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### ESTIMATES OF CYCLE TIMES IN STOCHASTIC PETRI NETS

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# Estimates of Cycle Times in Stochastic Petri Nets

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## Abstract

This paper focuses on the derivation of bounds and estimates for cycle times of strongly connected stochastic event graphs with i.i.d. holding times. We use association properties satisfied by partial sums of the holding times in order to prove that the firing epochs compare for stochastic ordering with the last birth in a multitype branching process, the structure of which is determined from the characteristics of the event graph using simple algebraic manipulations. Classical large deviation estimates are then used to compute the growth rate of this last birth epoch, following the method of Kingman and Biggins. The method allows one to derive a computable upper bound for the cycle time, and is exemplified on tandem queueing networks with communication blocking.

**Keywords:** Discrete Event Dynamic System, Stochastic Decision Free Petri Nets, Cycle Time, Throughput, Association, Large Deviations, Branching Processes.

# Estimées des temps de cycle dans les réseaux de Petri stochastiques

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## Résumé

Nous étudions dans cet article des bornes sur le temps de cycle de graphes marqués fortement connexes avec des 'temporisations' indépendentes et identiquement distribuées. Nous utilisons les propriétés d'association satisfaites par certaines sommes partielles des variables de temporisation pour montrer que les dates de tir d'un tel réseau sont bornées supérieurement par les dates de dernière naissance dans un processus de branchement multitype, dont la structure est déterminée à partir de la topologie du réseau de Petri. Des méthodes d'estimation dues à Kingman et Biggins, fondées sur les grandes déviations, sont alors utilisées pour calculer explicitement le taux de croissance des dernières naissances dans ce processus de branchement. Cette méthode permet donc de calculer simplement des bornes supérieures sur les temps de cycle, comme nous le montrons sur un exemple de réseau de files d'attente en tandem avec blocage.

**Mots-clés:** Systèmes à événements discrets, réseaux de Petri, temps de cycle, débit, association, grandes déviations, processus de branchement.

# 1 Introduction

This paper focuses on computational problems arising in the analysis of Stochastic Decision Free Petri Nets (SDFPN). The SDFPNs under consideration consist of a subclass of stochastic Petri nets, and are also called marked graphs or event graphs in the literature. A brief description of such networks together with the basic stochastic ordering concepts to be used in the paper are provided in §2 and §3.

The precise aim of this paper is the derivation of bounds for cycle times of strongly connected stochastic event graphs with i.i.d. holding times. Lower bounds based on convex ordering are already available (see [4]) under general statistical assumptions, and we will here focus on the derivation of computable upper bounds.

The derivation of our results is based on the evolution equations satisfied by firing times that were established in [1]. Section 4 summarizes related results on the association of the firing times that were obtained in [4], and that are of use in the estimates of the following sections

In §5, we first treat a simple example that allows one to introduce the techniques used in the paper. The general case is considered in §6. We use association properties satisfied by partial sums of the holding times in order to prove that the firing epochs are bounded from above for the stochastic ordering by the last birth in a multitype branching process, the structure of which is determined from the characteristics of the event graph. Classical large deviation estimates are then used to compute the growth rate of this last birth epoch, following the method of Kingman and Biggins. The method is exemplified on two simple examples of finite capacity tandem queueing networks with communication blocking.

This method also provides a way for analyzing the stability region of non strongly connected event graphs using the stability theorem based on the comparison of the cycle times of the strongly connected components of the graph proved in [1].

Large deviation results were already used in the literature for analyzing the growth of the longest branch in random graphs arising in computer science applications (see in particular [8]); Estimates of the type used in the uniform bounds of §6 were also shown to be useful for analyzing the stability region of certain models with infinitely many resources (see [2]). The contributions of the present paper are to point out that the method of Kingman ([12]) and Biggins ([7]) can be applied to the estimation of the cycle time (and hence to the stability region) of any finite stochastic event graph, and to provide the algebraic transformations that should be operated on the structure of the event graph in order to generate the relevant multitype branching process.

## 2 Notation and Definitions

### 2.1 Model Description

The basic model of this paper is a Stochastic Decision Free Petri Net with recycled transitions. We assume that tokens incur no sojourn times in places. The definition of this class of Petri net is sketched below (see [1] for more details on the matter).

- $\mathcal{T} = \{1, \dots, I\}$ : the set of transitions;
- $\pi(\cdot)$ : the predecessor function ( $\pi(j)$  is the set of transitions preceding  $j$ );
- $\sigma(\cdot)$ : the successor function ( $\sigma(j)$  the set of transitions that follow transition  $j$ );
- $\Gamma = (\mathcal{V}, \mathcal{E})$ : the directed graph defined by the precedence relation  $\pi$  on the set  $\mathcal{V} = \mathcal{T}$ ;
- $\mathcal{P}$ : the set of places. Each place is preceded and followed by exactly one transition (this is the so called decision free property). It is assumed that there is at most one place between two transitions. There is a place between  $j$  and  $j'$  iff  $(j, j') \in \mathcal{E}$ ; this place will be denoted  $(j, j')$ ;
- $\mu(j, j') \in \mathbb{N}$ : the initial marking in place  $(j, j') \in \mathcal{P}$ .
- $M = \max_{(j, j') \in \mathcal{P}} \mu(j, j')$  is the maximum initial marking;
- $\alpha_j(k) \in \mathbb{R}^+$  is the holding time of the  $k$ -th firing of transition  $j \in \mathcal{T}$ ,  $k \geq 1$ , namely the time it takes for transition  $j$  to fire when it is enabled for the  $k$ -th time.

All the transitions are assumed to be FIFO (see [1]). A simple condition for a transition to be FIFO is that it be *recycled* (vz., for all  $1 \leq j \leq I$ ,  $j \in \pi(j)$  and the place  $(j, j)$  has an initial marking  $\mu(j, j) = 1$ ). A transition with constant holding times is FIFO too. The evolution of the SDFPN is characterized by the circulation of *tokens*, which stay in places, and are consumed and created by transitions. A transition  $j$  is enabled to *fire* when there is at least one token in each of the places  $(i, j)$ ,  $i \in \pi(j)$ . The firing consumes one token of each of these places and creates, after some holding time  $\alpha_j(k)$ ,  $k \geq 1$ , one token into each of the places  $(j, j')$ ,  $j' \in \sigma(j)$ . We assume that the firing of a transition, takes place as soon as it is enabled. In the literature, SDFPN's are also called *marked graphs* ([10]) or *event graphs* ([1,9]).

Without loss of generality, we can assume that the SDFPN is connected. Moreover, in order to guarantee the *liveness* of the SDFPN (i.e., each transition fires infinitely many

times), we assume that for each cycle in the graph  $\Gamma$ , there is at least one place with a positive initial marking ([10]).

## 2.2 Statistical Assumptions and Definitions

Throughout this paper, we assume that the sequences  $\{\alpha_j(k)\}_{k=1}^{+\infty}$ ,  $j = 1, \dots, I$ , are mutually independent sequences of i.i.d. non-negative and integrable RV's (random variables) defined on a common probability space  $(\Omega, F, P)$ .

We will use the notion of association of random variables, and the notion of stochastic ordering between random variables, that will be denoted  $\leq_{st}$  (see [6] for the relevant definitions and for the basic properties that will be used in the sequel). The use that will be made of these two notions is essentially that advocated in [5]. Equivalence in law will be denoted  $=_{st}$ .

## 3 Evolution Equations

In this subsection, we summarize results that were obtained in [1] In comparison to the results of this reference, we limit ourselves to the case of holding times in transitions. Let  $X_j(k)$  denote the time when transition  $j$  starts firing for the  $k$ -th time. Whenever the initial condition is zero, these variables satisfy the evolution equation

$$X_j(k) = \max_{\{i \in \pi(j)\}} (X_i(k - \mu(i, j)) + \alpha_i(k - \mu(i, j))), \quad k = 1, 2, \dots, \quad (1)$$

where, by convention, the maximum over an empty set is  $-\infty$ . The way to define the initial condition of this equation is somewhat intricate, and we will skip this question for this formulation of the evolution equation.

Since the decision free net is live, the numbering of the transitions can be chosen in such a way that for all  $(j, k)$ ,  $j = 1 \dots, I$ ,  $k \geq 1$ , the variables  $X_{j'}(k')$  that are found in the R.H.S of (1) are always such that either  $k' < k$  or  $k' = k$ ,  $j' < j$ . Therefore, the state variables  $X_j(k)$  can be computed recursively in the order

$$X_1(1), X_2(1), \dots, X_I(1), X_1(2), X_2(2), \dots, X_I(2), \dots, X_1(k), X_2(k), \dots, X_I(k), \dots \quad (2)$$

Consider the semi-ring  $(\mathbb{R}, \oplus, \otimes)$ , where  $\oplus$  is max and  $\otimes$  is +. If the SDFPN under consideration is live, it is shown in [1] that one can rewrite this equation in matrix form in this semi-ring

$$X(k) = X(k - M) \otimes A(k - M, k) \oplus \dots \oplus X(k - 1) \otimes A(k - 1, k), \quad k = 1, 2, \dots \quad (3)$$

In this equation, the row vector  $X(k) = (X_1(k), \dots, X_J(k))$ ,  $J \leq I$ , is the restriction of  $(X_1(k), \dots, X_I(k))$  to the set of transitions followed by at least one place with at least one token in the initial marking.

The initial condition is  $X(0)$ . This initial condition can for instance be taken equal to  $(0, \dots, 0)$ . Any other licit initial condition  $X(0) \in \mathbb{R}^I$  will lead to the same cycle times. Hence, we will assume in the sequel without loss of generality that  $X(0) = 0$ .

In this equation, the matrix  $A(k-l, k)$  is defined as follows: let  $\mathcal{S}(j', j, l)$  be the set of paths in the graph  $\Gamma$  with at least two transitions, with initial vertex  $j'$  and final vertex  $j$ , and such that the first two transitions of the path are connected by a place with initial marking equal to  $l$ , while the other transitions are connected by places with zero initial marking. Then

$$A(k-l, k)_{j', j} = \bigoplus_{\{ (j'=i_0, i_1, j_2, i_2, \dots, i_{h-1}, i_h=j) \in \mathcal{S}(j', j, l) \}} \alpha_{j'}(k-l) \otimes \left( \bigotimes_{m=1}^{h-1} \alpha_{i_m}(k) \right), \quad (4)$$

with the usual convention if the set  $\mathcal{S}(j', j, l)$  is empty. The entry  $A(k-l, k)_{j', j}$  is hence simply the length of the longest path in  $\mathcal{S}(j', j, l)$ .

## 4 Basic Model

In this paper, we will consider a live stochastic event graph with all its transitions recycled. We will assume this event graph to be strongly connected. We will also assume that the initial marking is 0 – 1 valued. This last assumption is not a strong one as it was shown in [1] that for any stochastic event graph, one can always find an ‘equivalent’ event graph (with a possibly different topology, where the places and transitions that are added all have zero holding times), where the initial marking is 0 – 1-valued. The matrix  $A(k-1, k)$  will be denoted  $A(k)$  for sake of simplicity. In this case the entries of  $A(k)$  are found from equation (3):

$$X(k) = X(k-1) \otimes A(k), \quad (5)$$

with initial condition  $X(0)$ , and formula (4) is written as

$$A(k)_{j', j} = \max_{\{ (j'=i_0, i_1, j_2, i_2, \dots, i_{h-1}, i_h=j) \in \mathcal{S}(j', j, 1) \}} \alpha_{j'}(k-1) + \alpha_{i_1}(k) + \dots + \alpha_{i_{h-1}}(k). \quad (6)$$

## 5 Statistical Properties



**Lemma 5.1** *Under the foregoing statistical assumptions,*

$$\{A(k)_{i,j}, X_j(k), i, j = 1, \dots, J, k \geq 0\}$$

*forms a set of associated random variables.*

**Proof.** It is easy to prove that the i.i.d. assumption on the  $\alpha$ 's implies that the random variables  $\{A(k-1, k)_{i,j}, k \in \mathbb{N}, i, j = 1, \dots, J\}$  are associated in view of the fact that they are obtained as an increasing function of associated random variables (see (4)). The result for  $X_j(k)$  follows immediately from (1). ■

**Lemma 5.2** *For all  $j_0, j_1, \dots, j_n, \dots \in \{1, \dots, J\}$ , the random variables  $\{A(n)_{j_n, j_{n+1}}, n \in \mathbb{N}\}$  are mutually independent.*

**Proof.** The random variables  $A(n)_{j_n, j_{n+1}}$  and  $A(m)_{j_m, j_{m+1}}$  are clearly independent for  $m > n + 1$  in view of Equation (4) (for instance,  $A(n)$  is measurable with respect to the  $\sigma$ -field endowed by the random variables  $\alpha(n)$  and  $\alpha(n-1)$ , while  $A(n+2)$  is measurable with respect to the one endowed by the random variables  $\alpha(n+2)$  and  $\alpha(n+1)$ , and the independence follows from the independence assumptions on the  $\alpha$ 's).

We show that this independence property also holds for the random variables  $A(n)_{j_n, j_{n+1}}$  and  $A(n+1)_{j_{n+1}, j_{n+2}}$ . By looking at equation (6) we see that  $A(n)_{j_n, j_{n+1}}$  is a function of  $\alpha(n-1)$  and  $\alpha(n)$ , while  $A(n+1)_{j_{n+1}, j_{n+2}}$  is a function of  $\alpha_{j_{n+1}}(n)$  and  $\alpha(n+1)$ . By assumption  $\{\alpha(n)\}_{n=1}^\infty$  is an independent sequence, so the only possible source of dependency of  $A(n+1)_{j_{n+1}, j_{n+2}}$  and  $A(n)_{j_n, j_{n+1}}$  is the existence of the term  $\alpha_{j_{n+1}}(n)$  in the expression (6) for  $A(n)_{j_n, j_{n+1}}$ . Recalling the definition of  $\mathcal{S}(j_n, j_{n+1}, 1)$  we conclude that this is possible if and only if there exists a circuit from  $j_{n+1}$  to  $j_{n+1}$  with zero initial marking in all the places of the circuit, which contradicts the liveness assumption. ■

## 6 Simple Bounds on Cycle Times

Using sub-additive ergodic theory, the following lemma was shown in [1]:

**Lemma 6.1** *Under the foregoing assumptions, there exists a positive constant  $\gamma$  such that*

$$\lim_{k \rightarrow \infty} X^*(k)/k = \lim_{k \rightarrow \infty} E[X^*(k)]/k = \gamma \quad \text{a.s.}, \quad (7)$$

where  $X^*(k) = \max_j X_j(k)$ . In addition, if the event graph under consideration is strongly connected,

$$\lim_{k \rightarrow \infty} E[X_j(k)]/k = \lim_{k \rightarrow \infty} X_j(k)/k = \gamma \quad a.s., \quad \forall j = 1, \dots, J. \quad (8)$$

In the strongly connected case, the constant  $\gamma$  is called the cycle time of the Petri net. The reason for this terminology comes from the deterministic case that was studied in [9].

The basic data of the present section is a strongly connected event graph that satisfies the statistical assumptions of §2.2, and the assumption  $M = 1$ . Let

- $N$  be the maximal out-degree of the transitions that are followed by at least one place with a non-zero initial marking (the out-degree of a vertex is the number of edges starting from this vertex).
- $b$  be a random variable that is a  $\leq_{st}$  upper bound of each of the random variables  $A(1)_{i,j}$ , namely

$$A(1)_{i,j} \leq_{st} b, \quad \forall i, j = 1, \dots, J. \quad (9)$$

Let  $b^*(\theta) = E[e^{\theta b}]$  (since  $b \geq 0$ , this function is defined at least for  $Re(\theta) \geq 0$ ).

- Let  $M(x)$  be the Cramér-Legendre transform of the distribution function of the random variable  $b$ , namely

$$M(x) = \sup_{\theta \in \mathbb{R}} \{\theta x - \log(b^*(\theta))\}. \quad (10)$$

The present section is devoted to the proof of the following result:

**Proposition 6.2** *Under the foregoing assumptions, the cycle time of the event graph admits the upper bound*

$$\gamma \leq \inf\{x > Eb \text{ such that } M(x) > \log(N)\}. \quad (11)$$

We start with two preliminary lemmas.

**Lemma 6.3** *For all  $\epsilon > 0$ , and for all  $j = 1, \dots, J$ ,*

$$\lim_{k \rightarrow \infty} P\left[\frac{X_j(k)}{k} < \gamma + \epsilon\right] = 1, \quad (12)$$

and

$$\lim_{k \rightarrow \infty} P\left[\frac{X_j(k)}{k} < \gamma - \epsilon\right] = 0. \quad (13)$$

**Proof** The property follows immediately from the fact that a.s. convergence implies convergence in probability and from (8).  $\blacksquare$

**Lemma 6.4** *If  $\beta \in \mathbb{R}$  is such that*

$$\lim_{k \rightarrow \infty} P[X_j(k) - k\beta \leq 0] = 1, \quad (14)$$

*for some  $j = 1, \dots, J$ , then  $\beta \geq \gamma$ .*

**Proof** Under the assumption (14),

$$\lim_{k \rightarrow \infty} P\left[\frac{X_j(k)}{k} \leq \beta\right] = 1, \quad (15)$$

so that we cannot have  $\beta = \gamma - \epsilon$  for some  $\epsilon > 0$ , in view of Lemma 6.3. Therefore,  $\beta \geq \gamma$ .  $\blacksquare$

**Proof of Proposition 6.2** From Equation (5) with initial condition  $X(0) = 0$ , it is easily checked by induction that

$$\begin{aligned} X_j(k) &= \bigoplus_{j_0, \dots, j_{k-1} \in \{1, \dots, J\}} \bigotimes_{n=1}^k A(n)_{j_n, j_{n+1}} \\ &= \max_{j_0, \dots, j_k \in \{1, \dots, J\}} \sum_{n=1}^k A(n)_{j_n, j_{n+1}}, \end{aligned} \quad (16)$$

where  $j_k = j$ . Therefore,

$$P[X_j(k) - \beta k \leq 0] = P\left[\max_{j_0, \dots, j_k \in \{1, \dots, J\}} \sum_{n=1}^k \tilde{A}(n)_{j_n, j_{n+1}} \leq 0\right], \quad (17)$$

where  $\tilde{A}(k)_{i,j} = A(k)_{i,j} - \beta$ .

For  $k$  fixed, Lemma 6.3 implies that the variables  $\sum_{n=1}^k \tilde{A}(n)_{j_n, j_{n+1}}$ , where  $j_0, \dots, j_{k-1}$  varies over the set  $\{1, \dots, J\}^k$ , are associated. Therefore (see [6])

$$P\left[\max_{j_0, \dots, j_{k-1} \in \{1, \dots, J\}} \sum_{n=1}^k \tilde{A}(n)_{j_n, j_{n+1}} \leq 0\right] \geq \prod_{j_0, \dots, j_{k-1} \in \{1, \dots, J\}} P\left[\sum_{n=1}^k \tilde{A}(n)_{j_n, j_{n+1}} \leq 0\right]. \quad (18)$$

Since the random variables  $A(n)_{j_n, j_{n+1}}$  are independent (see Lemma 5.2), we have then

$$P\left[\sum_{n=1}^k \tilde{A}(n)_{j_n, j_{n+1}} \leq 0\right] \geq P\left[\sum_{n=1}^k (b(n) - \beta) \leq 0\right], \quad (19)$$

where  $\{b(n)\}$  is a sequence of i.i.d. random variables with the same distribution function as  $b$ . Observe first that if  $\beta < Eb$  then the R.H.S. of (18) converges to zero. This is not what we want, so we may as well assume at this point that  $\beta > Eb$ . Now, Chernoff's Theorem implies

$$P\left[\sum_{n=1}^k b(n) > \beta k\right] = e^{-M(\beta)k+o(k)}, \quad (20)$$

so that

$$P[X^j(k) - \beta k \leq 0] \geq (1 - e^{-M(\beta)k+o(k)})^{C_j(k)}, \quad (21)$$

where  $C_j(k)$  denotes the number of paths  $j_0, \dots, j_{k-1}$  such that  $\sum_{n=1}^k \tilde{A}(n)_{j_n, j_{n+1}} \neq -\infty$ . (It is interesting to observe at this point that  $C_j(k)$  can also be defined in terms of the adjacency matrix of the event graph in the usual path-counting way.) Therefore, if

$$C_j(k)e^{-kM(\beta)} \rightarrow 0, \quad (22)$$

when  $k$  goes to  $\infty$ , then

$$\lim_{k \rightarrow \infty} P[X^j(k) - \beta k \leq 0] = 1. \quad (23)$$

Clearly, the bound  $C_j(k) \leq N^k$  holds, so that a sufficient condition for (23) to hold is  $M(\beta) > \log(N)$ . In other words, for  $\beta$  such that  $M(\beta) + \log N < 0$ , (23) holds, so that  $\beta \geq \gamma$  in view of Lemma 6.4. ■

In fact, we proved the following and more general result

**Proposition 6.5** *If  $\log(C_j(k)) = Ck + o(k)$ , then*

$$\gamma \leq \inf\{\beta > Eb \text{ such that } M(\beta) > C\}. \quad (24)$$

In fact  $C$  is then necessarily the Perron-Frobenius eigenvalue of the adjacency matrix.

## 6.1 Example: 1-Buffer Blocking Queues

Consider the example of Figure 1, which represents a line of processors with blocking before service. Let  $J$  denote the number of processors, each of which is represented by a transition. In Figure 1,  $J=4$ . The first processor (on the left of the figure) has an infinite supply of items to serve. Between two successive processors, the buffer is of capacity two,

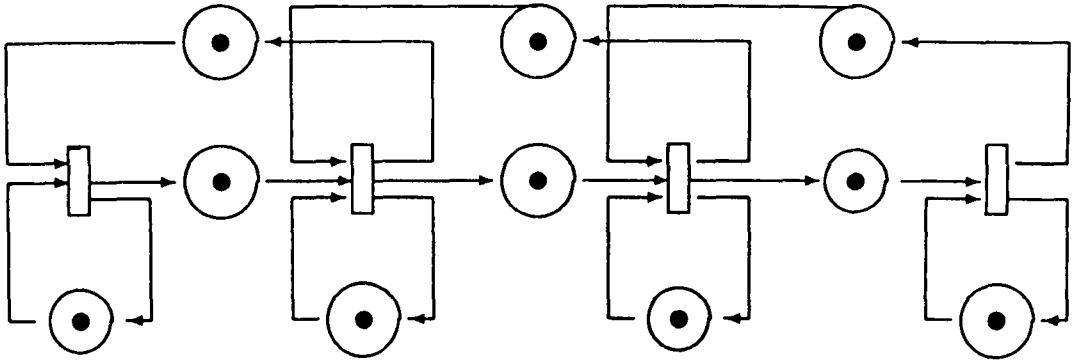


Figure 1: Communication Blocking: 4 Nodes, 1 Buffer

including the one in service (which is captured by the fact that there are two tokens in any of the upper circuits originating from a processor). The processors are single servers with a FIFO discipline (which is captured by the lower circuit associated with each transition) It is assumed that all transitions have exponentially distributed holding times with parameter 1. In this example, we have  $N = 3$ ,  $b^*(\theta) = (1 - \theta)^{-1}$  and

$$\frac{1}{k} \log C_j(k) = 1 + 2 \cos\left(\frac{\pi}{J+1}\right) + o(k) \leq 3 \quad (25)$$

(see the Appendix for the last formula). The Cramér-Legendre transform is given by

$$M(x) = \sup_{\theta \in \mathbb{R}} \theta x + \log(1 - \theta).$$

The derivative of the function  $\theta x + \log(1 - \theta)$  w.r.t.  $\theta$  vanishes for  $\theta = 1 - x^{-1}$  and this point is a minimum. Therefore

$$M(x) = x - \log(x) - 1.$$

As a direct application of Proposition 6.2, we get

$$\gamma \leq \inf\{x \mid x - \log(x) - 1 > \log(3)\}$$

which provides the following uniform bound in  $J$ :

$$\gamma \leq 3.33.$$

In other words, the throughput of the systems is always larger than .3, regardless of the number of processors. If we apply Proposition 6.5 using the more precise estimate of  $C_j(k)$  given in (25), we get

$$\gamma \leq \inf\{x \mid 1 - x + \log(x) < \log(1 + 2 \cos(\frac{\pi}{5}))\} \simeq 3.09$$

If the service times are Erlang-3 with mean 1, namely if  $b^*(\theta) = (3/(3 - \theta))^3$ , we get in the same way from Proposition 6.2 that

$$\gamma \leq \inf\{x \mid 3(x - 1 - \log(x)) > \log(3)\} \simeq 2.11,$$

that is a throughput of .48.

## 7 General Case

We start with Equation (3) with  $M = 1$ , which reads

$$X_j(k) = \max_{i \in \pi^*(j)} \{X_i(k-1) + A(k)_{i,j}\}, \quad (26)$$

where  $\pi^*(j)$  denotes the set of vertices  $i \in \{1, \dots, J\}$ , such that  $A(1)_{i,j} \neq -\infty$ .

An age dependent branching process with  $J$  types (see [7]) is characterized by random variables  $Z_{j,i}(t) \in \mathbb{R}$  counting the number of individuals of type  $i$  born in the first generation and by time  $t \in \mathbb{R}^+$ , from an individual of type  $j$ , where the offspring processes  $Z_j(t) = (Z_{j,1}(t), \dots, Z_{j,J}(t))$  associated with the individuals are assumed to be mutually independent in  $j$ .

Consider now the following specific age-dependent branching process associated with the stochastic event graph under consideration:

- there are as many types as transitions in the graph (more precisely as transitions followed by at least one place with a nonzero initial marking);
- the variables  $Z_{j,i}(t)$  are defined as follows
  - for all  $j$ , the  $J$  stochastic processes  $\{Z_{j,i}(t), t \in \mathbb{R}^+\}$ ,  $i \in \{1, \dots, J\}$  are mutually independent;
  - the law of  $Z_{j,i}(t)$  is defined by  $Z_{j,i}(t) \stackrel{st}{=} 1_{i \in \pi^*(j)} 1_{A(1)_{i,j} \leq t}$ , for all  $i, j \in \{1, \dots, J\}$ .

Finally, let  $\hat{X}_j(k)$  be the time of the latest birth of the  $k$ -th generation in this branching process when initiated at time 0 by an individual of type  $j$ . Taking  $\hat{X}_j(0) = 0$ , for all  $j$ , it is easily checked that these random variables satisfy the recurrence relations

$$\hat{X}_j(k) = \max_{i \in \pi^*(j)} \hat{X}_i(k-1) + \bar{A}(k)_{i,j}. \quad (27)$$

These relations hold only because each individual can have *at most one* child of each type. For any random vector  $B \in \mathbb{R}^K$ ,  $\bar{B}$  denotes the ‘product form version’ of  $B$ , namely

the vector defined in law by the relations  $\overline{B}_j =_{st} B_j$ , and such that its coordinates are mutually independent. For instance, the matrices  $\overline{A}(k)$  are such that  $\overline{A}(k)_{i,j} =_{st} A(k)_{i,j}$  and the random variables  $\overline{A}(k)_{i,j}, k \geq 1, i, j = 1, \dots, J$  are mutually independent.

**Lemma 7.1** *Under the foregoing statistical assumptions, for all  $j \in \{1, \dots, J\}$  and  $k \geq 1$ ,*

$$X_j(k) \leq_{st} \hat{X}_j(k). \quad (28)$$

**Proof** The proof is by induction on  $k$ . For  $k = 1$ , we have

$$\begin{aligned} X_j(1) &= \max_{i \in \pi^*(j)} A(1)_{i,j}, \\ &\leq_{st} \max_{i \in \pi^*(j)} \overline{A}(1)_{i,j}, \\ &=_{st} \hat{X}_j(1), \end{aligned} \quad (29)$$

where the first inequality comes from the association of the entries of  $A(1)$  (Lemma 5.1) and from the basic theorem on the maximum of associated random variables (see [6]). Assume now that the property holds up to rank  $k - 1$ . Then, using the association property of lemma 5.1, we get

$$\begin{aligned} X_j(k) &= \max_{i \in \pi^*(j)} X_i(k-1) + A(k)_{i,j}, \\ &\leq_{st} \max_{i \in \pi^*(j)} \overline{X_i(k-1) + A(k)_{i,j}}, \\ &=_{st} \max_{i \in \pi^*(j)} \overline{X_i(k-1)} + \overline{A(k)_{i,j}}, \end{aligned} \quad (30)$$

where the last equivalence comes from the independence between  $X_i(k-1)$  and  $A(k)_{i,j}$  (see Lemma 5.2 and (16)). Now, from the independence assumption,  $X_i(k-1) \leq_{st} \hat{X}_i(k-1)$  for all  $i \in \{1, \dots, J\}$ . Therefore, the monotonicity of the map  $x \in \mathbb{R}^J \rightarrow \max_{i \in \pi^*(j)} \{x_i + \overline{A}(k)_{i,j}\}$  implies

$$\begin{aligned} X_j(k) &\leq_{st} \max_{i \in \pi^*(j)} \hat{X}_i(k-1) + \overline{A}(k)_{i,j}, \\ &=_{st} \hat{X}_j(k). \end{aligned} \quad (31)$$

■

let  $F_{j,i}(t)$  be the monotone function defined by the relation

$$F_{j,i}(t) = E[Z_{j,i}(t)] = P[A_{i,j}(1) \leq t], \quad t \in \mathbb{R}^+, \quad (32)$$

(that is,  $F_{j,i}(t)$  is the distribution of the first birth time) and  $\Phi(\theta)$  be the  $J \times J$  matrix with entries

$$\Phi_{j,i}(\theta) = \int_0^\infty e^{\theta t} F_{j,i}(dt), \quad (33)$$

if  $i \in \pi^*(j)$  and zero otherwise. where  $\theta$  is some real valued parameter. Whenever the integrals defining the entries converge, the matrix  $\Phi(\theta)$  is positive, and we denote by  $\phi(\theta)$  its Perron-Frobenius eigenvalue. It should be observed at this point that  $\Phi(0)$  is simply the 0-1 adjacency matrix of the event graph. Its Perron-Frobenius eigenvalue,  $\phi(0)$ , is therefore strictly larger than 1 (except in trivial cases). Let  $M(x)$  be the Cramér-Legendre transform of  $\phi(\theta)$ :

$$M(x) = \sup_{\theta > 0} (\theta x - \log \phi(\theta)), \quad (34)$$

It is well known that  $M(x)$  is increasing for  $x \geq 0$ . Let  $\Gamma$  be defined by

$$\Gamma = \inf\{x \mid M(x) > 0\}. \quad (35)$$

We are now in a position to state the main theorem, which generalizes Proposition 6.2:

**Theorem 7.2** *Under the foregoing statistical assumptions, the cycle time  $\gamma$  of the event graph satisfies the bound*

$$\gamma \leq \Gamma. \quad (36)$$

**Proof** From Lemma 7.1, for all bounded and nondecreasing function  $f$ ,

$$E[f(\frac{X_j(k)}{k})] \leq E[f(\frac{\hat{X}_j(k)}{k})], \quad (37)$$

for all  $j = 1, \dots, J$ .

We first prove that

$$\limsup_k \frac{[\hat{X}_j(k)]}{k} \leq \Gamma, \quad a.s., \quad (38)$$

following closely the method proposed by Biggins [7] for analyzing this type of branching processes (in fact Biggins proves that  $\lim_k [\hat{X}_j(k)]/k = \Gamma$ , and we only summarize here the relevant part of his paper allowing one to establish (38)).

Denoting by  $Z_{j,i}^k(t) \in IN$  the total number of individuals of type  $i$  of the  $k$ -th generation born by time  $t$  from an initial individual of type  $j$ , we get the vectorial relation

$$E[\int_0^\infty e^{\theta t} dZ_j^{k+1}(t) | F_k dt] = (\int_0^\infty e^{\theta t} dZ_j^k(t) dt) \Phi(\theta),$$



where  $F_k$  denotes the  $\sigma$ -field of the events up to the  $k$ -th generation. Taking expectations in the last expression, we get the relation

$$E[\int_0^\infty e^{\theta t} dZ_{j,i}^k(t)] = \Phi_{j,i}^k(\theta), \quad (39)$$

where  $\Phi^k$  denotes the  $k$ -th power of  $\Phi$ . Let  $v(\theta)$  be the right eigenvector associated with the maximal eigenvalue  $\phi(\theta)$ . We get from (39) that

$$\langle E[\int_0^\infty e^{\theta t} dZ_j^k(t)dt], v(\theta) \rangle = \phi^k(\theta)v_j(\theta), \quad (40)$$

so that

$$\langle E[\int_0^\infty e^{\theta t} dZ_j^k(t)dt], 1 \rangle \leq \phi^k(\theta)v_j(\theta)u(\theta), \quad (41)$$

where  $u(\theta) = (\min_i v_i(\theta))^{-1}$  ( $v(\theta)$  is strictly positive due to Perron-Frobenius). Now, since  $\hat{X}_j(k) = \sup\{t \mid \exists i = 1, \dots, J, Z_{j,i}^k(t) = 0\}$  we have

$$E[e^{\theta \hat{X}_j(k)}] \leq \langle E[\int_0^\infty e^{\theta t} Z_j^k(dt)], 1 \rangle, \quad (42)$$

In addition, for  $\theta \geq 0$ , we have the obvious bound

$$P[\frac{\hat{X}_j(k)}{k} \geq \beta] \leq E[e^{\theta(\frac{\hat{X}_j(k)}{k} - \beta)}]. \quad (43)$$

This plus (41) and (42) in turn imply

$$\lim_k \frac{1}{k} \log P[\hat{X}_j(k) \geq k\beta] \leq \sup_{\theta > 0} (\theta\beta - \log(\phi(\theta))) = -M(\beta). \quad (44)$$

Therefore, for all  $\beta$  such that  $M(\beta) > 0$ ,  $\sum_{k \geq 1} P[\hat{X}_j(k) \geq k\beta] < \infty$ , so that the Borel-Cantelli Lemma immediately implies (38).

In view of (8), we get from Lebesgue dominated convergence theorem that for  $f$  bounded and continuous

$$\lim_k E[f(\frac{\hat{X}_j(k)}{k})] = f(\gamma). \quad (45)$$

For  $f$  continuous, monotone nondecreasing and bounded, we also have

$$\limsup_k E[f(\frac{\hat{X}_j(k)}{k})] \leq E[\limsup_k f(\frac{\hat{X}_j(k)}{k})] = E[f(\limsup_k \frac{\hat{X}_j(k)}{k})] \leq f(\Gamma), \quad (46)$$

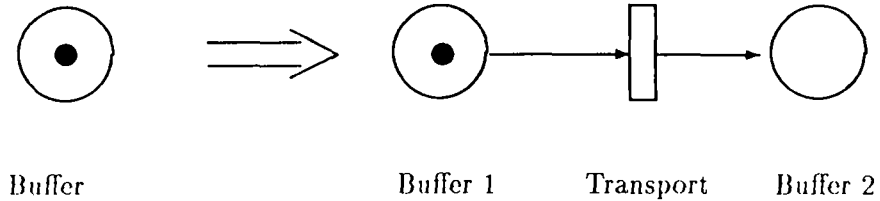


Figure 2: Blocking with Transportation Times

where we successively used Fatou's lemma, the monotonicity and continuity of  $f$  and (38). Therefore  $f(\gamma) \leq f(\Gamma)$  for all nondecreasing continuous and bounded  $f$ , which immediately implies (36). ■

Observe that in the particular case where all non  $-\infty$  entries of  $A(1)$  have the same distribution characterized by the function  $b^*(\theta)$ , the eigenvalue of interest is precisely

$$\phi(\theta) = b^*(\theta)C, \quad (47)$$

where  $C$  is the Perron-Frobenius eigenvalue of the adjacency matrix associated with the matrix  $A$ , namely the maximal eigenvalue of the matrix  $\Phi(0)$ .

### 7.1 Example: Blocking Queues with Transportation Times

The example is that of the line of processors described previously, but with a deterministic transportation time between processors. The associated event graph is obtained from that of Figure 1 by replacing each buffer by two buffers connected by a transportation transition with deterministic holding times  $\delta$  as shown in Figure 2. The statistical assumptions concerning the holding times of transitions associated with processors are those of the previous example. In this example, we have

$$\Phi(\theta) = \begin{pmatrix} \frac{1}{1-\theta} & \frac{1}{1-\theta} & 0 & 0 \\ \frac{e^{\delta\theta}}{1-\theta} & \frac{1}{1-\theta} & \frac{1}{1-\theta} & 0 \\ 0 & \frac{e^{\delta\theta}}{1-\theta} & \frac{1}{1-\theta} & \frac{1}{1-\theta} \\ 0 & 0 & \frac{e^{\delta\theta}}{1-\theta} & \frac{1}{1-\theta} \end{pmatrix}. \quad (48)$$

It is proved in the Appendix that the Perron-Frobenius eigenvalue of this matrix is

$$\phi(\theta) = \frac{1}{1-\theta} \left( 1 + 2e^{\frac{\delta\theta}{2}} \cos\left(\frac{\pi}{J+1}\right) \right). \quad (49)$$

The technique is then the same as above for deriving the upper bound  $\Gamma$ . The lower bound  $g$  given in the following arrays are those obtained by convex ordering following the method

indicated in [4]. For  $J = 4$ , one gets

$\delta$	0	1	2	3
$\Gamma$	3.1	3.3	3.7	4.2
$g$	1	1.5	2	2.5

For  $J$  large, the lower bound is unchanged, and we get the following upper bound

$\delta$	0	1	2	3
$\Gamma$	3.3	3.6	4.0	4.4

## 8 Appendix: 1-Buffer Blocking Queues

We give here a probabilistic solution to the Path Counting Argument of interest here. Let  $Q$  denote the adjacency matrix of the event graph, namely the  $J \times J$  matrix such that  $Q_{i,j} = 1$  if  $i \in \pi(j)$ , and zero otherwise. For  $J = 4$ , we get for instance

$$Q = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (50)$$

It is easily checked by induction that  $Q_{i,j}^k$  counts the number of paths of length  $k$  from  $i$  to  $j$ . Let  $M$  be the substochastic matrix defined by  $M = Q/3$ , and let  $\lambda_J$  denote the Perron-Frobenius eigenvalue associated with  $M$ . From the irreducibility of  $M$ , we get

$$C_j(k) = \sum_{i=1}^J Q_{i,j}^k = \Theta(3\lambda_J)^k.$$

In order to evaluate  $\lambda_J$ , we introduce the Markov chain  $Z_k$  with substochastic transition matrix  $M$  and uniform initial measure. We have then

$$P[Z_k = j] = \Theta(\lambda_J)^k$$

We can now evaluate  $P[Z_k = j]$  using the recurrence relations

$$\begin{aligned} P[Z_{k+1} = j] &= P[Z_k = j-1]/3 + P[Z_k = j]/3 + P[Z_k = j+1]/3, \quad 1 < j < J, \\ P[Z_{k+1} = 1] &= P[Z_k = 1]/3 + P[Z_k = 2]/3, \\ P[Z_{k+1} = J] &= P[Z_k = J-1]/3 + P[Z_k = J]/3. \end{aligned} \quad (51)$$

Let

$$P(x, y) = \sum_{k=0}^{\infty} \sum_{j=1}^J x^k y^j P[Z_k = j].$$

From (51), we get

$$P(x, y) = \frac{G(y) - F(x)(1 + y^{J+1})}{3 - x(y + 1 + y^{-1})}, \quad (52)$$

where  $F(x)$  is the function

$$F(x) = x \sum_{k=0}^{\infty} P[Z_k = 1] x^k, \quad (53)$$

and

$$G(y) = 1 + y + y^2 + \dots + y^J. \quad (54)$$

The denominator of (52) vanishes for  $x = x(y) = 3/(y + 1 + y^{-1}) \leq 1$ . Therefore, necessarily

$$F(x) = \frac{G(y)}{1 + y^{J+1}}, \quad (55)$$

for  $x = x(y)$ . The poles of  $F(x(y))$  are for  $y^{J+1} = -1$ , namely for

$$y(l) = e^{\frac{i\pi + 2il\pi}{J+1}}, \quad l = 0, \dots, J.$$

We have

$$x(y(l)) = \frac{3}{1 + 2 \cos \frac{\pi(1+2l)}{J+1}}, \quad (56)$$

the smallest of which is for  $l = 0$ . Therefore, from classical theorems on generating functions,

$$P[Z_k = 1] = \Theta(x(y(0)))^{-k},$$

or equivalently

$$\lambda_J = \frac{1 + 2 \cos \frac{\pi}{J+1}}{3} \quad (57)$$

which in turn implies

$$C_1(k) = 1 + 2 \cos \frac{\pi}{J+1}. \quad (58)$$

## 9 Appendix: Blocking Queues with Transportation Delay

The technique is the same as in the previous section. Let  $M$  be the substochastic matrix defined by  $M = \Phi(\theta)/K$ , where  $K = 2/(1 - \theta) + 1/(1 - \theta)^2$ . In order to compute the

Perron-Frobenius eigenvalue of  $M$ , we introduce the dissipative Markov chain  $Z_k$  with transition matrix  $M$ . This matrix has the form

$$M = \begin{pmatrix} \alpha & \alpha & 0 & 0 \\ \beta & \alpha & \alpha & 0 \\ 0 & \beta & \alpha & \alpha \\ 0 & 0 & \beta & \alpha \end{pmatrix}, \quad (59)$$

where  $\alpha = 1/K(1 - \theta)$  and  $\beta = 1 - 2\alpha$ . The associated functional equation with initial condition  $Z_0 = 1$  reads

$$P(x, y) = \frac{y - x(\alpha F(x) + \beta y^{J+1} G(x))}{1 - x(y\beta + \alpha + y^{-1}\alpha)}, \quad (60)$$

where  $F(x)$  and  $G(x)$  are the functions

$$\begin{aligned} F(x) &= x \sum_{k=0}^{\infty} P\{Z_k = 1\} x^k, \\ G(x) &= x \sum_{k=0}^{\infty} P\{Z_k = J\} x^k. \end{aligned} \quad (61)$$

The denominator of (60) vanishes for  $y = y(x)$ , where  $y(x)$  is the algebraic function defined by the equation

$$y^2 x \beta + y(x\alpha - 1) + x\alpha = 0. \quad (62)$$

This algebraic function has two real valued branch points and is analytic when cut between the two branch points. The cut is the interval of the real axis where the discriminant

$$\Delta(x) = (x\alpha - 1)^2 - 4x^2\alpha\beta \quad (63)$$

is negative. The algebraic function  $y(x)$  is two-valued with

$$\begin{aligned} y^+(x) &= \frac{1 - x\alpha + \sqrt{\Delta(x)}}{2x\beta}, \\ y^-(x) &= \frac{1 - x\alpha - \sqrt{\Delta(x)}}{2x\beta}. \end{aligned} \quad (64)$$

On the cut, let  $y^+(x) = \rho(x)e^{i\theta(x)}$  and  $y^-(x) = \rho(x)e^{-i\theta(x)}$ . We get immediately that  $\rho^2(x) = \alpha/\beta$  and that

$$\cos(\theta(x)) = \frac{1 - x\alpha}{2x\sqrt{\beta}\sqrt{\alpha}}. \quad (65)$$

From (60) we get that for  $x$  on the cut

$$\begin{aligned} y^+(x) &= x\alpha F(x) + \beta(y^+(x))^{J+1} G(x), \\ y^-(x) &= x\alpha F(x) + \beta(y^-(x))^{J+1} G(x), \end{aligned} \quad (66)$$

which implies that

$$\beta G(x) = \frac{y^+(x) - y^-(x)}{(y^+(x))^{J+1} - (y^-(x))^{J+1}}, \quad (67)$$

that is

$$\beta G(x) = \frac{\rho(x) \sin \theta(x)}{\rho(x)^{J+1} \sin((J+1)\theta(x))}. \quad (68)$$

This function is then continued analytically from the region of the cut. The poles of this function are located for  $x$  such that  $(J+1)\theta(x) = l\pi$ . In view of (65), this implies

$$\cos \frac{l\pi}{J+1} = \frac{1 - x\alpha}{2x\sqrt{\beta}\sqrt{\alpha}}, \quad (69)$$

that is

$$x = \frac{1}{\alpha + 2 \cos(\frac{l\pi}{J+1})\sqrt{\beta}\sqrt{\alpha}}. \quad (70)$$

The smallest of these poles is for  $l = 1$ . Hence

$$P[Z_k = 1] = \Theta(\lambda_k^1) \quad (71)$$

where

$$\lambda_J = \alpha + 2 \cos(\frac{\pi}{J+1})\sqrt{\beta}\sqrt{\alpha}. \quad (72)$$

Therefore,

$$\phi(\theta) = K(\alpha + 2 \cos(\frac{\pi}{J+1})\sqrt{\beta}\sqrt{\alpha}). \quad (73)$$

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