

$$\left. u \in L^2(t,\infty;U),\ w \in L^2(t,\infty;W) \; ; \; x' = Ax + B_2 u + B_1 w \text{ in } (t,\infty)\ x(t) = x_0 \right\} = \varphi(x_0).$$

Then by (3.25) and (3.26) we get

$$\lim_{n\to\infty} \sup |p_n(t)| \leq \beta^{-1}(\varphi(x^*(t) + \beta\theta) - \varphi(x^*(t))) \quad \forall t \geq 0 \tag{3.27}$$

where $|\theta| \leq 1$. Since the function φ is locally bounded the latter implies (3.23) as desired.

Lemma 5 *The solution (u^*, w^*) to problem (3.2) is given*

$$u^*(t) = P_{U_0}(B_2^* p(t)) \; ; \; w^*(t) = -\gamma^{-2} B_1^* p(t) \quad \forall t \geq 0 \tag{3.28}$$

where $p \in C(R^+; X)$ is a mild solution to

$$p' - A^* p + C_1^* C_1 x^* \text{ in } R^+ \; ; \; \lim_{t\to\infty} p(t) = 0 \tag{3.29}$$

i.e.,

$$p(t) = e^{A^*(T-t)} p(T) + \int_t^T e^{A^*(s-t)} C_1^* C_1 x^*(s) ds, \tag{3.30}$$

for all $0 \leq t \leq T < \infty$.

Proof . In virtue of estimate (3.23) we may assume that $p_n \to p$ weak star in $L^\infty(R^+; X)$. Clearly p is a weak solution to eq. (3.29) and since the map P_{U_0} is weakly-strongly closed in every $L^2(0,T;U)$ we may let n tend to $+\infty$ in (3.8) to get (3.28). To show that $\lim_{t\to\infty} p(t) = 0$ we come back to inequality (3.27) and note that it implies that

$$|p(t)| \leq \beta^{-1}(\varphi(x^*(t) + \beta\theta) - \varphi(x^*(t))) \quad \forall t \geq 0 \tag{3.31}$$

where $|\theta| = 1$. On the other hand, it is readily seen that (u^*, w^*) is the solution to problem

$$\begin{gathered}\sup_{w\in L^2(t,\infty;W)} \inf_{u\in L^2(t,\infty;W)} \{\int_t^\infty (|C_1 x|_Z^2 + h(u) - \gamma^2 |w|_W^2) ds \\ x' = Ax + B_2 u + B_1 w \text{ in } (t,\infty)\ ;\ x(t) = x^*(t)\}\end{gathered} \tag{3.32}$$

and therefore

$$\varphi(x^*(t)) = 2^{-1}\int_t^\infty (|C_1 x^*(s)|_Z^2 + h(u^*(s)) - \gamma^2 |w^*(s)|_W^2 ds), \quad t \geq 0. \tag{3.33}$$

Note also that by the detectability assumption (1.3) it follows by standard arguments involving Datco's theorem that,

$$x^* \in L^2(R^+; X)\ ;\ \lim_{t\to\infty} x^*(t) = 0 \tag{3.34}$$

Thus letting t tend to $+\infty$ in (3.31) we get

$$\lim_{t\to\infty} \sup |p(t)| \leq \beta^{-1} \sup\{\varphi(\beta\theta); |\theta| \leq 1\} \quad \forall \beta > 0 \tag{3.35}$$

Since by inequality (1.5) and (3.16) we have

$$0 \leq \inf_{u\in\mathcal{U}} \int_0^\infty (|C_1 x|_Z^2 + h(u)) dt \leq \varphi(x_0) \leq C|x_0|^2 \quad \forall x_0 \in X$$

letting $\beta \to 0$ in (3.35) see that $\lim_{t\to\infty} p(t) = 0$ as claimed.

Lemma 6 *The function φ in Gâteaux differentiable on X and*

$$\nabla\varphi(x_0) = -p(0) \tag{3.36}$$

where p is the solution to (3.29).

Proof. Let p be any mild solution to (3.29) such that $p(\infty) = \lim_{t\to\infty} p(t) = 0$. By a little calculation involving systems (1.1) and (3.29) it follows that

$$\varphi(x_0) - \varphi(y_0) \leq -(p(0), x_0 - y_0) \quad \forall y_0 \in X$$

i.e., $-p(0) \in \partial\varphi(x_0)$ ($\partial\varphi$ is the subdifferential of φ). In particular this implies that the mild solution p to (3.29) is unique.

Indeed if p_1, p_2 are two solutions to (3.29) then so is $p_1 + \lambda(p_1 - p_2)$ for all $\lambda \geq$ and $p_1(0) + \lambda(p_1(0) - p_2(0)) \in -\partial\varphi(x_0)$ for all $\lambda \geq 0$. Since $\partial\varphi(x_0)$ is bounded (see e.g. [2]) we arrived to a contradiction.

Hence the operator $P : X \to X$ defined by $Px_0 = -p(0)$ is single valued and $Px_0 \in \partial\varphi(x)$ $\forall x_0 \in X$. To prove that φ is Gâteaux differentiable it suffices to show that P is m-accretive in X i.e., $R(\lambda I + P) = X$ for some $\lambda > 0$. Let $y_0 \in X$ be arbitrary but fixed. The equation

$$\lambda x_0 + Px_0 = y_0 \tag{3.37}$$

reduces to the sup inf problem

$$\sup_{w\in\mathcal{W}} \inf_{u\in\mathcal{U}} \{\int_0^\infty (|C_1x|_Z^2 + h(u) - \gamma^2|w|_W^2)dt + \lambda|x(0)|^2 - 2(x(0), y_0)\} \tag{3.38}$$

where (x, u, w) are subject to system (1.1).

Arguing as in the proof of lemma 1, it follows that problem (3.38) has a unique solution $(\tilde{x}, \tilde{u}, \tilde{w})$. Since $(\tilde{u}, \tilde{w})$ is also a solution to problem (3.2) where $x_0 = \tilde{x}(0)$ we have by lemma 5 that

$$\tilde{u}(t) = P_{U_0}(B_2^*\tilde{p}(t)),\ \tilde{w}(t) = -\gamma^{-2}B_1^*\tilde{p}(t) \quad \forall t \geq 0 \tag{3.39}$$

where

$$\tilde{p}' = -A^*p + C_1^*C_1\tilde{x} \text{ in } R^+,\ p(\infty) = 0.$$

Moreover, as seen above we have

$$\tilde{p}(t) = w - \lim_{n\to\infty} \tilde{p}_n(t) \quad \forall t \geq 0$$

where $\tilde{p}_n$ is the solution to (3.20), i.e.,

$$\begin{aligned} &\tilde{p}_n^1 = -A^*\tilde{p}_n + C_1^*C_1\tilde{x}_n \text{ in } [0, n] \\ &\tilde{p}_n(n) = 0,\ \tilde{p}_n(0) = \lambda\tilde{x}_n(0) - y_0 \end{aligned} \tag{3.40}$$

and $(\bar{x}_n, \bar{u}_n, \bar{w}_n)$ is the solution to problem (3.19). Since $(\bar{u}_n, \bar{w}_n) \to (\tilde{u}, \tilde{w})$ strongly in $L^2(R^+; U) \times L^2(R^+; W)$ (Lemma (3.20)) and $\bar{x}_n(t) \to \tilde{x}(t)$ uniformlly on compact intervals we infer that $\tilde{p}$ is a solution to problem (3.39) and $\tilde{p}(0) = \lambda\tilde{x}(0) - y_0$. Hence $x_0 = \tilde{x}(0)$ is the solution to equation (3.37).

Proof of Theorem 1 (continued). By (3.33) and (3.28) we see that

$$\frac{d}{dt}\varphi(x^*(t)) = -2^{-1}((|C_1x^*(t)|_Z^2 - |P_{U_0}(B_2^*p(t))|_U^2 - \gamma^{-2}|B_1^*p(t)|_W^2) \quad \forall t \geq 0 \tag{3.41}$$

Note also that for $x_0 \in D(A)$, we have

$$\frac{d}{dt}\varphi(x^*(t)) \mid_{t=0} = (\nabla\varphi(x_0), Ax_0 + B_2u^*(0) + B_1w^*(0)) \tag{3.42}$$

Indeed since φ is convex, we have

$$\varphi(x_0) - \varphi(x^*(t)) \leq (\nabla\varphi(x_0), x_0 - x^*(t)) = (\nabla\varphi(x_0^*), x_0 - e^{At}x_0 + \int_0^t e^{A(t-s)}(B_2u^*(s) + B_1w^*(s))ds)$$

and this yields

$$\frac{d}{dt}\varphi(x^*(t)) \mid_{t=0} \geq (\nabla\varphi(x_0), Ax_0 + B_2u^*(0) + B_1w^*(0)).$$

The opposite inequality follows similarly using the fact that $\nabla\varphi$ is demicontinuous, i.e., strongly weakly continuous.

On the other hand, since as seen earlier (u^*, w^*) is the solution to problem (3.2) on (t, ∞) with the initial value condition $x = x^*(t)$, it follows by lemma 6 that

$$p(t) = -\nabla\varphi(x^*(t)) \quad \forall t \geq 0. \tag{3.43}$$

Combining (3.41), (3.42), (3.43) it follows that φ satisfies the Hamilton-Jacobi equation (2.2). Finally by (3.28) and (3.43) it is clear that $x = x^*$ is a mild solution to the Cauchy problem (2.3). Moreover, as seen earlier, it follows by assumption (1.3) that $x^* \in L^2(R^+; X)$ and therefore $\lim_{t\to\infty} x^*(t) = 0$. This completes the proof of the first part of Theorem 1.

Let us assume now that : either $B_2 \in L(U, X)$ or e^{At} are compact for all $t > 0$ and that eq. (2.2) has a convex solution φ which is Gâteaux differentiable and $0 \leq \varphi(x_0) \leq C|x_0|^2 \quad \forall x_0 \in X$. We must prove that the feedback $F = P_{U_0}(-B_2^*\nabla\varphi)$ belongs to $\mathcal{F}$ and

$$\|S_F(x_0, w)\|^2_{L^2(R^+;U)} \leq \gamma^2\|w\|^2_{L^2(R^+;W)} + C|x_0|^2 \tag{3.44}$$

for all $(x_0, w) \in X \times L^2(R^+; W)$.

Consider the Cauchy problem

$$x' = Ax + B_2P_{U_0}(-B_2^*\nabla\varphi(x)) + B_1w \; ; \; x(0) = x_0. \tag{3.45}$$

Since $\nabla\varphi$ is demicontinuous (i.e., strongly - weakly continuous) the feedback F is admissible and upper - semicontinuous and so by compacity assumption, the Cauchy problem (3.45) has least one local solution $x = x(t, x_0, w)$ see e.g. [1], [7]) defined on some interval $[0, T_0)$. Without any loss of generality we may assume that $x = x(t, x_0, w)$ is differentiable on $[0, T_0)$. Multiplying (3.45) by $\nabla\varphi(x(t))$ and using eq. (2.2) we get after some calculation that

$$\begin{aligned}
\frac{d}{dt}\varphi(x(t)) &= (Ax(t) + B_2P_{U_0}(-B_2^*\nabla\varphi(x(t)), \nabla\varphi(x(t)) + (w(t), B_1^*\nabla\varphi(x(t)))_W \\
&= -2^{-1}|C_1x(t)|^2_Z + \gamma^{-2}|B_1^*\nabla\varphi(x(t))|^2_W - |P_{U_0}(-B_2^*\nabla\varphi(x(t))|^2_U) \\
&\quad +(w(t), B_1^*\nabla\varphi(x(t)))_W \quad \forall t \in [0, T_0).
\end{aligned}$$

Hence

$$\begin{aligned}
&\varphi(x(t)) + 2^{-1}\int_0^t(|C_1x(s)|^2_Z + |P_{U_0}(-B_2^*\nabla\varphi(x(s)))|^2_U + \gamma^{-2}|B_1^*\nabla\varphi(x(s))|^2_W)ds \\
&= \varphi(x_0) + \int_0^t(w(s), B_1^*\nabla\varphi(x(s)))_W ds \quad \forall t \in [0, T_0).
\end{aligned} \tag{3.46}$$

Finally,

$$\begin{aligned}
&\int_0^t(|C_1x(s)|^2_Z + |P_{U_0}(-B_2^*\nabla\varphi(x(s))|^2_U + |B_1^*\nabla\varphi(x(s))|^2_W)ds \\
&\leq C \quad \forall t \in [0, T_0).
\end{aligned} \tag{3.47}$$

On the other hand, we may write equation (3.45) as

$$\begin{aligned}
&x' = (A + KC_1)x - KC_1x + B_2P_{U_0}(-B_2^*\nabla\varphi(x)) + B_1w \\
&x(0) = x_0
\end{aligned} \tag{3.48}$$

where $e^{(A+KC_1)t}$ is exponentially stable. Then by estimate (3.47) we see that

$$\int_0^t |C_1x(s)|_Z^2 ds + |x(t)| \le C \quad \forall t \in [0, T_0)$$

and therefore $x = x(t, x_0, w)$ can be extended as mild solution to (3.45) on the whole of R^+. Moreover, it follows by (3.47) and (3.48) that $x(., x_0, w) \in L^2(R^+; W)$ and $\lim_{t\to\infty} x(t, x_0, w) = 0$. Since in the previous argument $B_1 w$ can be replaced by any L^2 function f we conclude that $F \in \mathcal{F}$.

Now letting t tend to $+\infty$ in inequality (3.46) we get

$$\int_0^\infty (|C_1x(t)|_Z^2 + |P_{U_0}(-B_2^*\nabla\varphi(x(t))|_U^2)dt$$
$$\le \varphi(x_0) + \gamma^2 \int_0^\infty |w(t)|_W^2 dt \quad \forall w \in L^2(R^+; W)$$

for any mild solution to the Cauchy problem (3.45). The latter implies (3.44) as desired. The proof of Theorem 1 is complete.

4 Final remarks and examples

It is clear from the previous proof that the compacity hypothesis has been imposed in order to assure that the closed loop system (1.1) with the feedback $u = Fx = P_{U_0}(-B_2^*\nabla\varphi(x))$ has at least one local solution for every $x_0 \in X$ and $w \in L^2(R^+; W)$.

This assumption can be dispensed with if eq. (2.2) has a more regular solution. For instance if it has a solution $\varphi \in C^1(X)$ with locally Lipschitzian gradient $\nabla\varphi$ then by the above argument it follows that the feedback $F = P_{U_0}(-B_2^*\nabla\varphi)$ belongs to $\mathcal{F}$ and guarantees inequality (1.7) with $0 < \rho < \gamma$.

The next remark reffers to definition of the H_∞-suboptimal control F. If inequality (1.7) is weakened to

$$\begin{aligned} &|S_F(x_0, w)|^2 \le \rho^2 \|w\|^2_{L^2(R^+;W)} + C|x_0|^{2m} \\ &\forall (x_0, w) \in X \times L^2(R^+; W) \end{aligned} \tag{4.1}$$

then Theorem 1 remains valid if condition (2.1) is replaced by

$$0 \le \varphi(x) \le C|x|^{2m} \quad \forall x \in X \tag{4.2}$$

with $m \ge 1$. This follows by an easy modification of the proof of Lemma 3. Namely one replace equation (3.18) by

$$\lambda x_0 |x_0|^{m-2} + T_n x_0 = y_0$$

i.e., one proves that $R(T_n + \lambda |x|^{m-2}x) = X$ which clearly implies that T_n is maximal monotone.

Theorem 1 extends by a slight modification of the proof to H_∞-constrained control problems with the controlled output $z = C_1x + D_{11}w + D_{12}u$ where $\gamma^2 I - D_{11}^* D_{11}$ is positive definite.

Example 1 . Consider the system

$$x' = -x + u + w \text{ in } R^+; \quad z = \{x, u\}, \quad u(t) \ge 0, \quad \forall t \ge 0. \tag{4.3}$$

Here $X = U = W = R$, $Z = R \times R$ and $U_0 = \{u \in R; u \geq 0\}$. Equation (2.2) has therefore in this case the following form

$$2x\varphi'(x) - |(\varphi'(x))^-|^2 - (\gamma^2)^{-1}|\varphi'(x)|^2 - x^2 = 0 \tag{4.4}$$

By a little calculation we see that for $\gamma > 1$ this equation has a unique convex solution φ satisfying (2.1) and which is given by

$$\varphi(x) = \begin{cases} 2^{-1}(\gamma^2 + \gamma(\gamma^2 - 1)^{1/2})x^2 & \text{for} \quad x \geq 0 \\ 2^{-1}(\gamma^2 - 1)^{-1/2}((2\gamma^2 - 1)^{1/2} - \gamma)x^2 & \text{for} \quad x < 0 \end{cases} \tag{4.5}$$

and the suboptimal H_∞ feedback control F is given by

$$u = Fx = \begin{cases} 0 \quad \text{for} \quad x \geq 0 \\ -(\gamma^2 - 1)^{-1/2}((2\gamma^2 - 1)^{1/2} - \gamma)x & \text{for} \quad x < 0. \end{cases} \tag{4.6}$$

Example 2. Consider the intput-output system

$$\begin{aligned} & y_{tt} - \Delta y = \sum_{i=1}^{m} a_i(x)u_i + w \text{ in } \Omega \times R^+ \\ & y = 0 \text{ in } \partial\Omega \times R^+ \\ & z = \{y, y_t, u\} \in H_0^1(\Omega) \times L^2(\Omega) \times R^m \\ & u(t) = (u_1(t), \ldots, u_m(t)) \in U_0 = \{u \in R^m; |u_i| \leq 1, \quad i = 1, \ldots, m\} \end{aligned} \tag{4.7}$$

Here $a_i \in L^\infty(\Omega)$ are given and $w \in L^2(R^+; L^2(\Omega))$ (Ω is a bounded, open subset of $\mathbb{R}^N$ with a sufficiently smooth boundary). In this case $X = H_0^1(\Omega) \times L^2(\Omega)$, $U = R^m$, $W = L^2(\Omega)$,

$$A = \begin{Vmatrix} 0 & I \\ -\Delta & 0 \end{Vmatrix}, B_2 u = \begin{Vmatrix} 0 \\ \sum_{i=1}^{m} a_i u_i \end{Vmatrix}, B_1 w = \begin{Vmatrix} 0 \\ w \end{Vmatrix}.$$

Clearly assumptions of Theorem 1 are satisfied and equation (2.2) has the following form

$$\begin{aligned} & 2\int_\Omega (y_2\varphi_{y_1}(y_1, y_2) - \Delta y_1 \varphi_{y_2}(y_1, y_2))dx \\ & + \sum_{i=1}^{m}((\int_\Omega a_i\varphi_{y_2}(y_1, y_2)dx)^2 + ((1 - |\int_\Omega a_i\varphi_{y_2}(y_1, y_2)dx|)^+)^2) \\ & + \int_\Omega \varphi_{y_2}^2(y_1, y_2)dx + \int_\Omega (|\nabla y|^2 + |y_t|^2)dx = 0, \\ & \forall (y_1, y_2) \in (H_0^1(\Omega) \cap H^2(\Omega)) \times H_0^1(\Omega). \end{aligned} \tag{4.8}$$

whilist the corresponding H_∞-suboptimal feedback F is given by

$$u_i = \begin{cases} -\int_\Omega a_i(x)\varphi_{y_2}(y_1, y_2)dx \text{ if } |\int_\Omega a_i(x)\varphi_{y_2}(y_1, y_2)dx| \leq 1 \\ -sign \int_\Omega a_i(x)\varphi_{y_2}(y_1, y_2)dx \text{ if } |\int_\Omega a_i(x)\varphi_{y_2}(y_1, y_2)dx| > 1 \end{cases} \tag{4.9}$$

for $i = 1, \ldots, m$.

We mention also that Theorem 1 is applicable to intput-output systems defined by the delay equation

$$y' = A_0 y + A_1 y(t - h) + B_2 u + B_1 w \text{ in } R^+$$

where $A, A_1 \in L(R^N, R^N)$, $B_2 \in L(R^m, R^N)$, $B_1 \in L(R^P, R^N)$.

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