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Un problème combinatoire dans les graphes de Hamming et sa solution en Scratchpad

A combinatorial problem in Hamming graphs and its solution in Scratchpad

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Résumé

Nous présentons ici un problème combinatoire qui est apparu dans la détermination des énumérateurs de poids complets des translatés des codes linéaires. Dans la résolution de ce problème par les séries formelles exponentielles à coefficients dans un anneau de polynômes à plusieurs variables, nous tombons sur un système d'équations différentielles à coefficients dans un corps de fractions rationelles. Grâce aux possibilités d'abstraction de Scratchpad ce système peut être résolu simplement et naturellement, ce qui ne semble pas être le cas pour d'autre systèmes de calcul formel aujourd'hui disponibles.

Abstract

We present a combinatorial problem which arises in the determination of the complete weight coset enumerators of error-correcting codes [1]. In solving this problem by exponential power series with coefficients in a ring of multivariate polynomials, we fall on a system of differential equations with coefficients in a field of rational functions. Thanks to the abstraction capabilities of Scratchpad this differential equation may be solved simply and naturally, which seems not to be the case for the other computer algebra systems now available.

1 A combinatorial problem in Hamming graphs

Let $\mathbf{F} = GF(q)$ be a finite field with q elements, let $m = q - 1$ and fix an ordering $\mathbf{F}^* = [a_1, \ldots, a_m]$ of the nonzero elements of $\mathbf{F}$. For x in the vector space $\mathbf{F}^n$ the *(Hamming) weight* of x is defined as $w(x)$ = number of nonzero components of x and the *complete weight* of x as the list $w^c(x) = [w_{a_1}(x), \ldots, w_{a_m}(x)]$ where $w_a(x)$ = number of components of x which are equal to $a \in \mathbf{F}^*$. The (Hamming) *distance* between x and y is $d(x, y) = w(y - x)$ and the *gap* between x and y is $g(x, y) = w^c(y - x)$.

If Ω is the set of weight one vectors in $\mathbf{F}^n$, then the *Hamming graph* $\Gamma = \Gamma(n, q)$ is the Cayley graph $C(\mathbf{F}^n, \Omega)$. This means that the vertex set is $\mathbf{F}^n$ and that (x, y) is an oriented edge (arrow) iff $y - x \in \Omega$. Set $\Omega_i = \{x \in \Omega |$ the only nonzero component of x is $a_i\}$. An arrow (x, y) in Γ will be called of *color* i if $y - x \in \Omega_i$. A *path* of length j joining x to y is a sequence $\gamma = (x^{(0)}, x^{(1)}, \ldots, x^{(j)})$ where $x^{(0)} = x, x^{(j)} = y$ and $x^{(i)} - x^{(i-1)} \in \Omega, i = 1, \ldots, j$. Set $\mathrm{Path}_j(x, y)$ to be the set of all these paths and

$$\mathrm{Path} = \bigcup_{j \geq 0} \{\mathrm{Path}_j(x, y) | x, y \in \mathbf{F}^n\}$$

We are interested in the various color distributions of the paths in Γ. For this it is convenient to work in the multivariate polynomial ring $\mathbf{Z}[T_{a_1}, \ldots, T_{a_m}]$.

Definition 1 *The* weight function $\phi : \mathrm{Path} \to \mathbf{Z}[T_{a_1}, \ldots, T_{a_m}]$ *is defined as follows*

1. *if (x, y) is an arrow and if $y - x \in \Omega_i$, then $\phi(x, y) = T_{a_i}$*
2. *if $\gamma = (x^{(0)}, x^{(1)}, \ldots x^{(j)})$ is a path, then $\phi(\gamma) = \Pi_{i=1}^{j} \phi(x^{(i-1)}, x^{(i)})$. This weight function ϕ is extended to subsets U of* Path *by the formula*

$$\phi(U) = \sum \{\phi(\gamma) \mid \gamma \in U\}$$

$\phi(U)$ is called the inventory *of U.*

Problem 1 Determine the inventories $\phi(\mathrm{Path}_j(x, y))$ for all j:

$$\phi(\mathrm{Path}_j(x, y)) = \sum_{j_1 + \cdots + j_m = j} S_{j_1 \ldots j_m}(x, y) T_{a_1}^{j_1}, \ldots, T_{a_m}^{j_m}$$

where $S_{j_1 \ldots j_m}(x, y)$ is the number of paths of length $j = j_1 + \ldots + j_m$ joining x to y with j_1 arrows of color 1, j_2 arrows of color 2, etc.

Proposition 1 *If $g(x, y) = g(x', y')$, then $\phi(\mathrm{Path}_j(x, y)) = \phi(\mathrm{Path}_j(x', y')) = \phi(\mathrm{Path}_j(0, y - x))$. In fact $S_{j_1 \ldots j_m}(x, y) = S_{j_1 \ldots j_m}(x', y')$.*

Proof: It is evident that the translation by $-x$ establishes a bijection between $\mathrm{Path}_j(x, y)$ and $\mathrm{Path}_j(0, y-x)$ that preserves coloration. Moreover if $g(x, y) = g(x', y') = w^c(y-x) = [i_1, \ldots, i_m]$, then take a bijection of the set of coordinate places sending the i_1 places where $y-x$ has component a_1 to the corresponding i_1 places in $y'-x'$ etc. This establishes a bijection preserving coloration between $\mathrm{Path}_j(0, y-x)$ and $\mathrm{Path}_j(0, y'-x')$.

By this proposition we may reformulate our problem as follows.

Problem 2 *If a complete weight $\vec{\imath} = [i_1, \ldots, i_m]$ of some $x \in \mathbf{F}^n$ is given, determine the inventories*

$$\begin{aligned} S_{\vec{\imath},j} &= \phi(\mathrm{Path}_j(0, x)) \\ &= \sum_{|\vec{\jmath}|=j} S_{\vec{\imath},\vec{\jmath}} T^{\vec{\jmath}} \end{aligned}$$

where $T = [T_{a_1}, \ldots, T_{a_m}], \vec{\jmath} = [j_1, \ldots, j_m], |\vec{\jmath}| = j_1 + \ldots + j_m$ and $T^{\vec{\jmath}} = T_{a_1}^{j_1}, \ldots, T_{a_m}^{j_m}$. The number $S_{\vec{\imath},\vec{\jmath}}$ counts the paths of length $|\vec{\jmath}| = j$ and color distribution $\vec{\jmath}$ joining 0 to a vertex x of complete weight $\vec{\imath}$.

2 Analysis of the problem by exponential generating power series with coefficients in the ring $\mathbf{Z}[T_{a_1}, \ldots, T_{a_m}]$.

Definition 2 *Let $f_s(j_1, \ldots, j_m)$ be the number of sequences in F^* containing j_1 elements equal to a_1, j_2 elements equal to $a_2, \ldots, j_m$ elements equal to a_m and whose sum is equal to $s \in \mathbf{F}$. We define the power series $f_s(X)$ by*

$$f_s(X) = \sum_{j \geq 0} \left[\sum_{j_1 + \cdots + j_m = j} f_s(j_1, \ldots, j_m) T_{a_1}^{j_1} \ldots T_{a_m}^{j_m} \right] X^j / j!.$$

The relationship between these exponential generating power series and our problem follows from classical results on shuffle product or "composé partionnel" [2]. However, to be self contained, we prefer to give a direct proof which may also serve as an illustration of the ideas involved.

Proposition 2 *If $\vec{i} = [i_1, \ldots, i_m]$ is the complete weight of some $x \in \mathbf{F}^n$ and j is a natural number, then $S_{\vec{i},j}$ is the coefficient of $X^j/j!$ in the expansion of $f_{a_1}^{i_1}(X) \ldots f_{a_m}^{i_m}(X) f_0^{n-|\vec{i}|}(X)$.*

Proof: Without loss of generality we may take

$$x=[\underbrace{a_1,\ldots,a_1}_{i_1};\ldots;\underbrace{a_m,\ldots,a_m}_{i_m};\underbrace{0,\ldots,0}_{n-|\vec{i}|}].$$

We have to count the paths of length j joining 0 to x paying attention to the various color distributions of these paths.

In any path and for any position in x, the contribution of pertinent arrows has to sum up to the right element of $\mathbf{F}$: a_1 if the considered position is among the first i_1, a_2 if it is among the next i_2, etc.

Consider the elements $a_{ik}^{(l)} \in \mathbf{F}^*$ for $l=0,1,\ldots,m$ in the following figure:

$$\begin{array}{rclllllll}
 & & \overbrace{}^{i_1} & & \overbrace{}^{i_l} & & \overbrace{}^{i_m} & \overbrace{}^{n-|\vec{i}|} & \\
0 & = & [0,\ldots,0; & \ldots & 0,\ldots,0; & \ldots; & 0,\ldots,0 & 0,\ldots,0] & \\
 & & a_{11}^{(1)} & & a_{i1}^{(l)} & & & a_{11}^{(0)} & \\
 & & a_{12}^{(1)} & \ldots & a_{i2}^{(l)} & \ldots & & a_{12}^{(0)} & \ldots \\
 & & \vdots & & \vdots & & & \vdots & \\
 & & a_{1,j(1,1)}^{(1)} & & a_{i,j(l,i)}^{(l)} & & & a_{1,j(0,1)}^{(0)} & \\
x & = & [a_1,\cdots,a_1; & \ldots & \ldots,a_l,\ldots & \ldots; & a_m,\ldots,a_m; & 0,\ldots,0] &
\end{array}$$

Each of these $a_{ik}^{(l)}$ defines an element of Ω_l affecting the coordinate number $i_1+\ldots+i_{l-1}+i$ and we must have

$$\sum_{k=1}^{j(l,i)} a_{ik}^{(l)} = a_l$$

so that x will be the endpoint of the path of length $j=\sum_l\sum_i j(l,i)$ so defined. In the algebra $\mathbf{Z}[T_{a_1},\ldots,T_{a_m}]$ the collective contribution to this coordinate is $T_{a_l}^{j(l,i)}$.

Now we can express the generating power series $f_s(x)$ in the more convenient form

$$f_s(X)=\sum_{j\geq 0}[\sum_{b_1+\ldots+b_j=s} T_{b_1}\ldots T_{b_j}]\frac{X^j}{j!}.$$

If k is the number of coordinates of x that are equal to s, then

$$\begin{aligned}
f_s^k(X) &= \sum_{j_1,\ldots,j_k}[\left(\sum_{a_{11}}\cdots T_{a_1j(1)}\right)\cdots\left(\sum_{a_k1+\ldots=s} T_{a_k1}\ldots T_{a_kj(k)}\right)]\frac{X^{j_1}\ldots X^{j_k}}{j_1!\ldots j_k!}\\
&= \sum_{j\geq 0}[\sum T_{a_{11}}\ldots T_{a_1j(1)}\ldots T_{a_k1}\ldots T_{a_kj(k)}\frac{j!}{j_1!\ldots j_k!}]\frac{X^j}{j!}
\end{aligned}$$

where in the inner sum $a_{11} + \ldots + a_{1j(1)} = s, \ldots, a_{k1} + \ldots + a_{kj(k)} = s$.

This corresponds in shuffling the j arrows affecting these k different coordinates in such a way that the endpoint of the various paths so obtained is s at those k coordinates.

By multiplying all these powers we obtain the result.

Example 1 Take $\mathbf{F} = \{0, 1, 2\}$, $n = 4$, $\vec{\imath} = [2, 1]$. We seek the paths joining $0 = [0, 0, 0, 0]$ to $x = [1, 1, 2, 0]$. We have

$$
\begin{aligned}
f_0(X) &= 1 + 2T_1T_2\frac{X^2}{2!} + (T_1^3 + T_2^3)\frac{X^3}{3!} + \ldots \\
f_1(X) &= T_1X + T_2^2\frac{X^2}{2!} + 3T_1^2T_2\frac{X^3}{3!} + \cdots \\
f_2(X) &= T_2X + T_1^2\frac{X^2}{2!} + 3T_1T_2^2\frac{X^3}{3!} + \cdots \\
f_1^2(X) &= T_1^2X^2 + 6T_1T_2^2\frac{X^3}{3!} + (24T_1^3T_2 + 6T_2^4)\frac{X^4}{4!} + \cdots \\
f_1^2f_2f_0 &= 6T_1^2T_2\frac{X^3}{3!} + (12T_1^4 + 24T_1T_2^3)\frac{X^4}{4!} + \\
&\quad +(360T_1^3T_2^2 + 30T_2^5)\frac{X^5}{5!} + \cdots
\end{aligned}
$$

In the following figures 1, 2 and 3 we give a detailed account of what is going on.

type	T_1	T_1	T_2	
0	0	0	0	0
path transitions	1	0	0	0
	0	1	0	0
	0	0	2	0
x	1	1	2	0
number of permutations	6			
coefficients of $X^3/3!$	$6T_1^2T_2$			

Figure 1: Paths of length 3

type	T_1	T_1	T_1^2		T_1	T_2^2	T_2		T_2^2	T_1	T_2	
0	0	0	0	0	0	0	0	0	0	0	0	0
path transitions	1	0	0	0	1	0	0	0	2	0	0	0
	0	1	0	0	0	2	0	0	2	0	0	0
	0	0	1	0	0	2	0	0	0	1	0	0
	0	0	1	0	0	0	2	0	0	0	2	0
x	1	1	2	0	1	1	2	0	1	1	2	0
number of permutations	12				12				12			
coefficients of $X^4/4!$	$12T_1^4$				$24T_1T_2^3$							

Figure 2: Paths of length 4

type	T_1	T_1	T_2	T_1T_2	T_1	T_1	$T_1T_2^2$		T_1	T_2^2	T_1^2		T_2^2	T_1	T_1^2	
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
path transitions	1	0	0	0	1	0	0	0	1	0	0	0	2	0	0	0
	0	1	0	0	0	1	0	0	0	2	0	0	2	0	0	0
	0	0	2	0	0	0	1	0	0	2	0	0	0	1	0	0
	0	0	0	1	0	0	2	0	0	0	1	0	0	0	1	0
	0	0	0	2	0	0	2	0	0	0	1	0	0	0	1	0
x	1	1	2	0	1	1	2	0	1	1	2	0	1	1	2	0
number of permutations	120				60				30				30			
coefficients of $X^5/5!$	$360T_1^3T_2^2$															

	$T_1^2T_2$	T_1	T_2		T_1	$T_1^2T_2$	T_2		T_2^2	T_2^2	T_2	
	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	1	0	0	0	2	0	0	0
	1	0	0	0	0	1	0	0	2	0	0	0
	2	0	0	0	0	1	0	0	0	2	0	0
	0	1	0	0	0	2	0	0	0	2	0	0
	0	0	2	0	0	0	2	0	0	0	2	0
	1	1	2	0	1	1	2	0	1	1	2	0
	60				60				30			
									$30T_2^5$			

Figure 3: Paths of length 5

3 A differential equation

To determine the inventories $S_{\bar{i},j}$ it is sufficient, by proposition 2, to calculate the exponential generating power series $f_0(X), f_{a_1}(X), \ldots, f_{a_m}(X)$. This will be done by solving a system of differential equations with coefficients in the field of rational functions in m variables $T_{a_1}, \ldots, T_{a_m}$ with rational coefficients.

We first observe the recurrence

$$f_s(j_1, \ldots, j_m) = \sum_{k=1}^{m} f_{s-a_k}(j_1, \ldots, j_k - 1, \ldots, j_m)$$

for all $s \in \mathbf{F}$.

This is because we obtain a sequence σ summing to s containing j_1 times $a_1, \ldots, j_m$ times a_m from a sequence summing to $s-a_k$ containing j_1 times $a_1, \ldots, (j_k - 1)$ times $a_k, \ldots$ just by adding an a_k. Moreover all sequences σ are obtained in this fashion.

In differential terms this gives

$$Df_s(X) = \sum_{k=1}^{m} T_{a_k} f_{s-a_k}(X), s \in \mathbf{F}$$

because the derivative $Df_s(X)$ of the series of definition 2, defined formally as usual, gives here

$$\begin{aligned} Df_s(X) &= \sum_{j\geq 1}[\sum_{j_1+\cdots+j_m=j} f_s(j_1, \ldots, j_m) T_{a_1}^{j_1} \cdots T_{a_m}^{j_m}] \frac{X^{j-1}}{(j-1)!} \\ &= \sum_{j\geq 1}[\sum_{j_1+\cdots+j_m=j} \sum_{k=1}^{m} f_{s-a_k}(j_1, \ldots, j_k - 1, \ldots, j_m) T_{a_1}^{j_1} \cdots T_{a_m}^{j_m}] \frac{X^{j-1}}{(j-1)!} \\ &= \sum_{j\geq 1}[\sum_{k=1}^{m} T_{a_k} \sum_{j_1+\cdots+j_m=j} f_{s-a_k}(j_1, \ldots, j_k - 1, \ldots, j_m) T_{a_1}^{j_1} \cdots T_{a_k}^{j_k - 1} \cdots T_{a_m}^{j_m}] \frac{X^{j-1}}{(j-1)!} \\ &= \sum_{k=1}^{m} T_{a_k} \sum_{j\geq 0} \left[\sum_{j_1+\cdots+j_m=j} f_{s-a_k}(j_1, \ldots, j_m) T_{a}^{j_1} \cdots T_{a_m}^{j_m} \right] \frac{X^j}{j!} \end{aligned}$$

This proves the following result.

Proposition 3 *The vector $[f_0(X), f_{a_1}(X), \ldots, f_{a_m}(X)]$ consisting of the exponential generating power series of definition 2.1 is the unique solution of the linear system*

$$** \quad Df_s = \sum_{k=1}^{m} T_{a_k} f_{s-a_k}, \quad s \in \mathbf{F} \tag{1}$$

with initial condition vector $[1, 0, \ldots\ 0]$.

Remark 1 We may consider this differential equation as having coefficients in the field $K = \mathbf{Q}(T_{a_1}, \dots, T_{a_m})$ and the solution we seek has components in the differential ring $K[[X]]$. Thanks to its abstraction capabilities, Scratchpad is able to solve easily and naturally such a problem whereas other computer algebra systems available nowaday seem not.

As an illustration, we give in the next two sections a manipulation by hand and a short Scratchpad session yielding the solution of our problem in the particular case where the alphabet is $\mathbf{F} = \mathbf{Z}/3\mathbf{Z}$.

4 A partial solution by hand in the ternary case $F = \{0, 1, 2\}$

Only the differential equation ** may be handled in conventional manner, the use of proposition 2 implying too much computations.

In the particular case under investigation the differential equation ** is

$$\left\{\begin{array}{lcll} Df_0 & = & T_2 f_1 + T_1 f_2 & \quad (2) \\ Df_1 & = & T_1 f_0 + T_2 f_2 & \quad (3) \\ Df_2 & = & T_2 f_0 + T_1 f_1 & \quad (4) \end{array}\right.$$

with initial condition vector $[1, 0, 0]$.

We also observe that, by construction, we have

$$f_0 + f_1 + f_2 = e^{(T_1+T_2)X} \tag{5}$$

because the RHS is the generating function of all the colored sequences of elements in $\{1, 2\}$ and the LHS corresponds to a partition of these.

Thanks to the relation (5) we shall obtain a scalar second order differential equation to determine f_2.

First by differentiating (4), multiplying (2) by T_2 and adding we obtain

$$D^2 f_2 = T_1 D f_1 + T_1 T_2 f_2 + T_2^2 f_1 \quad \text{or} \tag{6}$$

$$T_1 D f_1 + T_2^2 f_1 = D^2 f_2 - T_1 T_2 f_2.$$

Now multiplying (4) by T_1, (3) by $-T_2$ and adding we have

$$T_1 D f_2 - T_2 D f_1 = T_1^2 f_1 - T_2^2 f_2 \quad \text{or} \tag{7}$$

$$T_2 D f_1 + T_1^2 f_1 = T_1 D f_2 + T_2^2 f_2.$$

which together with (6) implies immediately

$$(T_2^3 - T_1^3)f_1 = T_2D^2f_2 - T_1^2Df_2 - 2T_1T_2^2f_2. \tag{8}$$

On the other hand, substituting f_0 from (5) into (3), we have

$$Df_1 = T_1(e^{(T_1+T_2)X} - f_1 - f_2) + T_2f_2 \tag{9}$$

and differentiating (5) taking into account (2), we obtain

$$Df_1 + Df_2 + T_2f_1 + T_1f_2 = (T_1 + T_2)e^{(T_1+T_2)X}. \tag{10}$$

Substituting Df_1 from (9) into (10) yield

$$(T_2 - T_1)f_1 = T_2e^{(T_1+T_2)X} - Df_2 - T_2f_2 \quad \text{or}$$

$$(T_2^3 - T_1^3)f_1 = (T_1^2 + T_1T_2 + T_2^2)(T_2e^{(T_1+T_2)X} - Df_2 - T_2f_2). \tag{11}$$

Finally comparing (11) with (8) we obtain

$$D^2f_2 + (T_1 + T_2)Df_2 + (T_1^2 - T_1T_2 + T_2^2)f_2 = (T_1 + T_2)e^{(T_1+T_2)X} \tag{12}$$

which is a second order differential equation to determine f_2 as a power series with polynomial coefficients in the indeterminates T_1 and T_2. The initial conditions are here $f_2(0) = 0$ and $Df_2(0) = T_2$.

Once f_2 determined, the series f_0 and f_1 are calculated by the relations (4) and (5).

5 Scratchpad solution in the ternary case $\mathbf{F} = \{0, 1, 2\}$

5.1 Solution of the differential equation **

Creation of the coefficient field

```
> K := QF P[T1, T2]I
```

Specification of the solution

```
> s := List UPS(X, K)
```

Specification of the right member of (1)

```
> (F1, F2, F3) : List UPS(X, K) -> UPS(X, K)
> F1 u == T2*u.2 + T1*u.3
> F2 u == T1*u.1 + T2*u.3
> F3 u == T2*u.1 + T1*u.2
```

Call to Scratchpad command to solve (1)

```
> s := mpsode([1$K, 0$K, 0$K], [F1, F2, F3])
```

Verification (must give zero)

```
> pderiv(s.0) - T2*s.1 - T1*s.2
> pderiv(s.1) - T1*s.0 - T2*s.2
> pderiv(s.2) - T2*s.0 - T1*s.1
```

5.2 Determination of the numbers $S_{\vec{i},j}$

Input data

```
> vi := [i.0,i.1]
> The user or program assigns positive integer values to n and N
```

Calculation of the product power series as in proposition 2

```
> S := s.1^i.0*s.2^i.1*s.0^(n-i.0-i.1)
```

List of the numbers $S_{vi,j}$ for $j \leq N$

```
> S.vi := [j!*coefficient (S,j) for j in 0..N]
```

References

[1] Camion P., Courteau B., Montpetit A., R-partition designs relative to a weight function in Hamming spaces, in preparation.

[2] Foata D., Schützenberger M.P., Théorie géométrique des polynômes eulériens, Lecture Notes in Mathematics N° 138, Springer Verlag, 1970.

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