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**ON A STOCHASTIC DIFFERENTIAL
GAME AND ITS RELATIONSHIP
WITH MIXED H_2 / H_∞ CONTROL**

**Ramine NIKOUKHAH
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Septembre 1991



On a Stochastic Differential Game and its Relationship with Mixed H_2/H_∞ Control

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Abstract In this paper, we study a stochastic differential game motivated from a control problem in which two types of disturbance are considered: disturbance with respect to which we have to be robust and disturbance which can be modeled as random processes. We show that the solution can be obtained by solving three coupled Riccati equations with mixed boundary conditions; an algorithm is presented. Finally, we show that, in the time-invariant, infinite-horizon case, the controller that we obtain is a mixed-norm H_2/H_∞ controller.

Un problème de jeu différentiel stochastique et ses liens avec le problème de commande mixte H_2/H_∞

Résumé On étudie un problème de jeu différentiel stochastique motivé par un problème de commande robuste. Deux types de perturbations apparaissent: celles vis à vis desquelles on cherche à être robuste et celles qui peuvent être modélisées par un processus stochastique. On montre que la solution s'obtient par la résolution de trois équations de Riccati couplées avec des conditions aux limites aux deux bouts. Un algorithme est donné. Dans le cas stationnaire, on montre le contrôleur obtenu est de type mixte H_2/H_∞ .

1 Introduction

In the past few years, many researchers have studied the H_∞ control problem from the point of view of differential games [2],[7],[8]. This approach, which amounts to considering disturbance as an opponent who is trying to maximize certain cost function that the controller is trying to minimize, allows for a simple derivation of the H_∞ sub-optimal controllers and their generalization to the time-varying, finite horizon case.

For many applications, the H_∞ controller turns out to be too conservative. From the game theoretic point of view this can be explained by the fact that all disturbance is considered as an opponent, i.e. inputs with respect to which we would like to be robust; this is to be contrasted with H_2 controllers in which disturbance is modeled as random processes and the controller minimizes the average of some cost function. In this report we consider a control problem in which we allow for two types of disturbance: one with respect to which we like to be robust and one which can be reasonably modeled by random processes. This problem is formulated as a stochastic differential game problem where the opponent (disturbance with respect to which we like to be robust) maximizes some average cost that the controller minimizes. It turns out that the controller that we obtain in the time-invariant, infinite-horizon case has mixed-norm H_2/H_∞ property and is closely related to the H_2/H_∞ controller introduced in [3],[5].

The outline of the report is as follows. In Section 2, we formulate the stochastic differential game problem and establish the notations. In Section 3, we present the solution, consider the time-invariant, infinite-horizon case and its relationship with the mixed H_2/H_∞ control problem and present an algorithm for the construction of the controller. All the proofs are given in Section 4.

2 Problem formulation

Consider the following system:

$$\dot{x} = A(t)x + B_1(t)w + B_2(t)u + B_3(t)v \quad (2.1)$$

$$y = C(t)x + D(t)w + E(t)v \quad (2.2)$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{m_1}$, $u(t) \in \mathbf{R}^{m_2}$, $y(t) \in \mathbf{R}^{p_1}$ and v is a q -dimensional zero-mean random process with unit covariance independent of $x(0)$ which is a zero-mean random vector with covariance Λ . Matrices $A(t)$, $B_1(t)$, $B_2(t)$, $B_3(t)$, $C(t)$, $D(t)$ and $E(t)$ have appropriate dimensions and verify $B_1(t)^T D(t) = 0$, $B_3(t)^T E(t) = 0$, and $R(t) \triangleq E(t)E(t)^T > 0$ for all t .

The stochastic differential game problem that we consider is:

$$J^* = \inf_u \sup_w E\{J(u, w)\} \quad (2.3)$$

where

$$J(u, w) = \frac{1}{t_f} \left(\int_0^{t_f} (z^T z - \gamma^2 w^T w) dt + x(t_f)^T \Phi x(t_f) \right) \quad (2.4)$$

with

$$z = \bar{C}(t)x + \bar{D}(t)u \quad (2.5)$$

where $\bar{C}(t)$ and $\bar{D}(t)$ are respectively $p_2 \times n$ and $p_2 \times m_2$ matrices and Φ a constant nonnegative-definite matrix. We shall suppose that $\bar{D}(t)^T \bar{D}(t) = I$, $\bar{C}(t)^T \bar{D}(t) = 0$ and denote $Q(t) \triangleq \bar{C}(t)^T \bar{C}(t)$ so that

$$z^T z = x^T Q(t)x + u^T u. \quad (2.6)$$

All entries of time varying system matrices are bounded piecewise continuous functions of time.

As stated, (2.3) does not characterize completely the problem; we need to specify what kind of information is available to the two parties u and w . Clearly, to obtain the robustness property that we are seeking with respect to w , we need to provide him with a lot of information. Different formulations are possible depending on the kind of information we make available to u and w . Here, we suppose that u has access to y of past and present and that w has access to the strategy of u , and the past and present values of x and v (and consequently y) and consider only linear, finite-dimensional strategies for u of the type:

$$\dot{\xi} = F(t)\xi + G(t)y, \quad (2.7)$$

$$u = H(t)\xi \quad (2.8)$$

where $\xi(t) \in \mathbf{R}^{n_c}$ and $F(t)$, $G(t)$ and $H(t)$ have appropriate dimensions. Moreover, for the purpose of robustness, we only consider $(F(t), G(t), H(t))$'s for which

$$\text{For all } x(0) \text{ and } \xi(0), \sup_w J(u, w) < \infty \quad (2.9)$$

where u is obtained from (2.7)-(2.8).

For obvious reasons, we refer to $F(t)$, $G(t)$, $H(t)$ and $\xi(0)$ as the controller.

Thanks to the assumption on the information available to w , problem (2.3) corresponds to a simple game in a sense that we do not have to worry about any saddle points. Since w has knowledge of the strategy of u , we first have to find the cost associated with the optimal strategy of w for every controller and then optimize it over the set of controllers. This means that, given $x(0)$, $\xi(0)$, $F(t)$, $G(t)$ and $H(t)$, w should first compute

$$\bar{J}(F(t), G(t), H(t), x(0), \xi(0)) = \sup_w \mathbb{E}_v J(u, w) \quad (2.10)$$

where u satisfies (2.7)-(2.8) and then the optimal controller is obtained by computing

$$J^* = \inf_{F(t), G(t), H(t), \xi(0)} \mathbb{E}_{x(0)} \bar{J}(F(t), G(t), H(t), x(0), \xi(0)). \quad (2.11)$$

Lemma 2.1 *Suppose an optimal controller $(F(t), G(t), H(t), \xi(0))$ exists. Then the controller $(F(t), G(t), H(t), 0)$ is also an optimal controller, i.e. without any loss of generality we can restrict the class of controllers to those satisfying*

$$\xi(0) = 0. \quad (2.12)$$

This can be explained by the fact that at time 0, u has no other knowledge but the a-priori mean of $x(0)$ which is assumed to be zero. A proof of Lemma 2.1 is given in Section 4. From here on we shall only consider controllers satisfying (2.12).

Lemma 2.2 *Two controllers having the same input-output transfer function correspond to the same strategy of u .*

This is obvious since $\xi(0) = 0$ and thus the dependence of u on y depends only on the input-output transfer function of the controller. In view of this result, we consider two controllers to be just one as long as they have the same input-output transfer function.

Before ending this section, let us make a few comments concerning the formulation of the problem.

System (2.7)-(2.8) is a realization of some controller \mathcal{K} but \mathcal{K} has many other realizations. In fact, \mathcal{K} may have realizations with smaller state-space dimension than n_c . A realization of \mathcal{K} having smallest state-space dimension is called a minimal realization and its dimension, the order of \mathcal{K} . System (2.7)-(2.8) can realize controllers of order less than or equal to n_c .

Assumptions $B_1(t)^T D(t) = 0$, $B_3(t)^T E(t) = 0$ and $\bar{C}(t)^T \bar{D}(t) = 0$ are made because they allow considerable simplification of some of the expressions; they are by no means essential. These assumptions are commonly made in the LQG and H_∞ literature.

Our problem reduces to the LQG problem if $w = 0$. This can be done by setting $B_1(t)$ and $D(t)$ to zero (in which case the sup over w has for trivial solution $w = 0$) or by letting γ go to infinity.

There is a choice as to what information is available to w . A variation of our problem would be to consider that w has access to future values of ν as well. This would lead to an even "more robust" controller. We do not investigate this problem here.

For simplicity of notations, we shall drop the dependence on t of the system matrices; that is in the rest of the report, except for Section 3.2 where we consider the time-invariant case. A should be interpreted as $A(t)$, B_1 as $B_1(t)$, etc...

3 Main results

3.1 Solution of the problem

The following result shows that there is no point considering realizations (F, G, H) of dimension n_c larger than n .

Lemma 3.1 *There exists no optimal controller of order strictly larger than n .*

We shall thus consider only the case where $n_c = n$. This means that we look for optimal controllers in the class of controllers of order less than or equal to n .

The approach in this report can also be used to study reduced order controllers by choosing n_c strictly less than n . In that case, we would be looking for optimal controllers in the class of controllers having at most order n_c . This problem is however not considered here.

Theorem 3.1 *Suppose an optimal controller \mathcal{K} exists. Then there exist matrix functions X , P , \bar{P} and M on the interval $[0, t_f]$ satisfying:*

$$\dot{P} + A^T P + P A + Q - P(B_2 B_2^T - \gamma^{-2} B_1 B_1^T)P = 0, \quad P(t_f) = \Phi \quad (3.1)$$

$$\begin{aligned} \dot{\bar{P}} + A^T \bar{P} + \bar{P} A + Q - (\bar{P} - P)X C - C^T X^T (\bar{P} - P) + \\ \gamma^{-2} (\bar{P} B_1 B_1^T \bar{P} + (\bar{P} - P)X D D^T X^T (\bar{P} - P)) = 0, \quad \bar{P}(t_f) = \Phi \end{aligned} \quad (3.2)$$

$$-\dot{M} + (A + \gamma^{-2} B_1 B_1^T \bar{P})M + M(A + \gamma^{-2} B_1 B_1^T \bar{P})^T + B_3 B_3^T - X R X^T = 0, \quad M(0) = \Lambda \quad (3.3)$$

$$X R - M C^T + \gamma^{-2} M(\bar{P} - P) X D D^T = 0. \quad (3.4)$$

Moreover, \mathcal{K} has a realization (F, G, H) given by

$$G = X \quad (3.5)$$

$$H = -B_2^T P \quad (3.6)$$

$$F = A + B_2 H - G C + \gamma^{-2} B_1 B_1^T P, \quad (3.7)$$

and

$$J^* = \frac{1}{t_f} \left(\int_0^{t_f} \text{Tr}(\bar{P} B_3 B_3^T + (\bar{P} - P) X R X^T) dt + \text{Tr}(\Lambda \bar{P}(0)) \right). \quad (3.8)$$

We think that the converse of this theorem also holds, i.e., if there exist matrix functions X , P , \bar{P} and M on the interval $[0, t_f]$ satisfying (3.1)-(3.4), (F, G, H) given by (3.5)-(3.7) is a realization of the optimal controller. No satisfactory proof is as yet available and the problem is under investigation. In any case, the controller obtained from (3.1)-(3.7) has the good performance and robustness properties that we are looking for. Let us denote this controller by \mathcal{K}^* .

Lemma 3.2 *Let \mathcal{K}^* be obtained from (3.1)-(3.7). Then in the closed-loop system (2.1)-(2.2), (2.7)-(2.8):*

1- if we set $x(0)$ and v to zero, we obtain

$$\sup_w \frac{\int_0^{t_f} z^T z dt + x(t_f)^T \Phi x(t_f)}{\int_0^{t_f} w^T w dt} \leq \gamma^2, \quad (3.9)$$

2- if we set $w = 0$, we obtain

$$E\left\{ \int_0^{t_f} z^T z dt + x(t_f)^T \Phi x(t_f) \right\} \leq \int_0^{t_f} \text{Tr}(\bar{P} B_3 B_3^T + (\bar{P} - P) X R X^T) dt + \text{Tr}(\Lambda \bar{P}(0)). \quad (3.10)$$

Inequalities (3.9) and (3.10) illustrate respectively the robustness and the performance properties of \mathcal{K}^* .

Equation (3.4) is linear in X and we can solve X in terms of M , P and \bar{P} and eliminate it from all the equations; (this is done numerically during the actual computation of the solution). This means that (3.1)-(3.4) correspond really to three coupled Riccati equations with boundary conditions.

3.2 Time-invariant, infinite-horizon problem

In this section, we consider the case where all system matrices are constant, $\Phi = 0$ and t_f goes to infinity. In this case, even if for all t_f , a \mathcal{K}^* exists, there is no guarantee that as t_f is increased towards infinity, \mathcal{K}^* would converge. That is not even true in the LQG case, i.e. when $\gamma = \infty$ unless some stabilizability and detectability conditions are fulfilled.

Definition 3.1 Let $\mathcal{K}^*(t_f)$ denote the optimal controller for problem (2.3)-(2.4) over the interval $[0, t_f]$ and $J^*(t_f)$ the associated cost. Then we call the time-invariant controller \mathcal{K}_∞^* the optimal steady-state solution to the infinite horizon problem, if, as t_f goes to infinity, $\mathcal{K}^*(t_f)$ converges to \mathcal{K}_∞^* over the interval $[\frac{t_f}{\alpha}, \frac{t_f}{\beta}]$ where $1 < \beta < \alpha < \infty$, and $J^*(t_f)$ converges to $J_\infty^* < \infty$.

Clearly, if the solutions P, \bar{P}, M and X to (3.1)-(3.4) converge far from the boundaries as t_f goes to infinity, \mathcal{K}_∞^* exists. If we denote by a subscript s the limiting solutions far from the boundaries, we must have

$$A^T P_s + P_s A + Q - P_s (B_2 B_2^T - \gamma^{-2} B_1 B_1^T) P_s = 0 \quad (3.11)$$

$$A^T \bar{P}_s + \bar{P}_s A + Q - (\bar{P}_s - P_s) X_s C - C^T X_s^T (\bar{P}_s - P_s) + \gamma^{-2} (\bar{P}_s B_1 B_1^T \bar{P}_s + (\bar{P}_s - P_s) X_s D D^T X_s^T (\bar{P}_s - P_s)) = 0 \quad (3.12)$$

$$(A + \gamma^{-2} B_1 B_1^T \bar{P}_s) M_s + M_s (A + \gamma^{-2} B_1 B_1^T \bar{P}_s)^T + B_3 B_3^T - X_s R X_s^T = 0 \quad (3.13)$$

$$X_s R - M_s C^T + \gamma^{-2} M_s (\bar{P}_s - P_s) X_s D D^T = 0, \quad (3.14)$$

with

$$P_s \geq 0 \quad (3.15)$$

$$\bar{P}_s - P_s \geq 0 \quad (3.16)$$

$$M_s \geq 0, \quad (3.17)$$

and a realization of the (time-invariant) controller \mathcal{K}_∞^* is given by

$$G = X_s \quad (3.18)$$

$$H = -B_2^T P_s \quad (3.19)$$

$$F = A + B_2 H - G C + \gamma^{-2} B_1 B_1^T P_s \quad (3.20)$$

and

$$J_\infty^* = \text{Tr} (\bar{P}_s B_3 B_3^T + (\bar{P}_s - P_s) X_s R X_s^T). \quad (3.21)$$

However, the convergence of all these matrices is not a necessary condition for the existence of \mathcal{K}_∞^* . For example, consider the case where M goes to infinity in the middle of the interval as t_f goes to infinity. In that case, from (3.4) follows that,

$$X_s = \gamma^2 (\bar{P}_s - P_s)^{-1} C^T (D D^T)^{-1}. \quad (3.22)$$

Thanks to which (3.12) becomes

$$A^T \bar{P}_s + \bar{P}_s A + Q - \gamma^2 C^T (D D^T)^{-1} C + \gamma^{-2} \bar{P}_s B_1 B_1^T \bar{P}_s = 0. \quad (3.23)$$

For M to go to infinity, $A + \gamma^{-2} B_1 B_1^T \bar{P}$ must be unstable, i.e., \bar{P} must be the antistabilizing solution of the Riccati equation (3.23). On the other hand, with $\Phi = 0$, P in (3.1) converges to P_s , the stabilizing solution of (3.11). But (3.11) and (3.23) are exactly the Riccati equations derived in [4] (in their notations, \bar{P}_s would be X_∞ and $\bar{P}_s \cdot \gamma^2 Y_\infty^{-1}$) and (F, G, H) corresponds to their central controller in this case.

We cannot study the steady-state solution of the infinite-horizon problem simply by computing the solution of (3.11)-(3.17) because the solution is not unique. Moreover, the steady-state solution cannot always be constructed from (3.11)-(3.17), as we have seen above in the case where M does not converge. It turns out however that we can construct "good" controllers from the solutions of (3.11)-(3.17).

Theorem 3.2 *Let P_s , \bar{P}_s , X_s and M_s satisfy (3.11)-(3.17) and let \mathcal{K}_s be the controller obtained from (3.18)-(3.20). Then*

$$J^*(\mathcal{K}_s) \leq \text{Tr}(\bar{P}_s B_3 B_3^T + (\bar{P}_s - P_s) X_s R X_s^T). \quad (3.24)$$

Note that \mathcal{K}_s may or may not equal \mathcal{K}_∞^* .

Corollary 3.1 *Let \mathcal{K}_s be as in Lemma 3.2. Then in the closed-loop system (2.1)-(2.2), (2.7)-(2.8):*

1- if we set $x(0)$ and ν to zero, we obtain

$$\sup_w \frac{\|z\|_2^2}{\|w\|_2^2} \leq \gamma^2, \quad (3.25)$$

2- if we set $w = 0$, we obtain

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} E \left\{ \int_0^\tau z^T z dt \right\} \leq \text{Tr}(\bar{P}_s B_3 B_3^T + (\bar{P}_s - P_s) X_s R X_s^T). \quad (3.26)$$

Thus, if the closed-loop system (2.1)-(2.2), (2.7)-(2.8) is internally stable,

$$\|T_{zw}\|_\infty \leq \gamma \quad (3.27)$$

where T_{zw} denotes the transfer function from w to z and

$$\|T_{z\nu}\|_2 \leq (\text{Tr}(\bar{P}_s B_3 B_3^T + (\bar{P}_s - P_s) X_s R X_s^T))^{1/2}, \quad (3.28)$$

where $T_{z\nu}$ denotes the transfer function from ν to z (see Fig. 1).

We can see that in this case the problem considered reduces to the one, or more specifically to the dual of the one, presented in [3]. Note that the technical assumption of minimality of the controller made in [3] is not necessary.

3.3 Numerical Implementation

The solution to the coupled Riccati equations can be obtained using a continuation method on the parameter γ and fixed point iterations on the trajectory of X .

Let $\alpha = \gamma^{-2}$ then for $\alpha = 0$ (3.1)-(3.4) become

$$\dot{P} + A^T P + P A + Q - P B_2 B_2^T P = 0, \quad P(t_f) = \Phi \quad (3.1)$$

$$\begin{aligned} \dot{\bar{P}} + A^T \bar{P} + \bar{P} A + Q - (\bar{P} - P) M C^T R^{-1} C - \\ C^T R^{-1} C M (\bar{P} - P) = 0, \quad \bar{P}(t_f) = \Phi \end{aligned} \quad (3.2)$$

$$-\dot{M} + A M + M A^T + B_3 B_3^T - M C^T R^{-1} C M = 0, \quad M(0) = \Lambda. \quad (3.3)$$

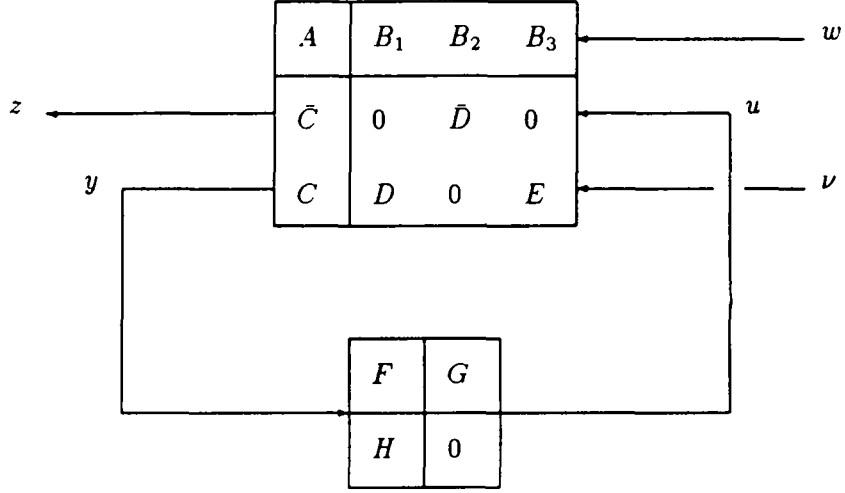


Figure 1: This mixed-norm H_2/H_∞ problem is the dual of the one presented in [3]; it also includes the H_∞ and the LQG problems as special cases.

Note that (3.1) and (3.3) are completely decoupled (in fact they correspond to the two LQG Riccati equations) and they can be solved by straightforward integration. Once P and M are computed, \bar{P} can be solved for using (3.2) and X using the fact that

$$X = MC^T R^{-1}. \quad (3.4)$$

The algorithm uses these values of P , \bar{P} , M and X for initial conditions.

Algorithm

- 1- Compute initial ($\alpha = 0$) values of P , \bar{P} , M and X . Let $X_{opt} = X$.
- 2- Choose an initial α .
- 3- Integrate (3.1) and (3.2) to obtain P and \bar{P} , and use the latter to integrate (3.3). If either of the integrations blows up, reduce α , let $X = X_{opt}$ and repeat.
- 4- Compute new X from (3.4).
- 5- If first time here with this α go to 3.
- 6- If X and previous X are sufficiently close, let $X_{opt} = X$, increase α and go to 3. If not, go to 3.

The algorithm should stop until α reaches the desired value. If α is increased by small steps, it seems that the fixed point iterations on X always converge but we have no proof of the contractive nature of this fixed point.

Note that we can always solve for X using (3.4) (step 4). To see this note that (3.4) is equivalent to

$$X + \gamma^{-2}M(\bar{P} - P)XDD^T R^{-1} = MC^T R^{-1}, \quad (3.5)$$

which can be expressed as

$$(I + \gamma^{-2}(M(\bar{P} - P)) \otimes (DD^T R^{-1})^T)\chi = \mu \quad (3.6)$$

where \otimes denotes the Kronecker product and, χ and μ are vectors constructed from the entries of X and $MC^T R^{-1}$, respectively. The eigenvalues of $M(\bar{P} - P)$ and $DD^T R^{-1}$ are all nonnegative (the eigenvalues of the product of two nonnegative matrices are real and nonnegative). This implies that the eigenvalues of $(M(\bar{P} - P)) \otimes (DD^T R^{-1})^T$, which are the product of the eigenvalues of $M(\bar{P} - P)$ and $DD^T R^{-1}$ are also nonnegative which in turn implies that the eigenvalues of $I + \gamma^{-2}(M(\bar{P} - P)) \otimes (DD^T R^{-1})^T$ are strictly positive and so the matrix is invertible.

For problems defined over long intervals or to study the limiting behavior as t_f goes to infinity, a continuation on the size of the interval can be used.

Example Figure 2 and 3 show the solution obtained for the following randomly generated simple scalar problem:

$$A = -0.76, \quad B_1 = 0.85, \quad B_2 = 0.72, \quad B_3 = 0.83, \quad C = -0.68, \quad D = 0.81, \quad (3.7)$$

$$Q = 2.17, \quad R = 0.06, \quad \Phi = 0, \quad \Lambda = 0, \quad (3.8)$$

$$\gamma = 2/3, \quad t_f = 1. \quad (3.9)$$

4 Proofs

Before presenting the proofs of the results of the previous sections, let us introduce a few notations.

By combining (2.1), (2.2), (2.7) and (2.8) and eliminating u , we obtain:

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \mathcal{A} \begin{pmatrix} x \\ \xi \end{pmatrix} + \mathcal{B}w + \mathcal{F}\nu \quad (4.1)$$

where

$$\mathcal{A} = \begin{pmatrix} A & B_2 H \\ GC & F \end{pmatrix} \quad (4.2)$$

$$\mathcal{B} = \begin{pmatrix} B_1 \\ GD \end{pmatrix} \quad (4.3)$$

$$\mathcal{F} = \begin{pmatrix} B_3 \\ GE \end{pmatrix}. \quad (4.4)$$

With this notation, we get that the cost associated with the controller \mathcal{K} ,

$$J^\dagger(\mathcal{K}) = \frac{1}{t_f} \sup_w \mathbb{E} \left\{ \int_0^{t_f} (\eta^T \mathcal{V} \eta - \gamma^2 w^T w) dt + \eta(t_f)^T \begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix} \eta(t_f) \right\} \quad (4.5)$$

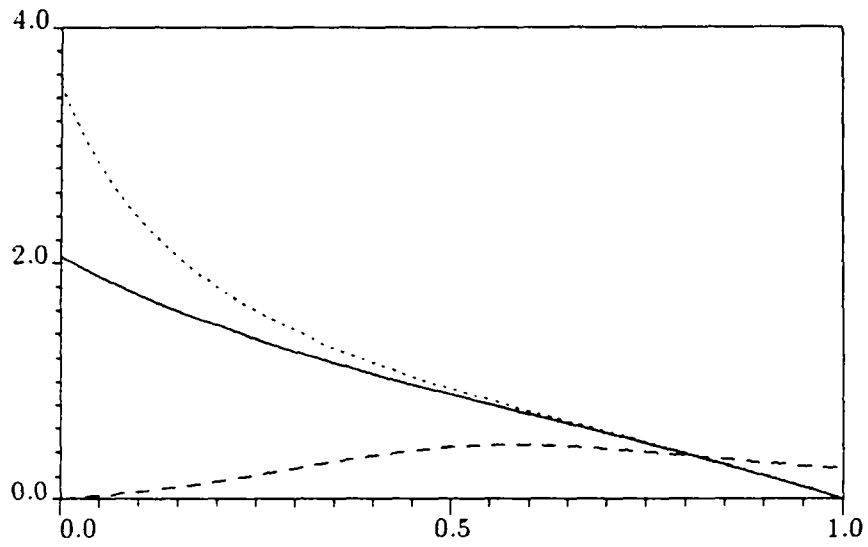


Figure 2: P , \bar{P} and M are represented respectively by solid, dotted and dashed lines.

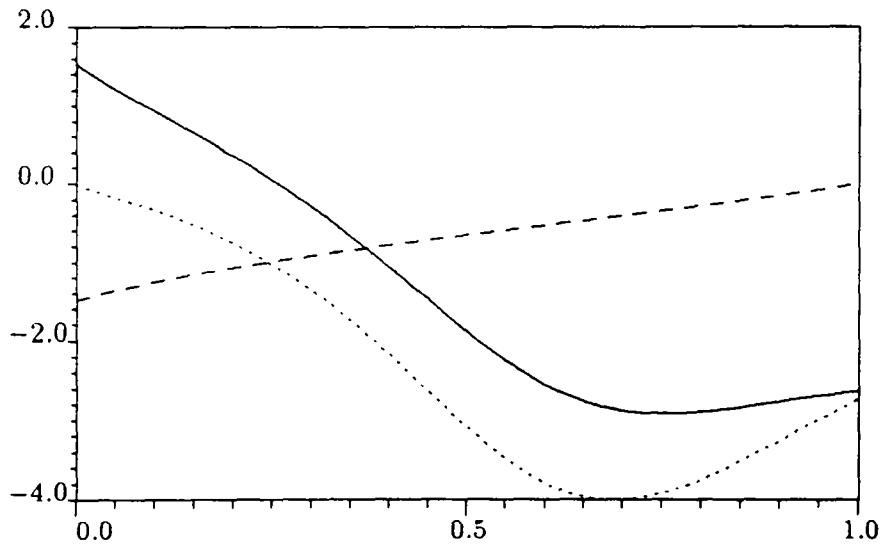


Figure 3: F , G and H are represented respectively by solid, dotted and dashed lines.

where

$$\eta = \begin{pmatrix} x \\ \xi \end{pmatrix} \quad (4.6)$$

and

$$\mathcal{V} = \begin{pmatrix} Q & 0 \\ 0 & H^T H \end{pmatrix}. \quad (4.7)$$

4.1 Proof of Lemma 2.1, Lemma 3.1 and Theorem 3.1

We start by finding the optimum strategy for w . This is a standard stochastic LQ problem with state feedback. In particular, the problem is

$$J^\dagger(\mathcal{K}) = \frac{1}{t_f} \sup_w \mathbb{E} \left\{ \int_0^{t_f} (\eta^T \mathcal{V} \eta - \gamma^2 w^T w) dt + \eta^T(t_f) \mathcal{P}_{t_f} \eta(t_f) \right\} \quad (4.8)$$

subject to

$$\dot{\eta} = \mathcal{A}\eta + \mathcal{B}w + \mathcal{F}\nu, \quad \eta(0) = \begin{pmatrix} x(0) \\ \xi(0) \end{pmatrix} \quad (4.9)$$

where

$$\mathcal{P}_{t_f} = \begin{pmatrix} \Phi & 0 \\ 0 & 0 \end{pmatrix}. \quad (4.10)$$

The solution to this problem is obtained by solving the following Riccati equation on the interval $[0, t_f]$

$$0 = \dot{\mathcal{P}} + \mathcal{A}^T \mathcal{P} + \mathcal{P} \mathcal{A} + \mathcal{V} + \gamma^{-2} \mathcal{P} \mathcal{B} \mathcal{B}^T \mathcal{P}, \quad \mathcal{P}(t_f) = \mathcal{P}_{t_f}. \quad (4.11)$$

If a solution exists, the optimal w is given by

$$w = \gamma^{-2} \mathcal{B}^T \mathcal{P} \eta \quad (4.12)$$

and

$$J^\dagger(\mathcal{K}) = \frac{1}{t_f} (\eta(0)^T \mathcal{P}(0) \eta(0) + \int_0^{t_f} \text{Tr}(\mathcal{P} \mathcal{F} \mathcal{F}^T) dt); \quad (4.13)$$

if not, \mathcal{K} is not an admissible controller since (2.9) fails. This means that the optimal controller can be found by computing

$$\begin{aligned} J^* &= \inf_{\mathcal{K}} \mathbb{E} J^\dagger(\mathcal{K}) \\ &= \inf_{\mathcal{K}} \frac{1}{t_f} \mathbb{E} (\eta(0)^T \mathcal{P}(0) \eta(0) + \int_0^{t_f} \text{Tr}(\mathcal{P} \mathcal{F} \mathcal{F}^T) dt). \end{aligned} \quad (4.14)$$

which is equivalent to

$$J^* = \inf_{\mathcal{K}} \frac{1}{t_f} (\text{Tr}(\Gamma \mathcal{P}(0)) + \int_0^{t_f} \text{Tr}(\mathcal{P} \mathcal{F} \mathcal{F}^T) dt) \quad (4.15)$$

where

$$\Gamma = \begin{pmatrix} \Lambda & 0 \\ 0 & \xi(0) \xi(0)^T \end{pmatrix}. \quad (4.16)$$

Clearly, since $\mathcal{P}(t)$ and in particular $\mathcal{P}(0)$ is nonnegative definite, an optimal choice for $\xi(0)$ is just zero. This proves Lemma 2.1.

The game problem is to solve (4.15) subject to constraint (4.11). Using the matrix minimum principle [1], we transform this problem into an unconstrained optimization problem by introducing a matrix Lagrange multiplier \mathcal{M} and the associated Lagrangian

$$\mathcal{L} = \text{Tr}(\Gamma\mathcal{P}(0)) + \int_0^{t_f} \text{Tr}(\mathcal{P}\mathcal{F}\mathcal{F}^T)dt + \int_0^{t_f} \text{Tr} \mathcal{M}(\dot{\mathcal{P}} + \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} + \mathcal{V} + \gamma^{-2}\mathcal{P}\mathcal{B}\mathcal{B}^T\mathcal{P})dt. \quad (4.17)$$

Using integration by part we obtain

$$\mathcal{L} = \text{Tr}(\Gamma\mathcal{P}(0) - \mathcal{M}(0)\mathcal{P}(0) + \mathcal{M}(t_f)\mathcal{P}(t_f)) + \quad (4.18)$$

$$\int_0^{t_f} \text{Tr}(\mathcal{P}\mathcal{F}\mathcal{F}^T + \mathcal{M}(\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} + \mathcal{V} + \gamma^{-2}\mathcal{P}\mathcal{B}\mathcal{B}^T\mathcal{P}) - \dot{\mathcal{M}}\mathcal{P})dt. \quad (4.19)$$

Setting to zero the variation with respect to $\mathcal{P}(0)$ yields

$$\text{Tr}(\Gamma - \mathcal{M}(0))\delta\mathcal{P}(0) = 0, \quad (4.20)$$

and with respect to \mathcal{P} , using the identity $\text{Tr}YZ = \text{Tr}ZY$,

$$\int_0^{t_f} \text{Tr}(-\dot{\mathcal{M}} + \mathcal{M}\mathcal{A}^T + \mathcal{A}\mathcal{M} + \mathcal{F}\mathcal{F}^T + \gamma^{-2}(\mathcal{M}\mathcal{P}\mathcal{B}\mathcal{B}^T + \mathcal{B}\mathcal{B}^T\mathcal{P}\mathcal{M}))\delta\mathcal{P}dt = 0. \quad (4.21)$$

So, we obtain the following Lyapunov equation for \mathcal{M} :

$$-\dot{\mathcal{M}} + \mathcal{M}(\mathcal{A}^T + \gamma^{-2}\mathcal{P}\mathcal{B}\mathcal{B}^T) + (\mathcal{A} + \gamma^{-2}\mathcal{B}\mathcal{B}^T\mathcal{P})\mathcal{M} + \mathcal{F}\mathcal{F}^T = 0 \quad \mathcal{M}(0) = \Gamma. \quad (4.22)$$

Note that \mathcal{P} and \mathcal{M} are $(n + n_c) \times (n + n_c)$ nonnegative-definite matrices so we can parameterize them, without any loss of generality, as follows

$$\mathcal{P} = \begin{pmatrix} P + TP_2T^T & TP_2 \\ P_2T^T & P_2 \end{pmatrix} \quad (4.23)$$

$$\mathcal{M} = \begin{pmatrix} M + SM_2S^T & SM_2 \\ M_2S^T & M_2 \end{pmatrix} \quad (4.24)$$

where M and P are $n \times n$, and M_2 and P_2 are $n_c \times n_c$ nonnegative-definite and S and T are $n \times n_c$ time-varying matrices. Notice that in this parameterization, S and T are not unique unless M_2 and P_2 are positive-definite.

Variation with respect to F : We obtain

$$\int_0^{t_f} \text{Tr}(\mathcal{M}\mathcal{P} \begin{pmatrix} 0 & 0 \\ 0 & \delta F \end{pmatrix})dt = 0 \quad (4.25)$$

which implies

$$P_2T^TSM_2 + P_2M_2 = 0. \quad (4.26)$$

Let us now prove Lemma 3.1. Suppose $n_c > n$ and that (F, G, H) is a minimal realization. In that case, we have to show that necessary condition (4.26) yields a contradiction. Expanding the (2,2)-blocks of (4.22) and (4.11) and using (4.26) we get

$$-\dot{M}_2 + M_2(F^T + S^T C^T G^T) + (F + GCS)M_2 + GRG^T = 0, \quad M_2(0) = 0 \quad (4.27)$$

$$\begin{aligned} \dot{P}_2 + P_2(F + T^T B_2 H) + (F^T + H^T B_2^T T)P_2 + H^T H + \\ \gamma^{-2} P_2(T^T B_1 B_1^T T + GDD^T G^T)P_2 = 0, \quad P_2(t_f) = 0. \end{aligned} \quad (4.28)$$

From (4.27), and the fact that state feedback does not alter the controllable space, we deduce that

$$\text{Im}M_2(t) = \text{Im}C(t) \quad (4.29)$$

where

$$C(t) = \int_0^t \Psi(t, s)B(s)B(s)^T \Psi(t, s)^T ds \quad (4.30)$$

is the controllability Grammian at time t associated with (F, G) . Here, $\Psi(t, s)$ is the state-transition matrix associated with F , i.e., Ψ satisfies the following equation

$$\frac{\partial}{\partial t} \Psi(t, s) = F(t)\Psi(t, s), \quad \Psi(t, t) = I. \quad (4.31)$$

Similarly, from (4.28) follows that $\text{Ker}P_2(t)$ is equal to the unobservable space at time t associated with (H, F) . i.e., $\text{Ker}\mathcal{O}(t)$, where

$$\mathcal{O}(t) = \int_t^{t_f} \Psi(s, t_f)^T H(s)^T H(s)\Psi(s, t_f) ds \quad (4.32)$$

is the observability Grammian at time t associated with (H, F) .

Since (F, G, H) is assumed to be minimal, for some t in $[0, t_f]$ the system must be controllable and observable, i.e., $P_2(t) > 0$ and $M_2(t) > 0$. But then from (4.26) we get

$$T(t)^T S(t) = -I. \quad (4.33)$$

This is a contradiction since $T(t)^T S(t)$ is $n_c \times n_c$ but its rank is at most n , thus $n \leq n_c$. This proves Lemma 3.1.

From this point on we shall suppose that $n_c = n$. Also note that if M_2 and P_2 are invertible, (4.26) implies that

$$T^T S = -I. \quad (4.34)$$

If not, this is not necessarily true. However, remembering that in that case, S and T are not uniquely determined in terms of \mathcal{P} and \mathcal{M} by inspecting the degree of freedom in S and T , we can easily see that we can use this degree of freedom to impose (4.34) without any loss of generality. This is particularly useful since thanks to $n = n_c$ we get

$$T^T = -S^{-1}. \quad (4.35)$$

Variation with respect to H : We obtain

$$\int_0^{t_f} \text{Tr} \left(\mathcal{M} \mathcal{P} \begin{pmatrix} 0 & B_2 \delta H \\ 0 & 0 \end{pmatrix} + \mathcal{M} \begin{pmatrix} 0 & 0 \\ 0 & H^T \delta H \end{pmatrix} \right) dt = 0 \quad (4.36)$$

which implies

$$M_2(S^T(P + TP_2T^T)B_2 + P_2T^TB_2 + H^T) = 0 \quad (4.37)$$

and thus

$$(H + B_2^T PS)M_2 = 0. \quad (4.38)$$

Variation with respect to G : We obtain

$$\int_0^T \text{Tr} \left(\mathcal{P} \begin{pmatrix} 0 & 0 \\ 0 & GR\delta G^T \end{pmatrix} + \mathcal{P} \mathcal{M} \begin{pmatrix} 0 & C^T \delta G^T \\ 0 & 0 \end{pmatrix} + \gamma^{-2} \mathcal{P} \mathcal{M} \mathcal{P} \begin{pmatrix} 0 & 0 \\ 0 & GDD^T \delta G^T \end{pmatrix} \right) dt = 0 \quad (4.39)$$

which after some algebra implies

$$P_2(GR + T^T MC^T + \gamma^{-2} T^T MTP_2GDD^T) = 0. \quad (4.40)$$

Now we have enough equations to solve for \mathcal{P} and \mathcal{M} . Let \bar{P} denote the (1,1)-block of \mathcal{P} , i.e.,

$$\bar{P} = P + TP_2T^T. \quad (4.41)$$

Then the (1,1)-block of (4.11) can be expressed as follows

$$\dot{\bar{P}} + A^T \bar{P} + \bar{P}A + C^T G^T P_2 T^T + TP_2 GC + Q + \gamma^{-2} (\bar{P} B_1 B_1^T \bar{P} + TP_2 GDD^T G^T P_2 T^T) = 0. \quad (4.42)$$

The (1,2)-block of (4.11) becomes

$$\dot{T} P_2 + T \dot{P}_2 + A^T T P_2 + C^T G^T P_2 + \bar{P} B_2 H + TP_2 F + \gamma^{-2} (\bar{P} B_1 B_1^T T P_2 + TP_2 GDD^T G^T P_2) = 0 \quad (4.43)$$

which using (4.28) yields

$$(\dot{T} - TF^T - TH^T B_2^T T + A^T T + C^T G^T + \gamma^{-2} P B_1 B_1^T T) P_2 + (P B_2 - TH^T) H = 0. \quad (4.44)$$

Subtracting (4.43) postmultiplied by T^T from (4.42) gives

$$\dot{P} + TP_2 \dot{T}^T + A^T P + PA + TP_2 T^T A + TP_2 GC + Q - \bar{P} B_2 H T^T - TP_2 F T^T + \gamma^{-2} \bar{P} B_1 B_1^T P = 0. \quad (4.45)$$

By adding (4.45) to the transpose (4.44) premultiplied by $-T$ we obtain

$$\dot{P} + A^T P + PA + Q - P(B_2 B_2^T - \gamma^{-2} B_1 B_1^T) P + (TH^T - P B_2)(HT^T - B_2^T P) = 0. \quad (4.46)$$

By performing same type of computations on (4.22) we obtain

$$(-\dot{S} - SF - SGCS + AS + B_2 H + \gamma^{-2} B_1 B_1^T PS) M_2 + (-SGR + MC^T + \gamma^{-2} MTP_2 GR) G^T = 0 \quad (4.47)$$

and

$$\begin{aligned}
& -\dot{M} + M(A^T + \gamma^{-2}\bar{P}B_1B_1^T) + (A + \gamma^{-2}B_1B_1^T\bar{P})M + \\
& (SGR - MC^T - \gamma^{-2}MTP_2GR)G^T S^T + B_3B_3^T + \\
& SG(RG^T S^T - CM - \gamma^{-2}RG^T P_2T^T M) - SGRG^T S^T = 0. \quad (4.48)
\end{aligned}$$

To summarize, we obtain the following set of necessary conditions:

$$\begin{aligned}
& \dot{P} + A^T P + PA + Q - P(B_2B_2^T - \gamma^{-2}B_1B_1^T)P + \\
& (TH^T - PB_2)(HT^T - B_2^T P) = 0, P(t_f) = \Phi \quad (4.49)
\end{aligned}$$

$$\begin{aligned}
& \dot{\bar{P}} + A^T \bar{P} + \bar{P}A + Q - (\bar{P} - P)SGC - C^T G^T S^T (\bar{P} - P) + \\
& \gamma^{-2}(\bar{P}B_1B_1^T \bar{P} + (\bar{P} - P)SGDD^T G^T S^T (\bar{P} - P)) = 0, \bar{P}(t_f) = \Phi \quad (4.50)
\end{aligned}$$

$$\begin{aligned}
& -\dot{M} + (A + \gamma^{-2}B_1B_1^T\bar{P})M + M(A + \gamma^{-2}B_1B_1^T\bar{P})^T + \\
& B_3B_3^T + S(GR + T^T MC^T + \gamma^{-2}T^T MTP_2GR)G^T S^T + \\
& SG(RG^T + CMTD + \gamma^{-2}RG^T P_2T^T MT)S^T - SGRG^T S^T = 0, M(0) = \Lambda \quad (4.51)
\end{aligned}$$

$$P_2(GR + T^T MC^T + \gamma^{-2}T^T MTP_2GDD^T) = 0, \quad (4.52)$$

$$(H + B_2^T PS)M_2 = 0, \quad (4.53)$$

$$\begin{aligned}
& P_2(F + S^{-1}\dot{S} - S^{-1}AS - S^{-1}B_2HS + GCS - \gamma^{-2}S^{-1}B_1B_1^T PS) \\
& + H^T(B_2^T PS + H) = 0, \quad (4.54)
\end{aligned}$$

$$\begin{aligned}
& (F + S^{-1}\dot{S} - S^{-1}AS - S^{-1}B_2HS + GCS - \gamma^{-2}S^{-1}B_1B_1^T PS)M_2 \\
& + (GR + T^T MC^T + \gamma^{-2}T^T MTP_2GDD^T)G^T = 0. \quad (4.55)
\end{aligned}$$

Note that if M_2 and P_2 are invertible for all t , Theorem 3.1 follows immediately from (4.49)-(4.55) by taking $S = I$. All other choice of S would correspond to a (possibly time-varying) change of coordinate system. Thus all (F, G, H) satisfying (4.49)-(4.55) are related by a change of coordinate system and thus \mathcal{K}^* is a unique function of \bar{P} , P , M and X .

If M_2 or P_2 are not invertible for all t , the situation is slightly more complicated and we need to use the following classical result.

Lemma 4.1 *Let (F, G, H) be a realization of a transfer function K . Then*

- (F, G', H) is also a realization of K iff $\forall t \in (0, t_f)$, $\text{Im}(G(t) - G'(t)) \subset \mathcal{O}(t)$,
- (F, G, H') is also a realization of K iff $\forall t \in (0, t_f)$, $\text{Ker}(H(t) - H'(t)) \supset \mathcal{C}(t)$,
- (F', G, H) is also a realization of K iff $\forall t \in (0, t_f)$, $\text{Im}(F(t) - F'(t)) \subset \mathcal{O}(t)$ or $\text{Ker}(F(t) - F'(t)) \supset \mathcal{C}(t)$.

Now suppose that (F, G, H) satisfies (4.49)-(4.55), then any other realization having the same transfer function also satisfies (4.49)-(4.55). So, thanks to Lemma 4.1, we can change H as long as HM_2 is not changed which means that thanks to (4.53) we can take

$$H = -B_2^T P S. \quad (4.56)$$

Applying again Lemma 4.1 but this time to G , from (4.52) we can see that without loss of generality we can suppose that

$$GR + T^T M C^T + \gamma^{-2} T^T M T P_2 G D D^T = 0. \quad (4.57)$$

Finally, using (4.56), (4.57), (4.54), (4.55) and Lemma 4.1, we can see that F can be chosen to satisfy

$$F + S^{-1} \dot{S} - S^{-1} A S - S^{-1} B_2 H S + G C S - \gamma^{-2} S^{-1} B_1 B_1^T P S = 0. \quad (4.58)$$

If we now use (4.56) and (4.57) to simplify (4.49)-(4.51), pick $S = I$ and let $X = G$. We recover all the equations in Theorem 3.1. Note that the particular choices that we have made concerning F , G and H had no effect on the transfer function of the controller. Same thing is true for the particular choice of the matrix S which corresponds simply to the choice of the basis. Thus there exists a unique controller \mathcal{K}^* associated with every P , \bar{P} , M and X satisfying (3.1)-(3.4). Finally, (3.8) follows from (4.15). This concludes the proof of Theorem 3.1.

4.2 Proof of Lemma 3.2

1- Suppose that there exists w_* such that

$$\frac{\int_0^{t_f} z^T z dt + x(t_f)^T \Phi x(t_f)}{\int_0^{t_f} w_*^T w_* dt} > \gamma^2 \quad (4.59)$$

which implies

$$\int_0^{t_f} z^T z dt + x(t_f)^T \Phi x(t_f) - \gamma^2 w_*^T w_* dt > 0. \quad (4.60)$$

But (4.13), with $\eta(0)$ and \mathcal{F} set to zero, implies that

$$\sup_w \int_0^{t_f} z^T z dt + x(t_f)^T \Phi x(t_f) - \gamma^2 w^T w dt = 0, \quad (4.61)$$

which is a contradiction.

2- Note simply that the left hand side of (3.10) equals $J(u_{\mathcal{K}^*}, 0)$ and the right hand side, $\sup_w J(u_{\mathcal{K}^*}, w)$.

4.3 Proof of Theorem 3.2

From (4.13) follows that

$$J^\dagger(\mathcal{K}_s) = \text{Tr } \mathcal{P}_* \mathcal{F} \mathcal{F}^T \quad (4.62)$$

where \mathcal{P}_* denotes the limiting solution of (4.11) with $\mathcal{P}_{t_f} = 0$. We already know that the algebraic version of this equation has a non-negative solution, (clearly

$$\mathcal{P}_s = \begin{pmatrix} \bar{P}_s & \bar{P}_s - P_s \\ \bar{P}_s - P_s & \bar{P}_s - P_s \end{pmatrix} \quad (4.63)$$

is a solution) and thus it has a least non-negative solution which is precisely \mathcal{P}_* . But the right hand side of (3.24) is just $\text{Tr } \mathcal{P}_s \mathcal{F} \mathcal{F}^T$ which since $\mathcal{P}_* \leq \mathcal{P}_s$, thanks to (4.62) implies (3.24).

The corollary follows from the theorem and Lemma 3.2.

5 Conclusion

In this report we have shown that the mixed H_2/H_∞ control problem is equivalent to a stochastic differential game problem. This has allowed us to extend the mixed-norm control ideas to time-varying problems over finite-horizons.

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