Controllability properties for elliptic systems, the fictitious domain method ans optimal shape design problems

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Propriétés de controlabilité pour les systèmes elliptiques, la méthode des domaines fictifs et problèmes de design optimal

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Résumé

La méthode des domaines fictifs est utilisée dans la résolution numérique d'équations aux dérivées partielles dans un domaine extérieur ou pour appliquer les différences finies aux systèmes définis dans des domaines avec une géométrie compliquée. Le principe de cette approche est très utile dans des problèmes avec des domaines inconnus au variables.

En tenant compte de certaines propriétés de controlabilité exacte pour les équations elliptiques linéaires ou nonlinéaires, on peut développer une méthode d'approximation pour les problèmes de design optimal par des problèmes de contrôle dans un domaine fixé.

Abstract

The fictitious domain method is used in the numerical solution of partial differential equations in exterior domains or in order to apply finite differences to systems defined in domains with a complicated geometry. The principle of this approach is very useful in problems with unknown or variable domains.

Taking into account certain exact controllability properties for linear or nonlinear elliptic equations, we develop an approximation method for optimal design problems by distributed control problems in a fixed domain.

Mots clés

Domaines fictifs. Equations elliptiques. Controlabilité exacte. Design optimal.

Key words

Fictitious domains. Elliptic equations. Exact controllability. Optimal design.

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1 Introduction

The fictitious domain approach is well known in the solution of partial differential equations by finite differences method in domains with a complicated geometry or in exterior domains, G.P. Astrakmantsev [1], W. Proskurowski, O. Windlund [15], R. Glowinski et al. [7]. One variant associates to the given equation a distributed control problem governed by the extension of the equation to a more advantageous larger domain. See P. Joly and C. Atamian [9], C. Atamian [2] for an analysis along these lines of the exterior Helmholtz problem.

A similar embedding of domains idea, with boundary control, was used independently by J. Blum [6] in free boundary problems related to the physics of plasma.

Naturally, the main numerical disadvantage of this approach of working "larger problems", disappears in applications concerning unknown or variable domain problems. In the first setting the Blum's work enters, while for the second we quote the work of Hofmann, Kocvara and Haslinger [8] devoted to the study of optimal shape design problems by a geometric-distributed control procedure. Moreover, a controllability-type argument for elliptic systems shows that the control problem may generate in an implicit manner a large class of variable domains which are considered in optimal design problems. In this way the geometric optimization problems may be discussed by a purely control approach, in a fixed domain. This was studied, in the case of boundary control and with a more limited range of applications, in the papers of Barbu and Tiba [5], Tiba [17], Neitaanmaki, Makinen and Tiba [13]. See also Tiba [18], ch. III.5 for a specific example along these lines.

One important advantage, by comparison with the standard boundary variation technique, is to avoid the new mesh generation at each step of the algorithm, which is extremely time consuming.

The aim of this paper is two fold : first, we establish some exact controllability properties for nonlinear elliptic systems and, next, we use them in general linear or nonlinear optimal design problems Pironneau [14], Barbu and Friedman [3], Barbu and Stojanovic [4].

Finally, we mention that a first approximate controllability-type result for elliptic equations may be found in the classical monograph of J.L. Lions [10], p. 85.

2 Exact distributed controllability properties

Assume that D is a subdomain of the domain $\Omega \subset R^2$, such that ∂D and $\partial\Omega$ are regular. We define the controlled variational inequality, in $\Omega - \bar{D}$:

$$-\Delta z + \beta(z) \ni u \qquad \Omega - \bar{D}, \tag{2.1}$$

$$z = 1 \qquad \partial\Omega, \tag{2.2}$$

$$z = 1 \qquad \partial D, \tag{2.3}$$

where $\beta \subset R \times R$ is a maximal monotone graph given by $\beta(r) = 0$, $r > 0$, $\beta(0) =]-\infty, 0]$, $\beta(r) = \emptyset$, $r < 0$. The boundary conditions are motivated by the subsequent applications in optimal design and don't play an essential role in the argument. The controllability problem we study is to find $u \in L^2(\Omega - D)$ such that

$$\frac{\partial z}{\partial n} = \varphi \text{ in } \partial D, \tag{2.4}$$

where φ is given in $H^{1/2}(\partial D)$.

Theorem 2.1 *The problem (2.4) has at least one exact solution $u \in L^2(\Omega - D)$.*

Proof Let $\tilde{z} \in H^2(\Omega - D) \cap C^1(\Omega - \bar{D})$ (not unique) be given by the trace theorem, such that

$$\frac{\partial \tilde{z}}{\partial n} = \varphi, \quad \tilde{z} = 1 \qquad \partial D, \tag{2.5}$$

$$\tilde{z} = 1 \qquad \partial \Omega. \tag{2.6}$$

The interior regularity of $\tilde{z}$ may be inferred by assuming it as the solution of a linear fourth order elliptic problem in $\Omega - \bar{D}$, with appropriate boundary conditions.

In a neighbourhood of ∂D, we have $\tilde{z} > 0$. We fix a regular curve Γ in this neighbourhood, surrounding ∂D, at some distance $c > 0$ from ∂D. To choose such a constant, we use the continuity of $\tilde{z}$ and a compactness argument. We define

$$\hat{z}(x) = \begin{cases} \tilde{z}(x) \text{ between } \partial D \text{ and } \Gamma, \\ 1 \text{ between } \Gamma \text{ and } \partial\Omega. \end{cases} \tag{2.7}$$

By a local regularization of $\hat{z}$ around Γ (see the next lemma), we construct z_ε associated to a regularization parameter $\varepsilon > 0$, which satisfies (2.2), (2.3), (2.4). We may take, obviously, $\beta(z_\varepsilon) = 0$ in $\Omega - D$ and compute u by (2.1).

Lemma 2.1 *Let $\bar{D} \hookrightarrow \Omega \subset R^n$ be regular domains and $\varphi \in C(\bar{\Omega}) \cap H^2(\Omega)$ be such that $\varphi \mid_{\Omega - \bar{D}} \in C^1(\Omega - \bar{D})$, $\varphi \mid_D \in C^1(D)$. There exists $\varphi_\varepsilon \in C^1(\Omega) \cap H^2(\Omega)$ such that*

$$\varphi_\varepsilon \to \varphi \quad in \quad C(\bar{\Omega}), \tag{2.8}$$

$$\varphi_\varepsilon > \varphi \quad in \quad \Omega, \tag{2.9}$$

$$\varphi_\varepsilon \mid_{\partial\Omega} = \varphi \mid_{\partial\Omega}, \qquad \nabla\varphi_\varepsilon \mid_{\partial\Omega} = \nabla\varphi \mid_{\partial\Omega} . \tag{2.10}$$

Proof Let Γ_ε^1, Γ_ε^2 be two regular surfaces surrounding ∂D, at distance less than ε from ∂D and such that $\Gamma_\varepsilon^1 \subset D$, $\Gamma_\varepsilon^2 \subset \Omega - \bar{D}$. We define, between Γ_ε^1 and Γ_ε^2, the mapping

$$\tilde{\varphi}_\varepsilon(x) = \int_{[0,1]^n} \varphi(x - \varepsilon\, dist^2(x, \Gamma_\varepsilon^1).dist^2(x, \Gamma_\varepsilon^2)\tau)\rho(\tau)d\tau, \tag{2.11}$$

where ρ is a Friedrichs mollifier, that is $\rho \in C_0^\infty(R^n)$, $\rho \geq 0$, supp $\rho \subset [0,1]^n$ and $\int_{[0,1]^n} \rho(\tau)d\tau = 1$.

If $x \to x_0 \in \Gamma_\varepsilon^i$, $i = 1, 2$, then $dist(x, \Gamma_\varepsilon^i) \to 0$ and, by the uniform continuity of φ, we get $\tilde{\varphi}_\varepsilon(x) \to \varphi(x_0)$, therefore $\tilde{\varphi}_\varepsilon \mid_{\Gamma_\varepsilon^i} = \varphi \mid_{\Gamma_\varepsilon^i}$.

By (2.11), for $x \notin \Gamma_\varepsilon^i$, we have :

$$\frac{\partial}{\partial x_i}\tilde{\varphi}_\varepsilon(x) = \int_{[0,1]^n} \frac{\partial \varphi}{\partial x_i}(\cdots)(1 - \varepsilon\chi(x)\tau_i)\rho(\tau)d\tau, \tag{2.12}$$

where $\lim_{x \to x_0} \chi(x) = 0$, $\forall x_0 \in \Gamma^i_\varepsilon$. Then (2.12) also gives $\lim_{x \to x_0} \frac{\partial}{\partial x_i} \tilde{\varphi}_\varepsilon(x) = \frac{\partial}{\partial x_i} \varphi(x_0)$, $x_0 \in \Gamma^i_\varepsilon$, that is $\nabla \tilde{\varphi}_\varepsilon |_{\Gamma^i_\varepsilon} = \nabla \varphi |_{\Gamma^i_\varepsilon}$.

We can define

$$\hat{\varphi}_\varepsilon(x) = \begin{cases} \tilde{\varphi}_\varepsilon(x), & x \text{ between } \Gamma^1_\varepsilon, \Gamma^2_\varepsilon, \\ \varphi(x) & \text{otherwise.} \end{cases} \tag{2.13}$$

Obviously, $\hat{\varphi}_\varepsilon \in C^1(\Omega)$ and (2.10) is satisfied. Moreover, again by the uniform continuity of φ, (2.8) is true as $\varepsilon \to 0$. However, (2.9) may be violated. In order to overcome this difficulty, we denote $K_\varepsilon = |\hat{\varphi}_\varepsilon - \varphi|_{C(\bar{\Omega})}$, $K_\varepsilon \to 0$, and the mapping

$$\eta(x) = \ dist\ (x, \partial\Omega)^2. \tag{2.14}$$

As the closure of the set $\{\hat{\varphi}_\varepsilon(x) \neq \varphi(x)\}$ is a compact in Ω, then

$$\eta(x) \geq K > 0 \tag{2.15}$$

on it. By (2.8), (2.14), (2.15), we see that

$$\varphi_\varepsilon(x) = \hat{\varphi}_\varepsilon(x) + \frac{2K_\varepsilon}{K} \eta(x) \tag{2.16}$$

will check (2.8), (2.9). It also satisfies (2.10) since $\eta \in H^2_0(\Omega)$.

Remark Higher order regularity, boundary conditions or convergence may be obtained working with higher order powers of the distance function.

Remark The local regularization, preserving inequalities and traces has a clear intuitive geometric contents. The above lemma has the advantage to provide an explicit construction in quite a general situation with respect to the singularities of φ.

As another application of this lemma, we give an exact distributed controllability result for the coincidence set, in the obstacle problem. This may be compared with the work of Barbu and Tiba [5], where the case of boundary control and approximate controllability is discussed.

Corollary 2.1 *Let $\bar{D} \hookrightarrow \Omega$ be regular domains. There is $u \in L^2(\Omega)$ (not unique) such that the variational inequality :*

$$-\Delta y + \beta(y) \ni u \qquad \Omega \tag{2.17}$$

$$y = 1 \qquad \partial\Omega \tag{2.18}$$

has the coincidence set $E_y = \{x \in \Omega\ ;\ y(x) = 0\}$ equal to $\bar{D}$.

Proof Consider in $\Omega - \bar{D}$ a regular surface Γ, at distance greater than $c > 0$ from ∂D and $\partial\Omega$, with c a suitable constant. Define the piecewise regular mapping ψ in $\Omega - D$, as follows :

$$\psi(x) = \begin{cases} dist^2(x, \partial D), & x \text{ between } \Gamma \text{ and } \partial D, \\ 1, & x \text{ between } \Gamma \text{ and } \partial\Omega. \end{cases}$$

Applying the regularization technique described in *lemma 2.2*, in the proof, we obtain a function $\psi_\varepsilon > 0$, satisfying the same boundary conditions. It remains only to define

$$y(x) = \begin{cases} \psi_\varepsilon & x \in \Omega - D \\ 0 & x \in \bar{D} \end{cases}$$

and $\beta(y) = 0$ in D (for instance). Then u is obtained by (2.17).

Remark The above result also ensures that $\beta(y)$ (the unknown reaction of the obstacle) may be taken 0 in Ω.

Remark Theorem 2.1 is also valid in the linear case ($\beta(y) = 0$). A stronger statement may be obtained, by the use of the HUM method Lions [11], allowing the control u to have the support in a neighbourhood of ∂D.

3 Optimal design in electrochemical machining process

We consider the regular domains $C \subset E \subset D \subset \Omega$, D variable. In $D - C$, we solve the obstacle problem

$$-\Delta y + \beta(y) \ni f \qquad D - C, \tag{3.1}$$
$$y = 0 \qquad \partial C, \tag{3.2}$$
$$y = 1 \qquad \partial D, \tag{3.3}$$

where β is the maximal monotone graph defined in section 2 and $f \in L^2(\Omega \backslash C)$. This models the electrochemical processing in industry : ∂C and ∂D are the electrodes and the boundary condition $y = 1$ on ∂D signifies that some constant tension is applied, Ockendon and Elliott [16]. The coincidence set

$$E_y = \{x \in D \backslash C \ ; \ y(x) = 0\} \tag{3.4}$$

gives the final shape of the metal piece obtained around C.

The optimal design problem discussed by Barbu and Friedman [3], Barbu and Stojanovic [5], is to find the domain D, $E \subset D \subset \Omega$, such that

$$E_y \supset E \backslash C. \tag{3.5}$$

Taking into account the definition of E_y, an obvious approach is

$$(P) \qquad \min_D \frac{1}{2} \int_{E \backslash C} y^2 dx$$

subject to (3.1)-(3.3).

We associate to (P) a sequence of optimal control problems and we prove, by the controllablity results established in section 2, that a strong approximation relationship is valid :

$$(P_n) \qquad \min_u \left\{ \frac{1}{2} \int_{E \backslash C} y^2 dx + n \int_{\Omega_y} (u - f)^2 dx \right\}$$

subject to

$$-\Delta y + \beta(y) \ni u \qquad \Omega - C, \tag{3.6}$$
$$y = 1 \qquad \partial\Omega, \tag{3.7}$$
$$y = 0 \qquad \partial C, \tag{3.8}$$

where $E \subset \Omega_y \subset \Omega$ is the smallest Lipschitzian domain with the property that $y = 1$ on $\partial\Omega_y$. The existence of Ω_y is ensured by (3.7).

Neither (P), nor (P_n) have ensured existence of optimal pairs since no compactness assumption is made.

If we assume the existence of a Lipschitz domain D, $E \subset D \subset \Omega$, which "solves" (3.1)-(3.5), then Theorem 2.1 allows to extend the variational inequality (3.1) to the whole domain $\Omega \backslash C$ and to find a control $u^* \in L^2(\Omega\backslash C)$ such that $u^* |_D \equiv f$ and the cost associated to it is null, therefore optimal.

The following result refines this remark

Theorem 3.1 *For any $n \in N$, the problem (P) is embedded in (P_n) and*

$$\inf(P) \geq \inf(P_n) \tag{3.9}$$

conversely, if $\delta_n > 0$ is small and $[y_n, u_n]$ is a δ_n optimal pair for (P_n), then Ω_{y_n} is an ε_n optimal subdomain of (P) with $\varepsilon_n > 0$ small, depending on δ_n.

Proof Let D be a Lipschitz subdomain of Ω, $E \subset \bar{D} \hookrightarrow \Omega$. By Theorem 2.1, there is $u_D \in L^2(\Omega\backslash D)$ such that the solution y_D of

$$-\Delta y_D + \beta(y_D) \ni u_D \qquad \Omega - \bar{D} \tag{3.10}$$

$$y_D = 1 \qquad \partial\Omega \cup \partial D \tag{3.11}$$

satisfies $\dfrac{\partial y_D}{\partial n} |_{\partial D} = -\dfrac{\partial \tilde{y}}{\partial \nu} |_{\partial D}$, where $\tilde{y}$ is the solution of (3.1)-(3.3) and n, ν are the normal vectors to ∂D in both directions.

Then the pair $[\bar{y}, \bar{u}]$

$$\bar{u}(x) = \begin{cases} u_D & \Omega - \bar{D}, \\ f & D - C, \end{cases} \tag{3.12}$$

$$\bar{y}(x) = \begin{cases} y_D & \Omega - \bar{D}, \\ \tilde{y} & D - C, \end{cases} \tag{3.13}$$

is admissible for (P_n) and we may take $\Omega_{\bar{y}} \subseteq D$, so $J_n(\bar{u}) = J(D)$.

By a variant of a result of Pironneau [14], Ch. III, p. 32, we may extend these considerations to all $E \subset D \subset \Omega$, which ends the proof of (3.9).

Conversely, by (3.9) we get

$$\int_{\Omega_{y_n}} (u_n - f)^2 dx \leq \frac{C + \delta_n}{n}, \quad C = \inf(P). \tag{3.14}$$

Let $\tilde{y}_n$ denote the solution of

$$-\Delta \tilde{y}_n + \beta(\tilde{y}_n) \ni f \qquad \Omega_{y_n} - C, \tag{3.15}$$

$$\tilde{y}_n = 1 \qquad \partial\Omega_{y_n}, \tag{3.16}$$

$$\tilde{y}_n = 0 \qquad \partial C. \tag{3.17}$$

Since, obviously, y_n satisfies :

$$-\Delta y_n + \beta(y_n) \ni u_n \quad \Omega_{y_n} \tag{3.18}$$

and (3.16), (3.17), then (3.14)-(3.18) and the continuous dependence on the right-hand side in variational inequalities shows that

$$|\tilde{y}_n - y_n|^2_{H^1(\Omega_{y_n})} \leq \varepsilon_n,$$

which ends the proof.

Remark We notice the generality of the argument developped in sections 2,3 which may be, similarly, applied to other types of equations or boundary conditions.

Remark Compared with the standard mapping method, Murat and Simon [12], we underline the simplicity of the above technique, the differential operator being maintained unchanged and the penalization term in the cost functional being very simple.

4 The algorithm

We detail in this section the algorithm suggested by the previous results, on a typical linear optimal design problem, Pironneau [14] :

$$(S) \qquad \min_D \left\{ \int_E |\nabla y - y_d|^2 dx \right\}$$

$$-\Delta y = f \qquad D, \tag{4.1}$$
$$y = 0 \qquad \partial D. \tag{4.2}$$

Above, we consider Lipschitzian domains $E \subset D \subset \Omega \subset R^N$, D variable, and $f \in L^2(\Omega)$, $y_d \in L^2(E)^N$.

The associated approximating control problems are

$$(S_n) \qquad \min_u \left\{ \int_E |\nabla y - y_d|^2 dx + n \int_{\Omega_y} (u - f)^2 dx \right\}$$

$$-\Delta y = u \qquad \Omega, \tag{4.3}$$
$$y = 0 \qquad \partial\Omega, \tag{4.4}$$

with $\Omega_y \subset \Omega$ being the smallest Lipschitzian domain such that $E \subset \Omega_y$ and $y = 0$ on $\partial\Omega_y$.

Remark If $u \geq 0$ in Ω_y, then the maximum principle shows that Ω_y is a connected component of $supp(y_+)$, y_+ being the positive part of y.

In the absence of specific assumptions nor the problem (S) neither the problem (S_n) have ensured the existence of optimal solutions. However, a result similar to Theorem 3.1 may be obtained, starting with the linear variant of Theorem 2.1

Theorem 4.1 *(S) is embedded in (S_n) and*

$$\inf(S_n) \leq \inf(S), \quad \forall n. \tag{4.5}$$

Conversely, let $u_n \in L^2(\Omega)$ be an δ_n-optimal control for (S_n), $\delta_n > 0$, a small parameter. Then Ω_{y_n} is an ε_n-optimal domain for (S) with $\varepsilon_n > 0$ small, depending on δ_n.

Let $[u, y]$ be an admissible pair for (S_n) (for any n). From now on, we assume that $\nabla y \neq 0$ in a neighbourhood of $\partial\Omega_y \subset \Omega \subset R^2$.

Remark If $\partial\Omega_y$ is regular, then $\nabla y \neq 0$ is equivalent with $\dfrac{\partial y}{\partial n} \neq 0$ on $\partial\Omega_y$ as the tangential component is null.

If $f \geq 0$ in Ω, applying the maximum principle and the Hopf lemma to the solution of

$$-\Delta\tilde{y} = f \qquad \Omega_y, \tag{4.6}$$
$$\tilde{y} = 0 \qquad \partial\Omega_y, \tag{4.7}$$

we get $\dfrac{\partial\tilde{y}}{\partial n} < 0$ in $\partial\Omega_y$. Taking into account the proof of Theorem 3.1, we see that the δ_n-optimal pairs for (S_n) are "close" for $\tilde{y}$ on Ω_y, thus justifying our assumption.

We take $u, v \in H^1(\Omega)+PC(\Omega)$ (piecewise continuous and bounded in Ω) and we compute the gradient of the cost functional for (S_n) :

$$I = \lim_{\lambda\to 0}\frac{1}{\lambda}\left\{\frac{1}{2}\int_E\{|\nabla y_\lambda - y_d|^2 - |\nabla y - y_d|^2\} + y\int_{\Omega_{y_\lambda}}(u+\lambda v - f)^2 - n\int_{\Omega_y}(u-f)^2\right\}$$

where y_λ is the solution associated to $u + \lambda v$ via (4.3), (4.4). Obviously $y_\lambda = y + \lambda r$ with

$$-\Delta r = v \qquad \Omega, \tag{4.8}$$
$$r = 0 \qquad \partial\Omega. \tag{4.9}$$

Proposition 4.2 *Under the above conditions*

$$I = \int_E(\nabla y - y_d, \nabla r)dx + 2n\int_{\Omega_y}(u-f)v - n\int_{\partial\Omega_y}\frac{(u-f)^2 r}{\frac{\partial y}{\partial n}}d\sigma. \tag{4.10}$$

Proof It is enough to study the limit

$$I_1 = \lim_{\lambda\to 0}\frac{1}{\lambda}\left\{\int_{\Omega_{y_\lambda}\cap w}(u-f)^2dx - \int_{\Omega_y\cap w}(u-f)^2dx\right\}, \tag{4.11}$$

and w is a neighbourhood of some arbitrary point $M \in \partial\Omega_y$. By the convergence properties of y_λ, we have $\nabla y_\lambda \neq 0$ in M, for λ small and the implicit function theorem will define C^1 curves $\Gamma_\lambda = \partial\Omega_{y_\lambda} \cap w$, $\Gamma = \partial\Omega_y \cap w$, for any M, starting with the equation $y_\lambda = 0$, respectively $y = 0$.

Choosing a new local system of coordinates in M, with $\vec{x}_1$ normal to $\partial\Omega_y$ and $\vec{x}_2$ tangent, then we can express

$$\Gamma \;:\; x_1 = \alpha(x_2) = \beta(x_2, 0), \tag{4.12}$$
$$\Gamma_\lambda \;=\; x_1 = \alpha_\lambda(x_2) = \beta(x_2, \lambda) \tag{4.13}$$

with some C^1 mappings $\alpha, \beta, \alpha_\lambda$.

Assume that $\Omega_{y_\lambda} \cap w \supset \Omega_y \cap w$ and (by further restricting it) $w = [a,b] \times [c,d]$ with a, b, c, d some constants. In the new coordinates, we have :

$$I_1 = \lim_{\lambda\to 0}\int_{(\Omega_{y_\lambda}-\Omega_y)\cap w}(u-f)^2dx = \lim_{\lambda\to 0}\frac{1}{\lambda}\int_c^d dx_2\int_{\alpha(x_2)}^{\alpha_1(x_2)}(u-f)^2dx_1 \tag{4.14}$$

Above, we suppose that $y > 0$, $r > 0$, $\lambda > 0$, in $\Omega_y \cap w$, the other cases being possible to discuss similarly.

Moreover, again by the implicit function theorem, we get

$$\lim_{\lambda \to 0} \frac{\alpha_\lambda(x_2) - \alpha(x_2)}{\lambda} = -\frac{r(\alpha(x_2), x_2)}{y_{x_1}(\alpha(x_2), x_2)}. \tag{4.15}$$

Then (4.14), (4.15) and a Leibniz-Newton type formula in the interior integral give :

$$\begin{aligned} I_1 &= -\int_c^d (u-f)^2(\alpha(x_2), x_2) \frac{r(\alpha(x_2), x_2)}{y_{x_1}(\alpha(x_2), x_2)} dx_2 \\ &= \int_c^d (u-f)^2(\alpha(x_2), x_2) \frac{r(\alpha(x_2), x_2))}{|y_{x_1}(\alpha(x_2), x_2)|} \frac{\sqrt{1+\alpha'(x_2)^2}}{\sqrt{1+\frac{y_{x_2}^2(\alpha(x_2),x_2)}{y_{x_1}^2(\alpha(x_2),x_2)}}} dx_2 \\ &= \int_{\partial\Omega_y \cap w} (u-f)^2 \frac{r}{|\nabla y|} d\sigma = -\int_{(\partial\Omega_y)\cap w} \frac{(u-f)^2 r}{\frac{\partial y}{\partial n}} d\sigma. \end{aligned} \tag{4.16}$$

The last form is independent of the system of axes and remains valid in all the possible configurations, thus ending the proof.

We apply the adjoint system method in order to obtain an explicit representation of (4.10). We define the adjoint equation, in variational formulation :

$$\begin{aligned} &\int_\Omega \nabla z . \nabla g - \int_E (\nabla y - y_d). \nabla g + n \int_{\partial\Omega_y} \frac{(u-f)^2}{\frac{\partial y}{\partial n}} g d\sigma = 0 \\ &\forall g \in H_0^1(\Omega), \quad z \in H_0^1(\Omega). \end{aligned} \tag{4.17}$$

The existence of a unique solution $z \in H_0^1(\Omega)$ for (4.17) follows from the minimization on $H_0^1(\Omega)$ of the functional

$$\frac{1}{2}\int_\Omega |\nabla z|^2 dx - \int_E (\nabla y - y_d). \nabla z dx + n \int_{\partial\Omega_y} \frac{(u-f)^2}{\frac{\partial y}{\partial n}} z d\sigma,$$

which is convex, lower semicontinuous, proper and coercive.

Taking $g = r$ in (4.17), we infer

$$\int_E (\nabla y - y_d) \nabla r = \int_\Omega \nabla z \nabla r + n \int_{\partial\Omega_y} \frac{(n-f)^2}{\frac{\partial y}{\partial n}} t d\sigma,$$

that is

$$I = \int_\Omega \nabla z \nabla r + 2n \int_{\Omega_y} (u-f) v = \int_\Omega z v + 2n \int_{\Omega_y} (u-f) v.$$

Finally, the gradient of the cost functional J_n, associated to (S_n) is

$$\nabla J_n(u) = z + 2n(u-f)\chi_{\Omega_y}, \tag{4.18}$$

where χ_{Ω_y} is the characteristic function of Ω_y.

Remark In a gradient algorithm, we notice that (4.18) will preserve the regularity of the control, required for the first iteration.

Remark Under regularity assumptions, the adjoint equation (4.17) may be formally interpreted as a transmission problem. If $p = z\,|_{\Omega-\bar{\Omega}_y}$ and $q = z\,|_{\Omega_y}$, then :

$$\begin{aligned} -\Delta p &= 0 && \Omega - \bar{\Omega}_y, \\ -\Delta q &= -div[\chi_E(\nabla y - y_d)] && \Omega_y, \\ p &= q && \partial\Omega_y, \\ p &= 0 && \partial\Omega, \\ \frac{\partial p}{\partial n} + \frac{\partial q}{\partial r} &= -\frac{n(u-f)^2}{\frac{\partial y}{\partial n}} && \partial\Omega_y. \end{aligned}$$

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