Numerical analysis for the heat flux in a mixed elliptic problem to obtain a discrete steady-state two-phase Stefan problem

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NUMERICAL ANALYSIS FOR THE HEAT FLUX IN A MIXED ELLIPTIC PROBLEM TO OBTAIN A DISCRETE STEADY-STATE TWO-PHASE STEFAN PROBLEM

ANALYSE NUMERIQUE DU FLUX DE CHALEUR DANS UN PROBLEME ELLIPTIQUE MIXTE POUR OBTENIR UN CAS STATIONNAIRE DISCRET DU PROBLEME DE STEFAN A DEUX PHASES

Domingo Alberto TARZIA (*)

Abstract

We consider a material $\Omega \subset \mathbb{R}^n$ which occupies a convex polygonal bounded domain, with regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\mathrm{meas}(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We assume, without loss of generality, that the melting temperature is 0°C. We apply a temperature b=Const. > 0 on Γ_1 and a heat flux q = Const. > 0 on Γ_2. We consider a steady-state heat conduction problem in Ω.

We consider a regular triangulation of the domain Ω with Lagrange triangles of type 1. We study sufficient (and/or necessary) conditions for the heat flux q on Γ_2 to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω.

Résumé

On considère un matériel $\Omega \subset \mathbb{R}^n$, un domaine polygonal borné et convexe avec une frontière réguliere $\Gamma = \Gamma_1 \cup \Gamma_2$ avec $\mathrm{mes}(\Gamma_1) = |\Gamma_1| > 0$ et $|\Gamma_2| > 0$. On suppose, sans perte de généralité, que la température de changement de phase est 0°C. On considère une température b=Const. > 0 sur Γ_1 et un flux de chaleur q = Const. > 0 sur Γ_2. On considère le cas stationnaire du problème de conduction de la chaleur dans Ω.

On considère une triangulation régulier de Ω avec des triangles de Lagrange de type 1. On étudie des conditions suffisantes (et/ou nécessaires) pour le flux de chaleur q sur Γ_2 pour obtenir un changement de phase (cas stationnaire discret d'un problème de Stefan à deux phases) dans le domaine discrète correspondante, c'est-à-dire une température discrète de signe non-constante dans Ω.

Key words : Steady-state Stefan problem, finite element method, mixed elliptic problem, numerical analysis, variational inequalities, error bounds.

Mots Clés : Problème de Stefan stationnaire, méthode d'éléments finis, problème elliptique mixte, analyse numérique, inéquations variationnelles, estimation de l'erreur.

AMS Subject Classification : 35R35, 35J85, 65N15, 65N30.

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I.– INTRODUCTION.

We consider a material $\Omega \subset \mathbb{R}^n$ which occupies a convex polygonal bounded domain, with a regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ with $\text{meas}(\Gamma_1) = |\Gamma_1| > 0$ and $|\Gamma_2| > 0$. We assume, without loss of generality, that the melting temperature is 0°C. We apply a temperature b=Const. > 0 on Γ_1 and a heat flux $q =$ Const. > 0 on Γ_2. We consider a steady-state heat conduction problem in Ω. Following [7], we study the temperature $\theta = \theta(x)$ for $x \in \Omega$. The set Ω can be written by

$$(1) \qquad \Omega = \Omega_1 \cup \Omega_2 \cup \mathcal{L} .$$

where

$$(2) \qquad \begin{aligned} \Omega_1 &= \left\{ x \in \Omega \;/\; \theta(x) < 0 \right\} \text{ (liquid phase)} , \\ \Omega_2 &= \left\{ x \in \Omega \;/\; \theta(x) > 0 \right\} \text{ (solid phase)} , \\ \mathcal{L} &= \left\{ x \in \Omega \;/\; \theta(x) = 0 \right\} \text{ (free boundary)} , \end{aligned}$$

are the liquid phase, the solid phase and the free boundary which separates them respectively. The temperature θ can be represented in Ω in the following way

$$(3) \qquad \theta(x) = \begin{matrix} \theta_1(x) < 0 , & x \in \Omega_1 , \\ 0 , & x \in \mathcal{L} , \\ \theta_2(x) > 0 , & x \in \Omega_2 , \end{matrix}$$

and satisfies the following conditions :

$$(4) \qquad \begin{aligned} &\Delta\theta_i = 0 \quad \text{in } \Omega_i \ (i = 1, 2) \quad , \\ &\theta_1 = \theta_2 = 0 , \ k_1 \frac{\partial\theta_1}{\partial n} = k_2 \frac{\partial\theta_2}{\partial n} \quad \text{on } \mathcal{L} \quad , \\ &\theta_2 \mid_{\Gamma_1} = b > 0 , \\ &- k_2 \frac{\partial\theta_2}{\partial n} \mid_{\Gamma_2} = q \qquad \text{if} \quad \theta \mid_{\Gamma_2} > 0 \quad , \\ &- k_1 \frac{\partial\theta_1}{\partial n} \mid_{\Gamma_2} = q \qquad \text{if} \quad \theta \mid_{\Gamma_2} < 0 \quad , \end{aligned}$$

where $k_i > 0$ is the thermal conductivity in Ω_i (i=1 : solid phase, i = 2 : liquid phase). If we introduce the new unknown function [2, 7]

$$(5) \qquad u = k_2\,\theta^+ - k_1\,\theta^- \quad \left(\theta = \frac{1}{k_2}\,u^+ - \frac{1}{k_1}\,u^- \right) \qquad \text{in } \Omega \quad ,$$

where θ^+ and θ^- represent the positive part and the negative part of the function θ respectively, then problem (4) is transformed into the following mixed elliptic problem

$$(6) \qquad \begin{aligned} &\Delta u = 0 \qquad \text{dans} \quad \Omega \,, \\ &u \mid_{\Gamma_1} = B \quad , \\ &-\frac{\partial u}{\partial n} \mid_{\Gamma_2} = q \quad , \end{aligned}$$

whose variational formulation is given by

$$(7) \qquad \begin{aligned} &a(u, v-u) = L(v-u) \quad , \qquad \forall\, v \in K \quad , \\ &u \in K \quad , \end{aligned}$$

where

$$(8) \qquad \begin{aligned} &V = H^1(\Omega) \quad , \quad B = k_2\, b > 0 \ \text{ on } \Gamma_1 \ , \\ &K = \left\{ v \in V \,/\, v \mid_{\Gamma_1} = B \right\} \,, \quad V_0 = \left\{ v \in V \,/\, v \mid_{\Gamma_1} = 0 \right\} \,, \\ &a(u, v) = \int_\Omega \nabla u \,.\, \nabla u \ dx \ , \quad L(v) = L_q(v) = -\int_{\Gamma_2} q\, v \ d\gamma \ . \end{aligned}$$

Moreover, the solution of (7) is characterized by the following minimum problem [4]

$$(9) \qquad \begin{aligned} &J(u) \le J(v) \quad , \qquad \forall\, v \in K \quad , \\ &u \in K \quad , \end{aligned}$$

where

$$(10) \qquad J(v) = J_q(v) = \frac{1}{2}\, a(v,v) - L_q(v) = \frac{1}{2}\, a(v,v) + \int_{\Gamma_2} q\, v\, d\gamma \ .$$

We can define the real fonction $f : \mathbf{R}^+ \to \mathbf{R}$ in the following way

$$(11) \qquad f(q) = J_q(u(q)) = \frac{1}{2}\, a(u(q), u(q)) + q \int_{\Gamma_2} u_q \ d\gamma \ ,$$

where $u = u(q)$ is the unique solution of the variational equality (7) for each heat flux $q > 0$ (for a given $B > 0$).

For the continuous problem (6) or (7), a sufficient condition to have a steady-state two-phase Stefan problem (i.e. the solution $u(q)$ of (7) is a function of non-constant sign in Ω) was obtained in [8, 9].

Theorem 1 .— (i) Function f is derivable. Moreover, f' is a continuous and strictly decreasing function, and it is given by the following expression

$$(12) \qquad f'(q) = \int_{\Gamma_2} u(q)\, d\gamma \ .$$

(ii) There exists a geometrical constant $C = C(\Omega, \Gamma_1, \Gamma_2) > 0$ such that

$$(13) \qquad a(u(q), u(q)) = C\, q^2 \ , \quad f(q) = -\frac{C}{2} q^2 + B\,|\,\Gamma_2\,|\, q \ , \quad \forall\, q > 0 \, .$$

Moreover, the constant $C > 0$ is given by

$$(14) \qquad C = a(u_3,u_3) = \int_{\Gamma_2} u_3\, d\gamma > 0 \, ,$$

where $u_3 \in V_o$ ($u(q) = B - q\, u_3$ in Ω) is the unique solution of the following mixed elliptic problem

$$(15) \qquad \begin{aligned} &\Delta u_3 = 0 \ \text{ in } \Omega \, , \\ &u_3 \,|\, _{\Gamma_1} = 0 \ , \quad \frac{\partial u_3}{\partial n} \,|\, _{\Gamma_2} = 1 \ , \end{aligned}$$

whose variational formulation is given by

$$(16) \qquad \begin{aligned} &a(u_3,v) = \int_{\Gamma_2} v\, d\gamma \ , \ \forall v \in V_o \, , \\ &u_3 \in V_o \ . \end{aligned}$$

(iii) If

$$(17) \qquad q > q_0\,(B)$$

then (6) or (7) represents a steady-state two-phase Stefan problem (i.e. the solution $u(q)$ of problem (7) is a fonction of non-constant sign in Ω), where $q_0 = q_0(B)$ is given by

$$(18) \qquad q_0(B) = \frac{B\,|\,\Gamma_2\,|}{C} \ , \quad \forall\, B > 0 \, .$$

(iv) If the function $u(q)$ is constant over Γ_2, then the sufficient condition (given by (17)) is also necessary.

Proof .— See [8, 9].

Now, we consider τ_h , a regular triangulation of the polygonal domain Ω with Lagrange triangles of type 1, constituted by affine-equivalent finite element of class C^0, where $h > 0$ is a parameter which goes to zero. We can take h equals to the longest side of the triangles $T \in \tau_h$ and we can approximated V_0 by [1] :

$$(19) \quad V_h = \left\{ v_h \in C^0(\bar{\Omega}) \,/\, v_h \mid_T \in P_1(T) \,,\ \forall T \in \tau_h \,,\ v_h \mid_{\Gamma_1} = 0 \right\},$$

where P_1 is the set of the polynomials of degree less or equals than 1. Let π_h be the corresponding linear interpolatation operator. Then, we can consider that there exists a constant $C_0 > 0$ (independent of the parameter h) such that

$$(20) \qquad \| v - \pi_h v \|_V \leq C_0 \; h^{r-1} \; \| v \|_{r,\Omega} \;, \quad \forall v \in H^r(\Omega) \;, \text{ with } 1 < r \leq 2 \,.$$

We consider the following finite dimensional approximate variational problem, corresponding to the continuous variational problem (7), given by :

$$(21) \qquad \begin{aligned} & a(u_h, v_h) = L(v_h) \,, \quad \forall v_h \in V_h \,, \\ & u_h \in K_h = B + V_h \,, \end{aligned}$$

and we can obtain the following results.

Lema 2 .— We have

$$(22) \qquad \lim_{h \to 0^+} \| u_h - u \|_V = 0 \,,$$

where u is the unique solution of the variational equality (7).

Proof .— Owing to $\mathrm{meas}(\Gamma_1) > 0$, we have that the bilinear form a is coercitivity over V_0 , that is [4] :

$$(23) \qquad \exists\, \alpha > 0 \,/\, a(v, v) = \| v \|^2_{V_0} \geq \alpha \| v \|^2_V \;, \ \forall v \in V_0 \;,$$

and therefore $\| \cdot \|_{V_0}$ and $\| \cdot \|_V$ are two equivalents norms in V_0 . We follow a similar method developped in [1].

Corollary 3 .— If we define

$$(24) \qquad \theta_h = \frac{1}{k_2} u_h^+ - \frac{1}{k_1} u_h^- \in V \ , \ \theta = \frac{1}{k_2} u^+ - \frac{1}{k_1} u^- \in V$$

then we have

$$(25) \qquad \lim_{h \to 0^+} \| \theta_h - \theta \|_H = 0 \ ,$$

where $H = L^2(\Omega)$.

Proof .— If we consider the scalar product in H, defined by

$$(26) \qquad (u, v) = \int_\Omega u \, v \, dx \ ,$$

then, we deduce

$$(27) \qquad \begin{aligned} \| u_h - u \|_H^2 &= \| u_h^+ - u^+ \|_H^2 + \| u_h^- - u^- \|_H^2 + 2\,(u_h^+, u^-) + \\ &+ 2\,(u_h^-, u^+) \geq \| u_h^+ - u^+ \|_H^2 + \| u_h^- - u^- \|_H^2 \ , \end{aligned}$$

that is

$$(28) \qquad \mathrm{Max}\,(\, \| u_h^+ - u^+ \| \, , \| u_h^- - u^- \| \,) \leq \| u_h - u \|_H \ .$$

From (24) we obtain :

$$(29) \qquad \begin{aligned} \| \theta_h - \theta \|_H &\leq \frac{1}{k_2} \| u_h^+ - u^+ \|_H + \frac{1}{k_1} \| u_h^- - u^- \|_H \leq \\ &\leq \left(\frac{1}{k_1} + \frac{1}{k_2} \right) \| u_h - u \|_H \ , \end{aligned}$$

i.e. (25).

The goal of this paper is to consider the numerical analysis of the inequality (17). We study sufficient (and/or necessary) conditions for the constant heat flux q on Γ_2 to obtain a change of phase (steady-state two-phase discretized Stefan problem) into the corresponding discretized domain, that is a discrete temperature of non-constant sign in Ω. We obtain that :

i) there exists a constant $C_h > 0$ (which depends only of the geometry of the domain Ω for each $h > 0$ and it is characterized by a variational problem) such that if $q > q_{0_h}(B) = B|\Gamma_2|/C_h$ then the steady-state discretized problem presents two phases.

ii) we have the estimations $C_h < C$ and $q_0(B) < q_{0h}(B)$ where C and $q_0(B)$ have been obtained for the continuous problem by (14) and (18) respectively [9].

iii) we deduce an error bounds for $C - C_h$ and $q_{0h}(B) - q_0(B)$ as a function of the parameter h.

In other words, we obtain for the mixed elliptic discretized problem, defined by u_h, analogous conditions to the ones obtained for the corresponding continuous problem [9], defined by u.

II.– INEQUALITY FOR THE HEAT FLUX IN THE DISCRETIZED PROBLEM.

For each $q > 0$ we consider the functions $u(q) \in K$ and $u_h(q) \in K_h$, as the unique solution of the variational equalities (7) (continuous problem) and (21) (discrete problem) respectively. We define the real function $f_h : \mathbb{R}^+ \to \mathbb{R}$, for each $h > 0$, in the following way

(30) $$f_h(q) = J_q(u_h(q)) = \frac{1}{2}\, a(u_h(q), u_h(q)) + q \int_{\Gamma_2} u_h(q)\, d\gamma \ , \ q > 0 .$$

Therefore, we obtain the following properties :

<u>Theorem 4</u> .– (i) If $u_i = u_h(q_i)$ is the solution of (21) for $q_i > 0$ $(i = 1, 2)$, then we have the following equalities :

(31) $$a(u_2 - u_1 , u_2 - u_1) = (q_1 - q_2) \int_{\Gamma_2} (u_2 - u_1)\, d\gamma ,$$

(32) $$a(u_2, u_2) - a(u_1, u_1) = a(u_2 + u_1, u_2 - u_1) = (q_1 + q_2) \int_{\Gamma_2} (u_1 - u_2)\, d\gamma .$$

(ii) For all real numbers $q > 0$ and Δ such that $(q + \Delta) > 0$, we obtain the following estimations :

(33) $$\left\| \frac{1}{\Delta}[u_h(q) - u_h(q + \Delta)] \right\|_V \le D_1 = \frac{\|\gamma_0\|}{\alpha} \ |\Gamma_2|^{1/2} ,$$

(34) $$\left\| \frac{1}{\Delta}[u_h(q) - u_h(q + \Delta)] \right\|_{L^2(\Gamma_2)} \le D_2 = D_1 \|\gamma_0\| ,$$

where γ_0 is the linear and continuous trace operator, defined over V. Moreover, the function

(35) $$q > 0 \ \to \int_{\Gamma_2} u_h(q)\, d\gamma \in \mathbb{R} ,$$

is a continuous and strictly decreasing function.

(iii) Function $f_h = f_h(q)$ is derivable. Moreover, $f_h{}'$ is a continuous and strictly decreasing function and given by the following expression

$$f_h{}'(q) = \int_{\Gamma_2} u_h(q)\, d\gamma \,, \tag{36}$$

Proof .— (i) If we take $v = u_2 - u_1 \in V_h$ in the variational equality corresponding to u_1 and $v = u_1 - u_2 \in V_h$ in the one corresponding to u_2, and we add up and subtract both equalities, then we obtain (31) and (32) respectively.

(ii) Taking into account (23), the Cauchy-Schwarz inequality and the continuity of the operator γ_0 we deduce (33). From (33) and the continuity of γ_0 we have (34). Therefore we have (35) because

$$\left| \int_{\Gamma_2} [u_h(q) - u_h(q+\Delta)]\, d\gamma \right| \le D_2\, |\Gamma_2|^{1/2}\, \Delta \,. \tag{37}$$

Moreover, the monotony property is a consequence of (31).

(iii) From (30) and elementary computations, we deduce

$$\frac{1}{\Delta}\,[\, f_h\,(q+\Delta) - f_h(q)\,] = \frac{1}{2} \int_{\Gamma_2} [u_h(q) + u_h(q+\Delta)]\, d\gamma \,, \tag{38}$$

that is (36), by using (35).

Lema 5 .— We have the following expressions :

$$a(u_h(q)\,,\, u_h(q)) = q\, B\, |\,\Gamma_2\,| - q \int_{\Gamma_2} u_h(q)\, d\gamma \,, \tag{39}$$

$$f_h(q) = q\, B\, |\,\Gamma_2\,| - \frac{1}{2}\; a(u_h(q)\,,\, u_h(q)) \,, \tag{40}$$

$$f_h(q) = \frac{q\, B\, |\,\Gamma_2\,|}{2} + \frac{q}{2} \int_{\Gamma_2} [u_h(q)\, d\gamma \,, \tag{41}$$

$$q\, f_h{}''(q) = f_h{}'(q) - B\, |\,\Gamma_2\,| \,, \tag{42}$$

$$\frac{d}{dq}[a(u_h(q)\,,\, u_h(q))] = \frac{2}{q}\, a(u_h(q)\,,\, u_h(q)) = 2\, B\, |\,\Gamma_2\,| - 2\, f_h{}'(q) \,. \tag{43}$$

Proof .— Taking $v_h = B - u_h(q) \in V_h$ in (21) we obtain (31). We deduce (40) and (41) by using (30) and (39). From (36), (39)—(41) we obtain (42) and (43) by derivating with respect to variable q.

<u>Theorem 6</u> .— (i) There exists a constant $C_h > 0$ such that

(44) $$f_h(q) = q\,B\,|\,\Gamma_2\,| - \tfrac{1}{2}\,C_h\,q^2 \ , \ \forall\, q > 0\ ,$$

(45) $$a(u_h(q)\ ,\ u_h(q)) = C_h\,q^2\ ,\ \ \forall\, q > 0\ .$$

(ii) If $q > q_{o_h}(B)$, then problem (21) represents a discretized steady-state two-phase Stefan problem (i.e. $u_h(q)$ is a funtion of non-constant sign in Ω), where

(46) $$q_{o_h}(B) = \frac{B\,|\,\Gamma_2\,|}{C_h}\ .$$

(iii) The constant $C_h > 0$ can be computed by the expression

(47) $$C_h = \frac{1}{q} \int_{\Gamma_2} [B - u_h(q)]\, d\gamma\ ,$$

for any $q > 0$. Moreover, C_h is given by

(48) $$C_h = a(u_{3_h}, u_{3_h}) = \int_{\Gamma_2} u_{3_h}\, d\gamma\ ,$$

where u_{3h} is the unique solution of the variational equality

(49) $$\begin{aligned} & a(u_{3h}\ ,\ v_h) = \int_{\Gamma_2} v_h\, d\gamma\ ,\quad \forall\, v_h \in V_h\ , \\ & u_{3h} \in V_h\ . \end{aligned}$$

<u>Proof</u> .— (i) Function $f_h = f_h(q)$ satisfies $f_h'''(q) = 0$, $\forall\, q > 0$ because (42). Owing to

(50) $$u_h(0^+) = B\ ,\quad f_h(0^+) = B\,|\,\Gamma_2\,|\,q\ ,\quad f_h'(0^+) = B\,|\,\Gamma_2\,|\ ,$$

there exists some constant $C_h > 0$ such that (44) and (45) are holds.
(ii) It follows taking into account

(51) $$f_h'\,(q_{o_h}(B) = 0\ ,$$

and the monotony property of function f_h' .
(iii) From (39) and (45) we obtain (47) for any $q > 0$. Moreover, if we define $w_h = w_h(q) \in V_h$ by the expression

(52) $$w_h(q) = \frac{B - u_h(q)}{q} \, ,$$

then, we have :

(53) $$a(w_h(q), v_h) = \frac{1}{q}\, a(B - u_h(q), v_h) = -\frac{1}{q}\, a(u_h(q), v_h) = \int_{\Gamma_2} v_h \, d\gamma \, , \quad \forall \, v_h \in V_h,$$

that is $w_h(q) = u_{3_h}$ by uniqueness of the solution of the variational equality (49). then, we deduce (48).

Theorem 7 .— (i) We have the following equality :

(54) $$a(u(q), u_h(q)) = C_h \, q^2 \, , \; \forall \, q > 0 \, .$$

(ii) We have the following inequalities :

(55) $$\text{(a)} \; C_h < C \, , \qquad \text{(b)} \; q_0(B) < q_{0_h}(B) \, .$$

Proof .— (i) If we take $v = u_h(q) \in K_h = B + V_h \subset B + V_0 = K$ in the variational equality (7), and we take into account the expressions (13) and (44), then we obtain (54).
(ii) On the other hand, from (23) and (54) we have

(56) $$\alpha \, \| \, u(q) - u_h(q) \, \|_V^{\,2} \leq a(u(q) - u_h(q), u(q) - u_h(q)) = (C - C_h) \, q^2,$$

that is (55a). Moreover, (55b) follows from (18), (46) and (55).

Now, we shall use the interpolation result (20) for the funtion $u_3 \in H^r(\Omega)$, as a hipotesis of regularity of the continuous problem (7) (in general, $1 < r < \frac{3}{2}$ [3, 5, 6]). In [8], we present three examples with explicit solution. In these cases, we have $u(q), u_3 \in C^{\infty}(\Omega)$.

Theorem 8 .— We have the following relations and estimations :

(57) $$a(u(q) - u_h(q), v_h) = 0 \;\; , \;\; \forall \, v_h \in V_h \, ,$$

(58) $$(C - C_h) \, q^2 = a(u(q) - u_h(q), u(q) - u_h(q)) \leq$$

$$\leq \inf_{v_h \in V_h} a(u(q) - v_h \, , u(q) - v_h) \, ,$$

$$(59) \qquad 0 < C - C_h \leq C_0^2 \; h^{2(r-1)} \, | \, u_3 \, |^2_{r,\Omega} \, ,$$

$$(60) \qquad 0 < q_{0_h}(B) - q_0(B) \leq \frac{C_0^2 \; h^{2(r-1)}}{C} \, | \, u_3 \, |^2_{r,\Omega} \; q_{0_h}(B) \, .$$

Proof .— If we take $v = v_h \in V_h \subset V_0$ in the variational equality (7) and we subtract it with the variational equality (21), we obtain (57). By using (54), (56) and (57) we deduce

$$(61) \qquad a(u(q) - u_h(q), u(q) - u_h(q)) = a(u(q) - u_h(q), u(q)) -$$

$$a(u(q) - u_h(q), u_h(q)) = a(u(q) - u_h(q), u(q)) - a(u(q) - u_h(q), v_h) =$$

$$= a(u(q) - u_h(q), u(q) - v_h) \leq [\, a(u(q) - u_h(q), u(q) - u_h(q))]^{\frac{1}{2}} \, .$$

$$. \, [\, a(u(q) - v_h, u(q) - v_h)]^{\frac{1}{2}} \, , \; \forall \; v_h \in V_h \, ,$$

because $a(.,.)$ is a scalar product in V_0 , then we obtain (58).

By using (58), the facts that

$$(62) \qquad \Pi_h(u(q)) \in B + V_h \quad , \qquad u(q) - \Pi_h(u(q)) \in V_h$$

and the interpolation result (20), we deduce (59). The relation (60) is obtained by using the definition of $q_{0_h}(B)$ and $q_0(B)$, and (59).

Remark 1 .— If we only have $u(q) \in V$ (i.e. $u_3 \in V$) we can obtain

$$(63) \qquad 0 < C - C_h \leq \frac{1}{q^2} \, \| \, u(q) - \Pi_h(u(q)) \, \|^2_V \; = \; \| \, u_3 - \Pi_h(u_3) \, \|^2_V \; ,$$

where the second term converges to zero when $h \to 0^+$ [1] , but we cannot give an order of convergence.

Remark 2 .— If the constant heat flux on Γ_2 verifies the inequality $q > q_{0_h}(B)$, then both discrete and continuous problem represent a steady-state two-phase Stefan problem, that is, their temperaturas are of non-constant sign in Ω.

Remark 3 .— When the function $u_h(q)$ is constant on Γ_2 (as a funtion of $x \in \Gamma_2$), then the sufficient condition, given by the Theorem 6—part (ii), is also a necessary condition to have a two-phase discrete problem, because

$$(64) \qquad \int_{\Gamma_2} u_h(q)\, d\gamma < 0 \iff u_h(q) < 0 \text{ on } \Gamma_2 .$$

Theorem 9 .— If we consider h, B > 0, and $0 < \epsilon_o < 1$ (ϵ_o is a parameter to be chosen arbitrarily), then we have the following estimations :

$$(65) \qquad q_o(B) < q_{o_h}(B) \le \frac{q_o(B)}{\epsilon_o} \quad \text{and} \quad C_h \ge C\ \epsilon_o \ , \quad \forall\, h \le h_r(\epsilon_o),$$

$$(66) \qquad 0 < q_{o_h}(B) - q_o(B) \le \frac{C_o^2\ \| u_3 \|^2_{r,\Omega}}{C\ \epsilon_o}\ q_o(B)\ h^{2(r-1)} \ , \quad \forall\, h \le h_r(\epsilon_o),$$

where

$$(67) \qquad h_r(\epsilon_o) = \left(\frac{C\,(1-\epsilon_o)}{C_o^2\, \| u_3 \|^2_{r,\Omega}} \right)^{\frac{1}{2(r-1)}} .$$

Proof .— From (60) we deduce

$$(68) \qquad A(h)\ q_{o_h}(B) \le q_o(B) ,$$

where

$$(69) \qquad A(h) = 1 - \frac{C_o^2\ \| u_3 \|^2_{r,\Omega}}{C}\ h^{2(r-1)} < 1 .$$

If we consider, for each parameter $0 < \epsilon_o < 1$, the following equivalence :

$$(70) \qquad 0 < \epsilon_o < A(h) < 1 \iff 0 < h < h_r(\epsilon_o) ,$$

we can deduce the inequalities (65) and (66).

Corollary 10 .— If B > 0, then we have the following limit

$$(71) \qquad \lim_{h \to 0^+} q_{o_h}(B) = q_o(B) .$$

Remark 4 .— If r = 2 then the convergence in Corollary 10 is of the order of h.

Remark 5 .— Every thing we proved in this paper is still valid if the boundary Γ of the bounded domain Ω is represented by the union of the portions ($\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$) such that they have the following characteristics :

(i) Γ_1 and Γ_2 have the same conditions as the ones previously described in (4).

(ii) Γ_3 is a wall impermeable to heat, i.e. we have $\frac{\partial \theta}{\partial n} \mid_{\Gamma_3} = 0$ in (4) and therefore $\frac{\partial u}{\partial n} \mid_{\Gamma_3} = 0$ in (6).

Moreover, the first example considered in [8] verifies this condition.

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REFERENCES

[1] P.G. CIARLET, "The finite element method for elliptic problems", North-Holland, Amsterdam (1978).

[2] G. DUVAUT, "Problèmes à frontière libre en théorie des milieux continus", Rapport de Recherche N^0 185, LABORIA—IRIA, Rocquencourt (1976).

[3] P. GRISVARD, "Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain", in Numerical Solution of Partial Differential Equations III, SYNSPADE 1975, B. Hubbard (Ed.), Academic Press, New York (1976), 207—274.

[4] D. KINDERLEHRER — G. STAMPACCHIA, "An introduction to variational inequalities and their applications", Academic Press, New York (1980).

[5] M.K.V. MURTHY — G. STAMPACCHIA, "A variational inequality with mixed boundary conditions", Israel J. Math., 13(1972), 188—224.

[6] E. SHAMIR, "Regularization of mixed second-order elliptic problems", Israel J. Math., 6 (1968), 150—168.

[7] D.A. TARZIA, "Sur le problème de Stefan à deux phases", C. R. Acad. Sc. Paris, 288A(1979), 941—944; See also "Aplicación de métodos variacionales en el caso estacionario del problema de Stefan a dos fases", Math. Notae, 27(1979/80), 145—156 .

[8] D.A. TARZIA, "The two-phase Stefan problem and some related conduction problems", Reuniões em Matemática Aplicada e Computacão Científica, Vol. 5, SBMAC, Rio de Janeiro (1987).

[9] D.A. TARZIA, "An inequality for the constant heat flux to obtain a steady-state two-phase Stefan problem", Engineering Analysis, 5(1988), 177—181.

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