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NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM TO OBTAIN A DISCRETE STEADY-STATE TWO-PHASE STEFAN PROBLEM

Domingo Alberto TARZIA

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NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM TO OBTAIN A DISCRETE STEADY-STATE TWO-PHASE STEFAN PROBLEM

ANALYSE NUMERIQUE DU FLUX DE CHALEUR DANS UN PROBLEME ELLIPTIQUE MIXTE POUR OBTENIR UN CAS STATIONNAIRE DISCRET DU PROBLEME DE STEFAN A DEUX PHASES

Domingo Alberto TARZIA (*)

Abstract

We consider a material $\Omega \subset \mathbb{R}^n$ which occupies a convex polygonal bounded domain, with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We assume, without loss of generality, that the melting temperature is 0°C . We apply a temperature $b = \text{Const.} > 0$ on Γ_1 and a heat flux $q = \text{Const.} > 0$ on Γ_2 . We consider a steady-state heat conduction problem in Ω .

We consider a regular triangulation of the domain Ω with Lagrange triangles of type 1. We study sufficient (and/or necessary) conditions for the heat flux q on Γ_2 to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω .

Résumé

On considère un matériel $\Omega \subset \mathbb{R}^n$, un domaine polygonal borné et convexe avec une frontière régulière $\Gamma = \Gamma_1 \cup \Gamma_2$ avec $\text{mes}(\Gamma_1) = |\Gamma_1| > 0$ et $|\Gamma_2| > 0$. On suppose, sans perte de généralité, que la température de changement de phase est 0°C . On considère une température $b = \text{Const.} > 0$ sur Γ_1 et un flux de chaleur $q = \text{Const.} > 0$ sur Γ_2 . On considère le cas stationnaire du problème de conduction de la chaleur dans Ω .

On considère une triangulation régulier de Ω avec des triangles de Lagrange de type 1. On étudie des conditions suffisantes (et/ou nécessaires) pour le flux de chaleur q sur Γ_2 pour obtenir un changement de phase (cas stationnaire discret d'un problème de Stefan à deux phases) dans le domaine discrète correspondante, c'est-à-dire une température discrète de signe non-constante dans Ω .

Key words : Steady-state Stefan problem, finite element method, mixed elliptic problem, numerical analysis, variational inequalities, error bounds.

Mots Clés : Problème de Stefan stationnaire, méthode d'éléments finis, problème elliptique mixte, analyse numérique, inéquations variationnelles, estimation de l'erreur.

AMS Subject Classification : 35R35, 35J85, 65N15, 65N30.

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I. - INTRODUCTION.

We consider a material $\Omega \subset \mathbb{R}^n$ which occupies a convex polygonal bounded domain, with a regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We assume, without loss of generality, that the melting temperature is 0°C . We apply a temperature $b = \text{Const.} > 0$ on Γ_1 and a heat flux $q = \text{Const.} > 0$ on Γ_2 . We consider a steady-state heat conduction problem in Ω . Following [7], we study the temperature $\theta = \theta(x)$ for $x \in \Omega$. The set Ω can be written by

$$(1) \quad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} .$$

where

$$(2) \quad \begin{aligned} \Omega_1 &= \left\{ x \in \Omega / \theta(x) < 0 \right\} \text{ (liquid phase) ,} \\ \Omega_2 &= \left\{ x \in \Omega / \theta(x) > 0 \right\} \text{ (solid phase) ,} \\ \mathcal{L} &= \left\{ x \in \Omega / \theta(x) = 0 \right\} \text{ (free boundary) ,} \end{aligned}$$

are the liquid phase, the solid phase and the free boundary which separates them respectively. The temperature θ can be represented in Ω in the following way

$$(3) \quad \begin{aligned} \theta_1(x) &< 0, & x \in \Omega_1, \\ \theta(x) &= 0, & x \in \mathcal{L}, \\ \theta_2(x) &> 0, & x \in \Omega_2, \end{aligned}$$

and satisfies the following conditions :

$$(4) \quad \begin{aligned} \Delta \theta_i &= 0 \quad \text{in } \Omega_i \quad (i = 1, 2) , \\ \theta_1 = \theta_2 = 0, \quad k_1 \frac{\partial \theta_1}{\partial n} &= k_2 \frac{\partial \theta_2}{\partial n} \quad \text{on } \mathcal{L} , \\ \theta_2 |_{\Gamma_1} &= b > 0 , \\ -k_2 \frac{\partial \theta_2}{\partial n} |_{\Gamma_2} &= q \quad \text{if } \theta |_{\Gamma_2} > 0 , \\ -k_1 \frac{\partial \theta_1}{\partial n} |_{\Gamma_2} &= q \quad \text{if } \theta |_{\Gamma_2} < 0 , \end{aligned}$$

where $k_i > 0$ is the thermal conductivity in Ω_i ($i=1$: solid phase, $i=2$: liquid phase). If we introduce the new unknown function [2, 7]

$$(5) \quad u = k_2 \theta^+ - k_1 \theta^- \quad \left(\theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \right) \quad \text{in } \Omega ,$$

where θ^+ and θ^- represent the positive part and the negative part of the function θ respectively, then problem (4) is transformed into the following mixed elliptic problem

$$(6) \quad \begin{aligned} \Delta u &= 0 \quad \text{dans } \Omega , \\ u |_{\Gamma_1} &= B , \\ -\frac{\partial u}{\partial n} |_{\Gamma_2} &= q , \end{aligned}$$

whose variational formulation is given by

$$(7) \quad \begin{aligned} a(u, v-u) &= L(v-u) \quad , \quad \forall v \in K , \\ u &\in K , \end{aligned}$$

where

$$(8) \quad \begin{aligned} V &= H^1(\Omega) \quad , \quad B = k_2 b > 0 \quad \text{on } \Gamma_1 , \\ K &= \left\{ v \in V / v |_{\Gamma_1} = B \right\} , \quad V_0 = \left\{ v \in V / v |_{\Gamma_1} = 0 \right\} , \\ a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad , \quad L(v) = L_q(v) = - \int_{\Gamma_2} q v \, d\gamma . \end{aligned}$$

Moreover, the solution of (7) is characterized by the following minimum problem [4]

$$(9) \quad \begin{aligned} J(u) &\leq J(v) \quad , \quad \forall v \in K , \\ u &\in K , \end{aligned}$$

where

$$(10) \quad J(v) = J_q(v) = \frac{1}{2} a(v, v) - L_q(v) = \frac{1}{2} a(v, v) + \int_{\Gamma_2} q v \, d\gamma .$$

We can define the real fonction $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ in the following way

$$(11) \quad f(q) = J_q(u(q)) = \frac{1}{2} a(u(q), u(q)) + q \int_{\Gamma_2} u_q \, d\gamma ,$$

where $u = u(q)$ is the unique solution of the variational equality (7) for each heat flux $q > 0$ (for a given $B > 0$).

For the continuous problem (6) or (7), a sufficient condition to have a steady-state two-phase Stefan problem (i.e. the solution $u(q)$ of (7) is a function of non-constant sign in Ω) was obtained in [8, 9].

Theorem 1 .— (i) Function f is derivable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

$$(12) \quad f'(q) = \int_{\Gamma_2} u(q) \, d\gamma .$$

(ii) There exists a geometrical constant $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$ such that

$$(13) \quad a(u(q), u(q)) = C q^2 , \quad f(q) = -\frac{C}{2} q^2 + B |\Gamma_2| q , \quad \forall q > 0 .$$

Moreover, the constant $C > 0$ is given by

$$(14) \quad C = a(u_3, u_3) = \int_{\Gamma_2} u_3 \, d\gamma > 0 ,$$

where $u_3 \in V_0$ ($u(q) = B - q u_3$ in Ω) is the unique solution of the following mixed elliptic problem

$$(15) \quad \begin{aligned} \Delta u_3 &= 0 \text{ in } \Omega , \\ u_3 |_{\Gamma_1} &= 0 , \quad \frac{\partial u_3}{\partial n} |_{\Gamma_2} = 1 , \end{aligned}$$

whose variational formulation is given by

$$(16) \quad \begin{aligned} a(u_3, v) &= \int_{\Gamma_2} v \, d\gamma , \quad \forall v \in V_0 , \\ u_3 &\in V_0 . \end{aligned}$$

(iii) If

$$(17) \quad q > q_0(B)$$

then (6) or (7) represents a steady-state two-phase Stefan problem (i.e. the solution $u(q)$ of problem (7) is a function of non-constant sign in Ω), where $q_0 = q_0(B)$ is given by

$$(18) \quad q_0(B) = \frac{B |\Gamma_2|}{C} , \quad \forall B > 0 .$$

(iv) If the function $u(q)$ is constant over Γ_2 , then the sufficient condition (given by (17)) is also necessary.

Proof .— See [8, 9].

Now, we consider τ_h , a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0 , where $h > 0$ is a parameter which goes to zero. We can take h equals to the longest side of the triangles $T \in \tau_h$ and we can approximate V_0 by [1] :

$$(19) \quad V_h = \left\{ v_h \in C^0(\bar{\Omega}) / v_h|_T \in P_1(T), \forall T \in \tau_h, v_h|_{\Gamma_1} = 0 \right\},$$

where P_1 is the set of the polynomials of degree less or equals than 1. Let π_h be the corresponding linear interpolation operator. Then, we can consider that there exists a constant $C_0 > 0$ (independent of the parameter h) such that

$$(20) \quad \|v - \pi_h v\|_V \leq C_0 h^{r-1} \|v\|_{r,\Omega}, \quad \forall v \in H^r(\Omega), \quad \text{with } 1 < r \leq 2.$$

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (7), given by :

$$(21) \quad \begin{aligned} a(u_h, v_h) &= L(v_h), \quad \forall v_h \in V_h, \\ u_h &\in K_h = B + V_h, \end{aligned}$$

and we can obtain the following results.

Lema 2 .— We have

$$(22) \quad \lim_{h \rightarrow 0^+} \|u_h - u\|_V = 0,$$

where u is the unique solution of the variational equality (7).

Proof .— Owing to $\text{meas}(\Gamma_1) > 0$, we have that the bilinear form a is coercitivity over V_0 , that is [4] :

$$(23) \quad \exists \alpha > 0 / a(v, v) = \|v\|_{V_0}^2 \geq \alpha \|v\|_V^2, \quad \forall v \in V_0,$$

and therefore $\|\cdot\|_{V_0}$ and $\|\cdot\|_V$ are two equivalent norms in V_0 . We follow a similar method

developed in [1].

Corollary 3 .- If we define

$$(24) \quad \theta_h = \frac{1}{k_2} u_h^+ - \frac{1}{k_1} u_h^- \in V, \quad \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \in V$$

then we have

$$(25) \quad \lim_{h \rightarrow 0^+} \|\theta_h - \theta\|_H = 0,$$

where $H = L^2(\Omega)$.

Proof .- If we consider the scalar product in H , defined by

$$(26) \quad (u, v) = \int_{\Omega} u v \, dx,$$

then, we deduce

$$(27) \quad \begin{aligned} \|u_h - u\|_H^2 &= \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2 + 2(u_h^+, u^-) + \\ &+ 2(u_h^+, u^+) \geq \|u_h^+ - u^+\|_H^2 + \|u_h^- - u^-\|_H^2, \end{aligned}$$

that is

$$(28) \quad \text{Max} (\|u_h^+ - u^+\|, \|u_h^- - u^-\|) \leq \|u_h - u\|_H.$$

From (24) we obtain :

$$(29) \quad \begin{aligned} \|\theta_h - \theta\|_H &\leq \frac{1}{k_2} \|u_h^+ - u^+\|_H + \frac{1}{k_1} \|u_h^- - u^-\|_H \leq \\ &\leq \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \|u_h - u\|_H, \end{aligned}$$

i.e. (25).

The goal of this paper is to consider the numerical analysis of the inequality (17). We study sufficient (and/or necessary) conditions for the constant heat flux q on Γ_2 to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω . We obtain that :

i) there exists a constant $C_h > 0$ (which depends only of the geometry of the domain Ω for each $h > 0$ and it is characterized by a variational problem) such that if $q > q_{0h}(B) = B|\Gamma_2|/C_h$ then the steady-state discretized problem presents two phases.

ii) we have the estimations $C_h < C$ and $q_0(B) < q_{0h}(B)$ where C and $q_0(B)$ have been obtained for the continuous problem by (14) and (18) respectively [9].

iii) we deduce an error bounds for $C - C_h$ and $q_{0h}(B) - q_0(B)$ as a function of the parameter h .

In other words, we obtain for the mixed elliptic discretized problem, defined by u_h , analogous conditions to the ones obtained for the corresponding continuous problem [9], defined by u .

II. - INEQUALITY FOR THE HEAT FLUX IN THE DISCRETIZED PROBLEM.

For each $q > 0$ we consider the functions $u(q) \in K$ and $u_h(q) \in K_h$, as the unique solution of the variational equalities (7) (continuous problem) and (21) (discrete problem) respectively. We define the real function $f_h : \mathbb{R}^+ \rightarrow \mathbb{R}$, for each $h > 0$, in the following way

$$(30) \quad f_h(q) = J_q(u_h(q)) = \frac{1}{2} a(u_h(q), u_h(q)) + q \int_{\Gamma_2} u_h(q) d\gamma, \quad q > 0.$$

Therefore, we obtain the following properties :

Theorem 4 .- (i) If $u_i = u_h(q_i)$ is the solution of (21) for $q_i > 0$ ($i = 1, 2$), then we have the following equalities :

$$(31) \quad a(u_2 - u_1, u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1) d\gamma,$$

$$(32) \quad a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2) d\gamma.$$

(ii) For all real numbers $q > 0$ and Δ such that $(q + \Delta) > 0$, we obtain the following estimations :

$$(33) \quad \left\| \frac{1}{\Delta} [u_h(q) - u_h(q + \Delta)] \right\|_V \leq D_1 = \frac{|\gamma_0|}{\alpha} |\Gamma_2|^{1/2},$$

$$(34) \quad \left\| \frac{1}{\Delta} [u_h(q) - u_h(q + \Delta)] \right\|_{L^2(\Gamma_2)} \leq D_2 = D_1 |\gamma_0|,$$

where γ_0 is the linear and continuous trace operator, defined over V . Moreover, the function

$$(35) \quad q > 0 \rightarrow \int_{\Gamma_2} u_h(q) d\gamma \in \mathbb{R},$$

is a continuous and strictly decreasing function.

(iii) Function $f_h = f_h(q)$ is derivable. Moreover, f_h' is a continuous and strictly decreasing function and given by the following expression

$$(36) \quad f_h'(q) = \int_{\Gamma_2} u_h(q) \, d\gamma ,$$

Proof .— (i) If we take $v = u_2 - u_1 \in V_h$ in the variational equality corresponding to u_1 and $v = u_1 - u_2 \in V_h$ in the one corresponding to u_2 , and we add up and subtract both equalities, then we obtain (31) and (32) respectively.

(ii) Taking into account (23), the Cauchy-Schwarz inequality and the continuity of the operator γ_0 we deduce (33). From (33) and the continuity of γ_0 we have (34). Therefore we have (35) because

$$(37) \quad \left| \int_{\Gamma_2} [u_h(q) - u_h(q + \Delta)] \, d\gamma \right| \leq D_2 |\Gamma_2|^{1/2} \Delta .$$

Moreover, the monotony property is a consequence of (31).

(iii) From (30) and elementary computations, we deduce

$$(38) \quad \frac{1}{\Delta} [f_h(q + \Delta) - f_h(q)] = \frac{1}{2} \int_{\Gamma_2} [u_h(q) + u_h(q + \Delta)] \, d\gamma ,$$

that is (36), by using (35).

Lema 5 .— We have the following expressions :

$$(39) \quad a(u_h(q), u_h(q)) = q B |\Gamma_2| - q \int_{\Gamma_2} u_h(q) \, d\gamma ,$$

$$(40) \quad f_h(q) = q B |\Gamma_2| - \frac{1}{2} a(u_h(q), u_h(q)) ,$$

$$(41) \quad f_h(q) = \frac{q B |\Gamma_2|}{2} + \frac{q}{2} \int_{\Gamma_2} [u_h(q) \, d\gamma ,$$

$$(42) \quad q f_h''(q) = f_h'(q) - B |\Gamma_2| ,$$

$$(43) \quad \frac{d}{dq} [a(u_h(q), u_h(q))] = \frac{2}{q} a(u_h(q), u_h(q)) = 2 B |\Gamma_2| - 2 f_h'(q) .$$

Proof .— Taking $v_h = B - u_h(q) \in V_h$ in (21) we obtain (31). We deduce (40) and (41) by using (30) and (39). From (36), (39)–(41) we obtain (42) and (43) by derivating with respect to variable q .

Theorem 6 .- (i) There exists a constant $C_h > 0$ such that

$$(44) \quad f_h(q) = q B |\Gamma_2| - \frac{1}{2} C_h q^2, \quad \forall q > 0,$$

$$(45) \quad a(u_h(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) If $q > q_{0h}(B)$, then problem (21) represents a discretized steady-state two-phase Stefan problem (i.e. $u_h(q)$ is a function of non-constant sign in Ω), where

$$(46) \quad q_{0h}(B) = \frac{B |\Gamma_2|}{C_h}.$$

(iii) The constant $C_h > 0$ can be computed by the expression

$$(47) \quad C_h = \frac{1}{q} \int_{\Gamma_2} [B - u_h(q)] d\gamma,$$

for any $q > 0$. Moreover, C_h is given by

$$(48) \quad C_h = a(u_{3h}, u_{3h}) = \int_{\Gamma_2} u_{3h} d\gamma,$$

where u_{3h} is the unique solution of the variational equality

$$(49) \quad \begin{aligned} a(u_{3h}, v_h) &= \int_{\Gamma_2} v_h d\gamma, \quad \forall v_h \in V_h, \\ u_{3h} &\in V_h. \end{aligned}$$

Proof .- (i) Function $f_h = f_h(q)$ satisfies $f_h'''(q) = 0, \forall q > 0$ because (42). Owing to

$$(50) \quad u_h(0^+) = B, \quad f_h(0^+) = B |\Gamma_2| q, \quad f_h'(0^+) = B |\Gamma_2|,$$

there exists some constant $C_h > 0$ such that (44) and (45) are holds.

(ii) It follows taking into account

$$(51) \quad f_h'(q_{0h}(B)) = 0,$$

and the monotony property of function f_h' .

(iii) From (39) and (45) we obtain (47) for any $q > 0$. Moreover, if we define $w_h = w_h(q) \in V_h$ by the expression

$$(52) \quad w_h(q) = \frac{B - u_h(q)}{q},$$

then, we have :

$$(53) \quad a(w_h(q), v_h) = \frac{1}{q} a(B - u_h(q), v_h) = -\frac{1}{q} a(u_h(q), v_h) = \int_{\Gamma_2} v_h \, d\gamma, \quad \forall v_h \in V_h,$$

that is $w_h(q) = u_{3h}$ by uniqueness of the solution of the variational equality (49). then, we deduce (48).

Theorem 7 .- (i) We have the following equality :

$$(54) \quad a(u(q), u_h(q)) = C_h q^2, \quad \forall q > 0.$$

(ii) We have the following inequalities :

$$(55) \quad (a) C_h < C, \quad (b) q_0(B) < q_{0h}(B).$$

Proof .- (i) If we take $v = u_h(q) \in K_h = B + V_h \subset B + V_0 = K$ in the variational equality (7), and we take into account the expressions (13) and (44), then we obtain (54).

(ii) On the other hand, from (23) and (54) we have

$$(56) \quad \alpha \|u(q) - u_h(q)\|_V^2 \leq a(u(q) - u_h(q), u(q) - u_h(q)) = (C - C_h) q^2,$$

that is (55a). Moreover, (55b) follows from (18), (46) and (55).

Now, we shall use the interpolation result (20) for the function $u_3 \in H^r(\Omega)$, as a hypothesis of regularity of the continuous problem (7) (in general, $1 < r < \frac{3}{2}$ [3, 5, 6]). In [8], we present three examples with explicit solution. In these cases, we have $u(q), u_3 \in C^\infty(\Omega)$.

Theorem 8 .- We have the following relations and estimations :

$$(57) \quad a(u(q) - u_h(q), v_h) = 0, \quad \forall v_h \in V_h,$$

$$(58) \quad (C - C_h) q^2 = a(u(q) - u_h(q), u(q) - u_h(q)) \leq \\ \leq \inf_{v_h \in V_h} a(u(q) - v_h, u(q) - v_h),$$

$$(59) \quad 0 < C - C_h \leq C_0^2 h^{2(r-1)} |u_3|_{r,\Omega}^2,$$

$$(60) \quad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_0^2 h^{2(r-1)}}{C} |u_3|_{r,\Omega}^2 q_{0_h}(B).$$

Proof .— If we take $v = v_h \in V_h \subset V_0$ in the variational equality (7) and we subtract it with the variational equality (21), we obtain (57). By using (54), (56) and (57) we deduce

$$(61) \quad \begin{aligned} a(u(q) - u_h(q), u(q) - u_h(q)) &= a(u(q) - u_h(q), u(q)) - \\ &a(u(q) - u_h(q), u_h(q)) = a(u(q) - u_h(q), u(q)) - a(u(q) - u_h(q), v_h) = \\ &= a(u(q) - u_h(q), u(q) - v_h) \leq [a(u(q) - u_h(q), u(q) - u_h(q))]^{\frac{1}{2}} \cdot \\ &\cdot [a(u(q) - v_h, u(q) - v_h)]^{\frac{1}{2}}, \quad \forall v_h \in V_h, \end{aligned}$$

because $a(.,.)$ is a scalar product in V_0 , then we obtain (58).

By using (58), the facts that

$$(62) \quad \Pi_h(u(q)) \in B + V_h, \quad u(q) - \Pi_h(u(q)) \in V_h$$

and the interpolation result (20), we deduce (59). The relation (60) is obtained by using the definition of $q_{0_h}(B)$ and $q_0(B)$, and (59).

Remark 1 .— If we only have $u(q) \in V$ (i.e. $u_3 \in V$) we can obtain

$$(63) \quad 0 < C - C_h \leq \frac{1}{q^2} \|u(q) - \Pi_h(u(q))\|_V^2 = \|u_3 - \Pi_h(u_3)\|_V^2,$$

where the second term converges to zero when $h \rightarrow 0^+$ [1], but we cannot give an order of convergence.

Remark 2 .— If the constant heat flux on Γ_2 verifies the inequality $q > q_{0_h}(B)$, then both discrete and continuous problem represent a steady-state two-phase Stefan problem, that is, their temperatures are of non-constant sign in Ω .

Remark 3 .— When the function $u_h(q)$ is constant on Γ_2 (as a function of $x \in \Gamma_2$), then the sufficient condition, given by the Theorem 6—part (ii), is also a necessary condition to have a

two-phase discrete problem, because

$$(64) \quad \int_{\Gamma_2} u_h(q) \, d\gamma < 0 \Leftrightarrow u_h(q) < 0 \text{ on } \Gamma_2 .$$

Theorem 9 .- If we consider $h, B > 0$, and $0 < \epsilon_0 < 1$ (ϵ_0 is a parameter to be chosen arbitrarily), then we have the following estimations :

$$(65) \quad q_0(B) < q_{0h}(B) \leq \frac{q_0(B)}{\epsilon_0} \quad \text{and} \quad C_h \geq C \epsilon_0 , \quad \forall h \leq h_r(\epsilon_0),$$

$$(66) \quad 0 < q_{0h}(B) - q_0(B) \leq \frac{C_0^2 |u_3|_{r,\Omega}^2}{C \epsilon_0} q_0(B) h^{2(r-1)} , \quad \forall h \leq h_r(\epsilon_0),$$

where

$$(67) \quad h_r(\epsilon_0) = \left(\frac{C(1-\epsilon_0)}{C_0^2 |u_3|_{r,\Omega}^2} \right)^{\frac{1}{2(r-1)}} .$$

Proof .- From (60) we deduce

$$(68) \quad A(h) q_{0h}(B) \leq q_0(B) ,$$

where

$$(69) \quad A(h) = 1 - \frac{C_0^2 |u_3|_{r,\Omega}^2}{C} h^{2(r-1)} < 1 .$$

If we consider, for each parameter $0 < \epsilon_0 < 1$, the following equivalence :

$$(70) \quad 0 < \epsilon_0 < A(h) < 1 \Leftrightarrow 0 < h < h_r(\epsilon_0) ,$$

we can deduce the inequalities (65) and (66).

Corollary 10 .- If $B > 0$, then we have the following limit

$$(71) \quad \lim_{h \rightarrow 0^+} q_{0h}(B) = q_0(B) .$$

Remark 4 .- If $r = 2$ then the convergence in Corollary 10 is of the order of h .

Remark 5 .— Every thing we proved in this paper is still valid if the boundary Γ of the bounded domain Ω is represented by the union of the portions ($\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$) such that they have the following characteristics :

- (i) Γ_1 and Γ_2 have the same conditions as the ones previously described in (4).
- (ii) Γ_3 is a wall impermeable to heat, i.e. we have $\frac{\partial \theta}{\partial n} |_{\Gamma_3} = 0$ in (4) and therefore $\frac{\partial u}{\partial n} |_{\Gamma_3} = 0$ in (6).

Moreover, the first example considered in [8] verifies this condition.

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