On r-partition designs in hamming spaces

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Sur les configurations de r-partition dans les espaces de Hamming

Version révisée

On r-partition designs in Hamming spaces

Revised version

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Résumé

Le concept de matrice combinatoire d'un code sans restriction et la notion de configuration de r-partition (on pourra dire aussi configuration cohérente) admise par un code sont introduits et discutés en détail. La théorie comprend une caractérisation des codes complètement réguliers et une interprétation combinatoire du fait que les lignes distinctes de la matrice de distribution des distances d'un code soient linéairement indépendantes. En général, il est possible de calculer la matrice de distribution des distances de tout code admettant une configuration de partition donnée en résolvant un système bien défini d'équations linéaires; c'est une technique efficace pourvu que le nombre de classes de la partition soit relativement petit.

Abstract

The concept of the combinatorial matrix of an unrestricted code and the notion of an r-partition design admitted by a code are introduced and discussed in detail. The theory includes a characterization of completely regular codes, and a combinatorial interpretation of the fact that the distinct rows of the distance distribution matrix of a code are linearly independent. In general, it is possible to compute the distance distribution matrix of any code admitting a given partition design by solving a well-defined system of linear equations; this is an efficient technique provided the number of classes in the partition is relatively small.

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1 Introduction

The distance distribution matrix B of a code C contains a good deal of useful information about C. Furthermore, for a large class of codes that enjoy some interesting combinatorial properties, this matrix can be determined, in a mechanical manner, from a relatively small set of data [6]. The concept of an r-partition design was introduced recently, in the case of linear codes, to enable one to determine the matrix B only with the knowledge of that partition whenever C coincides with one class of the partition, while the other classes are unions of cosets of C [2]. Such a partition with three classes ($r = 2$) appears when studying two-weight projective codes, because these codes yield partial difference sets with two parameters [1]. This notion generalizes naturally to the r-partition design concept, for any value of the parameter r.

The present contribution extends that approach to the case of unrestricted ("nonlinear") codes. (Further extensions to the theory of "codes in regular graphs" were investigated by Montpetit [10].) We are dealing with r-partition designs $\{E_0, E_1, ..., E_r\}$ of the ambient Hamming space F^n (over a q-ary alphabet F); they exactly are coherent configurations in Higman's terminology [8]. Assuming that we are given such a partition design (with a relatively small value of r), we are able to construct the distance distribution matrix B of any code C which is a union of some classes E_u. In this situation, C is said to admit the r-partition design in question. Conversely, it may happen that, knowing the matrix B of a code C, we are able to determine the r-partition design with the smallest r that is admitted by C.

The main results of the paper are concerned with combinatorial properties of classes of "remarkable codes"; these properties are closely related to r-partition designs, and are derived from some theorems about paths in Hamming spaces. Four characteristic integers, associated with any code C, play a major role in the theory. They are the covering radius of C, the rank of the distribution matrix B of C, the number of distinct rows in B, and the smallest r for which C admits an r-partition design. This last integer is denoted $\bar{r}$ and is called the regularity number of C.

Section 2 contains some preliminary results concerning paths in Hamming spaces, and it introduces the combinatorial matrix of a code. The integer s_{ij} which counts the length j paths joining two points in $\mathbf{F}^n$ at distance i apart is interpreted as the index i component of the jth power of the binomial $P_1(\xi) = n(q-1) - q\xi$, reduced modulo $\xi(\xi-1)...(\xi-n)$, in the basis of Krawtchouk polynomials $P_i(\xi)$. The combinatorial matrix A of a code $C \subset \mathbf{F}^n$ has as its (x, j) entry the number of length j paths joining the point $x \in \mathbf{F}^n$ to the code C. It is related to the distance distribution matrix B by the identity $A = BS$, with $S = (s_{ij})$. The columns of A are shown to satisfy a well-defined minimum order linear recurrence. The characteristic polynomial of this recurrence has degree $t+1$, where t is the external distance of C. More precisely, its zeros are the integers $P_1(w_i)$, where $w_0 = 0, w_1, ..., w_t$ are the dual weights of C. This material is an extention to unrestricted codes of notions and results given previously in [2], [3] and [4] for the case of linear codes.

Section 3 is devoted to the general theory of r-partition designs. By definition, they are $(r+1)$-class partitions $\{E_0, ..., E_r\}$ of $\mathbf{F}^n$ such that the number m_{uv} of points in E_v at distance 1 from any point in E_u is a constant. The square matrix $M = (m_{uv})$ is referred to as the associate matrix of the given r-partition design. Perfect codes [12], uniformly packed codes [13] and, more generally, completely regular codes [5] are then characterized in terms of partition designs. For example, we obtain the following result. A code C with covering radius ρ is completely regular if and only if it admits a ρ-partition design (which necessarily is the distance partition). In this case, we have $\rho = t = \bar{r}$, and the eigenvalues of the associate matrix M are the numbers $P_1(w_i)$, with $w_0, w_1, ..., w_t$ the dual weights of C. In general, if C admits an r-partition design, then we have $r \geq t$. The case where the bound is sharp, i.e., $\bar{r} = t$, is characterized as follows. A code C with external distance t admits a t-partition design if and only if its distance distribution matrix B contains exactly $t+1$ distinct rows. The result is an analogue of Theorem 6.11 in [5] for unrestriced codes.

Section 4 is concerned with the case of linear codes (over a field alphabet $\mathbf{F}$). We give a description of an r-partition design admitted by a linear code C, under the restrictive assumption that each class E_u is a union of cosets of C. In this interpretation, the column set of a parity check matrix for C appears as some kind of generalized difference set [1]. Furthermore, by making use of Theorems 6.10 and 6.11 in [5] we obtain the following results.

A linear code C with external distance t admits a t-partition design if and only if the orthogonal code $C^{\perp}$ carries an association scheme with respect to the Hamming distance relations. The character matrix of the dual of the $C^{\perp}$ scheme, which is the coset scheme of C, was determined by Montpetit in terms of the t-partition design admitted by C; the rows of this matrix are the left eigenvectors of the associate matrix M [11]. Finally, we give two illustrative examples, one with $\rho = t < \bar{r}$, and the other with $\rho < t < \bar{r}$.

2 The combinatorial matrix of code

Let F be a finite alphabet with q elements ($q \geq 2$), and let $H(n,q)$ denote the Hamming space of dimension n over F, that is, the nth Cartesian power F^n of F equipped with the Hamming distance d (for any positive integer n). By definition, the distance $d(x,y)$ between two points x and y in F^n equals the number of coordinate positions in which x and y differ. The space $H(n,q)$ has the combinatorial structure of a metric association scheme, called the Hamming scheme. The reader is referred to [5] for the relevant notions and results about that subject. The generating graph of the Hamming scheme has the elements of F^n as its points and the pairs of points $\{x,y\}$ with $d(x,y) = 1$ as its edges.

Definition 2.1 *A path of length j joining two points x and y in F^n is a sequence $x_0, x_1, ..., x_{j-1}, x_j$, with $x_0 = x$ and $x_j = y$, such that $d(x_{i-1}, x_i) = 1$ for $i = 1, ..., j$.*

It is clear that the Hamming distance $d(x,y)$ equals the length of the shortest path joining x and y. Let $D_0, D_1, ..., D_n$ denote the $q^n \times q^n$ distance relation matrices of the space $H(n,q)$. By definition, D_i has F^n as its row and column labelling set, and the (x,y) entry of D_i is given by $D_i(x,y) = 1$ if $d(x,y) = i$ and $D_i(x,y) = 0$ otherwise. In the Bose–Mesner algebra of $H(n,q)$, generated by the distance relation matrices, consider the identity

$$D_1^j = \sum_{i=0}^{n} s_{ij} D_i, \tag{1}$$

for any nonnegative integer j. This defines the real numbers $s_{0j}, s_{1j}, ..., s_{nj}$. If $d(x,y) = i$, then $D_1^j(x,y) = s_{ij}$. Therefore, s_{ij} counts the paths of length j between any two points at distance i apart.

Definition 2.2 *The* <u>*path matrix*</u> *of $H(n,q)$ is the $(n+1)\times\infty$ matrix $S=(s_{ij})$ where s_{ij} is the number of length j paths joining two points x and y in F^n such that $d(x,y)=i$, for $0\le i\le n$ and $0\le j<\infty$.*

Definition 2.3 *The* <u>*Krawtchouk polynomial*</u> *$P_i(\xi)$, of degree i in the variable ξ, with parameters n and q, is given by*

$$P_i(\xi)=\sum_{k=0}^{i}(-1)^k(q-1)^{i-k}\binom{\xi}{k}\binom{n-\xi}{i-k},\tag{2}$$

for $i=0,1,...,n$. The Krawtchouk matrix $P=(p_{\ell i})$ is the square matrix of order $n+1$ defined by $p_{\ell i}=P_i(\ell)$ for ℓ , $i=0,...,n$.

Proposition 2.1 *The path matrix S is upper triangular. Its diagonal entry s_{ii} equals $i!$, and divides all entries s_{ij} (with $0\le j<\infty$), for each i. It can be written as*

$$S=q^{-n}PV,\tag{3}$$

where P is the Krawtchouk matrix and V denotes the $(n+1)\times\infty$ matrix having $P_1(\ell)^j=(n(q-1)-q\ell)^j$ as its (ℓ,j) entry, for $0\le\ell\le n$ and $0\le j<\infty$.

Proof: The first property ($s_{ij}=0$ for $j<i$) is obvious from the definition. To prove the second statement, let us consider two points x and y in F^n, with $d(x,y)=i$. Without loss of generality we can assume $x=(0,...,0)$ and $y=(1,...,1,0,...,0)$, where 0 and 1 denote two distinguished symbols in F. The number of 1 components of y is equal to i. Let G denote the symmetric group of degree i acting on the first i coordinate positions. It is clear that G acts as a permutation group on the set of paths of a given length j (with $j\ge i$) joining x and y. Moreover, no such path is fixed by an element of G, except by the identity. This implies that the number s_{ij}, counting the length j paths, is divisible by $i!$ (the order of G). In particular, it is easily seen that G acts transitively on the length i paths, which means that s_{ii} equals $i!$.

The last property follows from the fact that the integers $P_i(\ell)$, with $\ell=0,1,...,n$, are the eigenvalues of D_i. (Recall that all matrices in the

Bose–Mesner algebra are diagonalized by the same orthogonal transformation). By use of (1) we then deduce the relation

$$P_1(\ell)^j = \sum_{i=0}^{n} s_{ij} P_i(\ell), \tag{4}$$

for $0 \leq \ell \leq n$ and $0 \leq j < \infty$. In matrix form, this gives $V = PS$, whence the result (3) since $P^2 = q^n I$. □

Let C be a code of length n over the q-ary alphabet F. This simply means that C is a nonempty subset of F^n. We shall describe some significant properties of C by use of the "combinatorial matrix" concept introduced below for unrestricted codes.

Definition 2.4 *For any point $x \in \mathsf{F}^n$, let $A_j(x)$ be the number of length j paths that join x to a given code C, for $0 \leq j < \infty$, and let $B_i(x)$ be the number of elements of C at distance i form x, for $0 \leq i \leq n$. The combinatorial matrix of C is the $q^n \times \infty$ matrix A having $A_j(x)$ as its (x, j) entry. The distance distribution matrix of C is the $q^n \times (n+1)$ matrix B having $B_i(x)$ as its (x, i) entry [5], [6].*

The next result is elementary; it shows the "equivalence" between the matrices A and B. We shall see in Section 3 that the combinatorial matrix A is especially suited to the study of partition designs. This is mainly due to the existence of a well-defined linear recurrence for the columns of A (see Theorem 2.1).

Proposition 2.2 *The combinatorial matrix A and the distance distribution matrix B of any code C have the same rank. More precisely, they are related by the identity $A = BS$, where S is the path matrix of $H(n, q)$.*

Proof: By the very definition of the numbers $A_j(x)$ and $B_i(x)$ we have the relation

$$A_j(x) = \sum_{i=0}^{n} s_{ij} B_i(x). \tag{5}$$

In matrix form, (5) yields $A = BS$. This implies rank A = rank B since, by Proposition 2.1, the first $n+1$ columns of S constitute a nonsingular matrix. □

Definition 2.5 *(see [5], [6]) The inner distance distribution of a code C is the rational $(n+1)$-tuple (row vector) $a = (a_0, a_1, ..., a_n)$, where a_i is the average number of code points at distance i from a given code point, i.e.,*

$$a_i = |C|^{-1}|\{(x,y) \in C^2 : d(x,y) = i\}|. \tag{6}$$

The MacWilliams transform of a is the rational $(n+1)$-tuple $a' = aP$, where P is the Krawtchouk matrix. Set $a' = (a'_0, a'_1, ..., a'_n)$ with $a'_0 = 1$, and let $w_0 = 0, w_1, ..., w_t$ denote the indices ℓ, with $0 \le \ell \le n$, such that $a'_\ell \ne 0$. The integer t is called the external distance of C, and the $t+1$ numbers w_i are the dual weights of C. The monic polynomial of degree $t+1$ having as its zeros the numbers $P_1(w_i) = n(q-1) - qw_i$ is referred to as the annihilator polynomial for C.

If C is a linear code over a field alphabet, then the dual weights w_i are nothing but the Hamming weights of the orthogonal code $C^\perp$. The choice of the name "external distance" for the parameter t (denoted r in [5] and s' in [6]) is motivated by the relation between t and the covering radius (see Proposition 2.3 below). The following result plays a major role in this paper. (It is closely related to Theorem 3.2 in [6]).

Theorem 2.1 *The external distance of any code C equals the smallest non-negative integer t for which the columns of the combinatorial matrix A of C satisfy a recurrence of order $t+1$, that is*

$$\sum_{j=0}^{t+1} c_j A_{j+k}(x) = 0, \tag{7}$$

for $x \in \mathsf{F}^n$ and $0 \le k < \infty$, where $c_0, c_1, ..., c_{t+1}$ are rational numbers with $c_{t+1} = 1$. Moreover, the minimum order recurrence is unique and its coefficients c_j are integers; they are determined via the identity

$$\sum_{j=0}^{t+1} c_j \xi^j = \prod_{i=0}^{t} (\xi - n(q-1) + qw_i), \tag{8}$$

where $w_0 = 0, w_1, .., w_t$ are the dual weights of C. In other words, the minimum degree recurrence polynomial is the annihilator polynomial for the code C.

Proof: Let us endow the alphabet $\mathbf{F}$ with the structure of an Abelian group (of order q). We shall use an additive notation for the group operation. Then the Hamming distance $d(x,y)$ between two points x and y in $\mathbf{F}^n$ can be written as $d(x,y) = \|x-y\|$, where $\|h\|$ denotes the Hamming weight of h, i.e., the number of nonzero components of h. We shall make some calculations in the group algebra $\mathbf{QF}^n$ of the Abelian group $\mathbf{F}^n$ over the field $\mathbf{Q}$ of rational numbers. It will be convenient to write any element $K \in \mathbf{Q}F^n$ as a "polynomial" in an indeterminate Z, that is

$$K = \sum_{x \in \mathbf{F}^n} K(x) Z^x, \quad \text{with } K(x) \in \mathbf{Q}. \tag{9}$$

In the sequel, we identify a given subset R of $\mathbf{F}^n$ with the element $R \in \mathbf{QF}^n$ defined by $R(x) = 1$ if $x \in R$ and $R(x) = 0$ otherwise.

Let Y denote the set of points h in $\mathbf{F}^n$ of weight $\|h\| = 1$. For any nonnegative integer j we compute the product CY^j in the group algebra. We obtain

$$CY^j = (\sum_{g \in C} Z^g)(\sum_{h \in Y} Z^h)^j = \sum_{x \in \mathbf{F}^n} A_j(x) Z^x, \tag{10}$$

since $A_j(x)$ counts the $(j+1)$-tuples $(g, h_1, ..., h_j)$, with $g \in C$ and $h_1 \in Y, ..., h_j \in Y$, such that $x = g + h_1 + ... + h_j$. Now consider any linear combination of the identities (10), that is

$$C(\sum_{j=0}^{\infty} c_j Y^j) = \sum_{x \in \mathbf{F}^n} (\sum_{j=0}^{\infty} c_j A_j(x)) Z^x, \tag{11}$$

with $c_j \in \mathbf{Q}$ and $c_j = 0$ except for a finite number of indices j. Let χ_u denote the complex valued irreducible group character associated with a given element u of $\mathbf{F}^n$. By a q-ary extension of Theorem 7 in chapter 5 of [9] (see also [5] and [6]) it can be shown that $\chi_u(C)$ is nonzero for at least one element u of a given weight $\|u\| = \ell$ if and only if ℓ is one of the dual weights w_i of C.

Hence, by applying the character χ_u to both sides of the relation (11), and making use of $\chi_u(Y) = P_1(\|u\|)$, we see that $\Sigma c_j A_j(x)$ vanishes for all $x \in \mathbf{F}^n$ if and only if $\Sigma c_j P_1(w_i)^j$ equals zero for $i = 0, 1, ..., t$. The latter condition exactly means that $\Sigma c_j \xi^j$ is divisible by the annihilator polynomial for C. This leads to the desired conclusions. $\square$

Corollary 2.1 *The rank of the combinatorial matrix A of a code C with external distance t is equal to $t + 1$.*

Remark : In view of Proposition 2.1 and 2.2, Corollary 2.1 also follows from the known fact that the rank of B is equal to $t + 1$ ([5], [6]).

Definition 2.6 *For a point $x \in \mathsf{F}^n$ and a code $C \subset \mathsf{F}^n$, let $d(x, C)$ denote the distance between x and C, that is*

$$d(x, C) = \min\{d(x, y) : y \in C\}. \tag{12}$$

The <u>covering radius</u> of C, denoted ρ, is defined as follows:

$$\rho = \max\{d(x, C) : x \in \mathsf{F}^n\}. \tag{13}$$

A code C is said to be <u>completely regular</u> [5] if the x-row $(B_i(x))_{i=0}^n$ of its distance distribution matrix depends only on $d(x, C)$, for all $x \in \mathsf{F}^n$. Let $\gamma + 1$ denote the number of distinct rows in the combinatorial matrix A of C (or, equivalently, in the distribution matrix B). The integer γ is called the <u>combinatorial number</u> of C.

Proposition 2.3 *The external distance of a code C is bounded from below by its covering radius and from above by its combinatorial number:*

$$\rho \leq t \leq \gamma. \tag{14}$$

The code C is completely regular if and only if both bounds are sharp, i.e., $\rho = t = \gamma$.

Proof: The right hand inequality in (14) follows immediately from Definition 2.6, in view of Corollary 2.1. To prove the left hand inequality, suppose we have $\rho \geq t+1$. Then there exists a point $x \in \mathsf{F}^n$ satisfying $d(x, C) = t+1$. This means $A_j(x) = 0$ for $j = 0, 1, ..., t$ and $A_{t+1}(x) \neq 0$, in contradiction with Theorem 2.1. Finally, we observe that the matrix A contains $\rho + 1$ distinct rows $(A_j(x))_{j=0}^{\infty}$ enjoying the properties $d(x, C) = 0, 1, ..., \rho$, respectively. Therefore, the equality $\rho = \gamma$ is a necessary and sufficient condition for the code C to be completely regular. □

3 Partition designs

Definition 3.1 *Let r be an integer, with $0 \leq r \leq q^n - 1$. An r-partition design for the Hamming space $H(n,q)$ is defined as a partition $\{E_0, E_1, ..., E_r\}$ of the set F^n into $r+1$ classes, with the following regularity property. For any point $x \in E_u$, the number of points $y \in E_v$ satisfying $d(x,y) = 1$ is a constant m_{uv}, independent of the choice of x. This holds for each u and v in $\{0, 1, ..., r\}$. (Thus a partition design is the same concept as a coherent partition [8]). The $(r+1) \times (r+1)$ matrix*

$$M = (m_{uv})_{u,v=0}^{r} \tag{15}$$

is called the associate matrix of the given r-partition design. Let C be a code that can be written in the form

$$C = \bigcup_{v \in R} E_v, \tag{16}$$

where R is a nonempty subset of $\{0, 1, ..., r\}$. Then C is said to admit the r-partition design $\{E_0, E_1, ..., E_r\}$. The special case $C = E_u$ (for a certain u) is referred to as the homogeneous case. For a given code C, the smallest nonnegative integer r for which C admits an r-partition design is denoted by $\bar{r}$ and is called the regularity number of C.

In particular, interesting examples of partition designs can be obtained by means of the following method (which could be referred to as the "group case"). Let G be any group of isometries of the Hamming space $H(n,q)$. It is easily seen that the set of orbits of G on F^n is a partition design. We will see an example of that type at the end of the paper.

It is clear that any code C admits the trivial partition design, where the classes E_u are the one-element subset of F^n. (In this situation, we have $r = q^n - 1$ and $M = D_1$). Therefore, the regularity number $\bar{r}$ is well defined. Note that a given r-partition design is admitted by $2^{r+1} - 1$ distinct codes C.

Let us make some preliminary observations about the associate matrix M and its eigenvalues. Counting in two different ways the pairs (x,y) in

$E_u \times E_v$ with $d(x,y) = 1$ we obtain the symmetry property

$$|E_u| m_{uv} = |E_v| m_{vu}. \tag{17}$$

As a result, M is equivalent to a symmetric real matrix under a diagonal similarity transformation (defined by the square roots of the cardinalities $|E_u|$). This implies that M is a diagonalizable matrix.

From the definition it follows that the eigenvalue spectrum of any associate matrix M is a subset of that of the distance relation matrix D_1, which consists of the numbers $P_1(\ell)$ with $\ell = 0, 1, ..., n$. In particular, $P_1(0)$ always is an eigenvalue of M (with the all one eigenvector), since we have

$$\sum_{v=0}^{r} m_{uv} = n(q-1), \tag{18}$$

for each $u \in \{0, 1, ..., r\}$. The present result about the spectrum of associate matrices will be tightly strenghtened by Theorem 3.3 in the sequel.

We now give one of the basic results of this paper. It shows the intimate connection that exists between the partition design concept and the combinatorial matrix concept.

Theorem 3.1 *Suppose that the code C admits the r-partition design $\{E_0, E_1, ..., E_r\}$. Then the entries of the combinatorial matrix A of C satisfy the identity $A_j(x) = A_j(y)$ for all points x and y belonging to the same class E_u, and for all integers $j \geq 0$. Furthermore the numbers $A_j(u)$, defined by $A_j(u) = A_j(x)$ for $x \in E_u$, satisfy the recurrence relation*

$$A_j(u) = \sum_{v=0}^{r} m_{uv} A_{j-1}(v). \tag{19}$$

Proof: We argue by induction on j. Let R denote the subset of $\{0, 1, ..., r\}$ characterized by (16). For a point $x \in E_u$ we have $A_0(x) = 1$ or 0 depending on whether $u \in R$ or not. So the first statement is true when $j = 0$.

Assume now that the numbers $A_{j-1}(y) = A_{j-1}(v)$ depend only on the class E_v that contains y. Given a point $x \in E_u$, consider the length j

paths joining x to a point in C (see Definition 2.1). Since any such path $x_0 = x, x_1, ..., x_j$ (with $x_j \in C$) can be viewed as the concatenation of the length 1 path x_0, x_1 with the length $j-1$ path $x_1, ..., x_j$, we can write the identity

$$A_j(x) = \sum_{x_1 \in \mathsf{F}^n} D_1(x, x_1) A_{j-1}(x_1). \tag{20}$$

By our induction assumption, $A_{j-1}(x_1)$ equals $A_{j-1}(v)$ for each $x_1 \in E_v$. Furthermore, it is seen from Definition 3.1 that the numbers $D_1(x, x_1)$ with $x_1 \in E_v$ add up to m_{uv}. Therefore, the right hand side of (20) coincides with that of (19). This proves the regularity property $A_j(x) = A_j(u)$ for all $x \in E_u$, together with the recurrence (19). □

Consider the following equivalence relation over the set F^n. Two points x and y in F^n are said to be combinatorially equivalent with respect to a given code $C \subset \mathsf{F}^n$ if the corresponding rows of the combinatorial matrix A of C are equal, i.e., if we have $A_j(x) = A_j(y)$ for all j (Note that, in view of Proposition 2.6, we obtain exactly the same equivalence relation if we use the distance distribution matrix instead of the combinatorial matrix.) According to Definition 2.6, this relation has $\gamma + 1$ equivalence classes, with γ denoting the combinatorial number of C.

Definition 3.2 *The combinatorial partition relative to a code C is the $(\gamma+1)$-class partition of F^n induced by the combinatorial equivalence relation (with respect to C). The distance partition relative to C is the $(\rho+1)$-class partition $C_0 = C, C_1, ..., C_\rho$ of F^n defined by*

$$C_i = \{x \in \mathsf{F}^n : d(x, C) = i\}. \tag{21}$$

As shown by the proof of Proposition 2.3, the combinatorial partition is a refinement of the distance partition, and coincides with it if and only if the code is completely regular. In general, neither the distance partition nor even the combinatorial partition have the regularity properties of a partition design. However, Theorem 3.1 states that any r-partition design admitted by a code C is a refinement of the combinatorial partition relative to C. Thus, we can extend the inequalities of Proposition 2.3 as follows.

Proposition 3.1 *For any code C, the covering radius ρ, the external distance t, the combinatorial number γ, and the regularity number $\bar{r}$ form a* *nondecreasing sequence:*

$$\rho \leq t \leq \gamma \leq \bar{r}. \tag{22}$$

In case $\bar{r} = \gamma$, the $\bar{r}$-partition admitted by C is unique; it is the combinatorial partition relative to C.

As explained below, the case of equality $t = \bar{r}$ is of special interest in the theory. The following result plays a significant role in our study. It states that the central equality in (22), i.e., $t = \gamma$, implies the property we are interested in.

Theorem 3.2 *A code C with external distance t admits a t-partition design (i.e., satisfies $\bar{r} = t$) if and only if the distinct rows of the combinatorial matrix A of C are linearly independent, i.e., if and only if the equality $\gamma = t$ is satisfied.*

Proof: The "only if" part is immediate in view of Proposition 3.1. To prove the converse statement we shall make use of relation (20). Let us denote by $E_0, E_1, ..., E_\gamma$ the classes of the combinatorial partition of F^n relative to C. For a point $x \in E_u$, define $m_{uv}(x)$ as the number of points $x_1 \in E_v$ satisfying $d(x, x_1) = 1$. Then, the identity (20) can be written as

$$A_j(u) = \sum_{v=0}^{\gamma} m_{uv}(x) A_{j-1}(v), \tag{23}$$

with $A_j(u) = A_j(x)$ and $A_{j-1}(v) = A_{j-1}(x_1)$. Our assumption is $\gamma = t$. Since the matrix A has rank $t + 1$ (Corollary 2.1), the system of linear equations (23), with $1 \leq j < \infty$, determines the "unknowns" $m_{uv}(x)$ in a unique manner. Hence, $m_{uv}(x)$ is independent of x, which proves that the combinatorial partition (with $\gamma = t$) is a t-partition design. □

As a straightforward consequence of Theorem 3.2, combined with Proposition 2.3, we obtain the following remarkable characterization of the complete regularity property. (Note that the ρ-partition design mentioned below belongs to the homogeneous case.)

Corollary 3.1 *A code C is completely regular if and only if the distance partition $\{C_0 = C, C_1, ..., C_\rho\}$ relative to C is a partition design or, equivalently, if and only if the regularity number $\bar{r}$ is equal to the covering radius ρ.*

Let us now examine the question of determining the eigenvalues of the associate matrix M, especially in the case $\bar{r} = t$. (See preliminary comments in the beginning of this section.) In general, we can state the following result.

Theorem 3.3 *Assume that the r-partition design* $\{E_0, E_1, ..., E_r\}$ *is admitted by a code* C *with dual weights* $w_0 = 0, w_1, ..., w_t$. *Then the* $t+1$ *numbers* $\lambda_0, \lambda_1, ..., \lambda_t$ *given by*

$$\lambda_i = P_1(w_i) = n(q-1) - qw_i \tag{24}$$

are eigenvalues of the associate matrix M *of this partition design.*

Proof: Set the vector $\mathcal{A}_j = (A_j(0), A_j(1), ..., A_j(r))^t$, with $A_j(u) = A_j(x)$ for $x \in E_u$ as in Theorem 3.1. Relation (19) can be written in matrix form as $\mathcal{A}_j = M\mathcal{A}_{j-1}$. Hence, we have

$$\mathcal{A}_j = M^j \mathcal{A}_0, \quad \text{for } 0 \leq j < \infty. \tag{25}$$

Consider the minimal polynomial $p(\xi) = \sum_{j=0}^{s} p_j \xi^j$ of the matrix M. Since M is diagonalizable, $p(\xi)$ has as its zeros (each with multiplicity 1) the eigenvalues of M. Combining the assumption $p(M) = 0$ with the formula (25) we readily deduce the relation

$$\sum_{j=0}^{s} p_j A_{j+k}(u) = 0, \tag{26}$$

for $0 \leq u \leq r$ and $0 \leq k < \infty$. In view of Theorem 2.1, this implies that $p(\xi)$ is divisible by the annihilator polynomial for C. Hence, the zeros $P_1(w_i)$ of the latter are among the zeros of the former, which proves the theorem. □

In the case of equality $\bar{r} = t$ and, more particularly, in the case of a completely regular code (see Corollary 3.1), the result above gives the whole spectrum of the associate matrix. Indeed, Theorem 3.3 has the following immediate consequence.

Corollary 3.2 *If a code* C *with external distance* t *admits a t-partition design, then the eigenvalues* λ_i *of the associate matrix* M *are the numbers* $\lambda_i = P_1(w_i)$, *for* $i = 0, 1, ..., t$, *where* $w_0, w_1, ..., w_t$ *are the dual weights of* C.

To conclude this section we give a characterization of perfect codes and of uniformly packed codes in the framework of partition design theory.

Recall that a code C with packing radius e and covering radius ρ is perfect if and only if $\rho = e$, and is uniformly packed if and only if $\rho = t = e + 1$. In the latter case it is known that the number μ_i counting the points in C at distance ρ from a given point $x \in \mathsf{F}^n$ with $d(x, C) = e + i$, is a constant for $i = 0, 1$ (see [13]); the numbers μ_0 and μ_1 are called the parameters of C.

Theorem 3.4 *A code C with packing radius e and regularity number $\bar{r}$ is perfect if and only if $\bar{r} = e$, and is uniformly packed if and only if $\bar{r} = e + 1$. In both cases ($\bar{r} = e$ or $e+1$), the $\bar{r}$-partition design admitted by C is unique; it is the distance partition relative to C. The associate matrix M is tridiagonal, i.e., $m_{uv} = 0$ for $| u - v | \geq 2$, and satisfies*

$$m_{u,u-1} = u \text{ for } 1 \leq u \leq e, \; m_{uu} = u(q-2) \text{ for } 0 \leq u \leq e-1, \tag{27}$$

together with (18). In the second case, i.e., for a uniformly packed code with parameters μ_0 and μ_1, the missing entries of M are given by

$$m_{ee} = (e+1)\mu_0 + e(q-2), \quad m_{e+1,e} = (e+1)\mu_1. \tag{28}$$

Proof: The first part of the theorem can be deduced from Corollary 3.1, since perfect codes and uniformly packed codes are completely regular. Let us however give a more explicit argument, and explain how to compute the associate matrix M. Consider the distance partition $\{C_i : 0 \leq i \leq \rho\}$ relative to C. For $x \in C_u$ define the integer $m_{uv}(x)$ as the number of points $y \in C_v$ satisfying $d(x, y) = 1$. If C is perfect or uniformly packed, then it can easily be verified that the relations given above are satisfied when $m_{uv}(x)$ is substituted for m_{uv}. Therefore, all numbers $m_{uv}(x)$ are independent of x (for each u), which proves that the distance partition is a ρ-partition design (with $\rho = e$ or $e + 1$). Conversely, if $\bar{r} = e$ or $e + 1$, then we have $\gamma = t = \rho = \bar{r}$, by Proposition 3.1, which implies that C is perfect or uniformly packed. □

By Corollary 3.2, the dual weights w_i of a perfect or uniformly packed code can be expressed very simply in terms of the eigenvalues λ_i of the tridiagonal associate matrix M described in Theorem 3.4. This result gives strong necessary conditions for the existence of such codes; they are equivalent to those obtained from the Lloyd theorem for perfect codes [12] and its analogue for uniformly packed codes [13].

Remark : As it stands, the theory of uniformly packed codes [13] is restricted to the case where the order q of the alphabet is a prime power. However, the basic concepts and results of this theory can be extended without difficulty to arbitrary alphabet orders.

4 The linear case, with some examples

Let us now concentrate on the interesting special case where C is a linear code over a field alphabet F, i.e., a linear subspace of the linear space F^n. In the first part of this section we shall see how to interpret the combinatorial matrix of C, and the concept of a partition design admitted by C (in a well-defined restrictive sense) in terms of a parity check matrix for C. The packing radius of the code C will be assumed to be strictly positive; this means that any two columns of the parity check matrix are linearly independent.

In the sequel, F^k denotes a k-dimensional linear space over the field F, and the elements of F^k are represented as column vectors of length k (with components in F). Let F^* be the set of nonzero elements of F. Given a subset Ω of F^k, we denote by $\mathsf{F}^*\Omega$ the subset of F^k that contains the vector αh with $\alpha \in \mathsf{F}^*$ and $h \in \Omega$. Recall that a parity check matrix H for a linear code C of length n and codimension k is a $k \times n$ matrix of rank k over F such that we have

$$C = \{x \in \mathsf{F}^n : Hx = 0\}. \tag{29}$$

Proposition 4.1 *Let H be a parity check matrix for a linear code C with packing radius $e \geq 1$, and let Ω denote the set of columns of H. If $h = Hx$ is the syndrome associated with a given point $x \in \mathsf{F}^n$, then the entry $A_j(x)$ of the combinatorial matrix A of C counts the $2j$-tuples $(\alpha_1, ..., \alpha_j, h_1, ..., h_j)$, with $\alpha_i \in \mathsf{F}^*$ and $h_i \in \Omega$ for $1 \leq i \leq j$, satisfying $h = \alpha_1 h_1 + ... + \alpha_j h_j$.*

Proof: Consider a length j path $x_0 = x, x_1, ..., x_{j-1}, x_j$, with $x_j \in C$, joining x to the code C. Since $d(x_{i-1}, x_i) = 1$ we have $H(x_{i-1} - x_i) = \alpha_i h_i$ for a unique pair (α_i, h_i) in $\mathsf{F}^* \times \Omega$. By use of the identity $x - x_j = \sum_{i=1}^{j}(x_{i-1} - x_i)$, together with $Hx_j = 0$, we obtain the relation $h = \sum_{i=1}^{j} \alpha_i h_i$. As the argument can also be applied backwards, this proves the proposition. □

Proposition 4.2 *Let C be the linear code defined from a given $k \times n$ parity check matrix H (of rank k) that has pairwise linearly independent columns. Suppose that C admits an r-partition design $\{E_0, E_1, ..., E_r\}$, with $E_0 = C$ (homogeneous case), such that each class E_u is a union of some cosets of C in F^n. For $u = 0, 1, ..., r$, define the subset Δ_u of the space F^k as follows:*

$$\Delta_u = \{Hx : x \in E_u\}. \tag{30}$$

Then the set $\{\Delta_0, \Delta_1, ..., \Delta_r\}$ is a partition of F^k, with $\Delta_0 = \{0\}$. The entries m_{uv} of the associate matrix M can be expressed in terms of this partition in the following manner. For any choice of an element $a \in \Delta_u$, the number m_{uv} counts the pairs $(b, h) \in \Delta_v \times \mathsf{F}^*\Omega$ satisfying $a = b + h$, where Ω is the set of columns of H.

Conversely, let $\{\Delta_0, \Delta_1, ..., \Delta_r\}$ be a partition of F^k, with $\Delta_0 = \{0\}$, and let Ω be an n-subset of F^k whose elements are pairwise linearly independent and span the whole space F^k. Assume that the number of pairs $(b, a - b)$ in $\Delta_v \times \mathsf{F}^*\Omega$ depends only on the class Δ_u that contains a. Let H be a $k \times n$ matrix having the elements of Ω as its columns. For $u = 0, 1, ..., r$, define the subset E_u of the space F^n as follows:

$$E_u = \{x \in \mathsf{F}^n : Hx \in \Delta_u\}. \tag{31}$$

Then the set $\{E_0, E_1, ..., E_r\}$ is an r-partition design admitted by the linear code $C = E_0$ with parity check matrix H. Moreover, each class E_u is a union of cosets of C.

Proof: The first statement, saying that the sets Δ_u in (30) form a partition of F^k, is proved by straightforward verification. To prove the second statement we argue as follows. By definition, m_{uv} counts the points $y \in E_v$ satisfying $d(x, y) = 1$, for a given $x \in E_u$. Setting the syndrome vectors $a = Hx$ and $b = Hy$, we see that the condition $d(x, y) = 1$ can be expressed exactly by $a - b \in \mathsf{F}^*\Omega$. This leads to the desired conclusion. The converse result can be proved in a similar manner. □

It is interesting to notice that the set $\mathsf{F}^*\Omega$ is a union of some classes Δ_u with $1 \leq u \leq r$. This expresses the fact that the set C_1 in (21) is a union of some classes E_u, since any partition design admitted by C is a refinement of the distance partition relative to C (see Theorem 3.1 and Proposition 2.3).

The characterization of linear type partition designs stated in Proposition 4.2 is somewhat more general than the definition given in [2]. Note also that a 2-partition design of that type gives rise to a partial difference set with 2 parameters [1]. So we may consider the sets Ω underlying r-partition designs (as explained in Proposition 4.2) as some kind of "generalized difference sets".

In the second part of this section we give a characterization of linear subschemes of a Hamming scheme in terms of partition designs. Let C be any linear code of length n over the finite field F. For a given point $x \in \mathsf{F}^n$, consider the x-row $B(x) = (B_i(x))_{i=0}^n$ of the distance matrix B of C. It is clear that $B(x)$ depends only on the coset of C that contains x. More precisely, $B(x)$ is the weight distribution of the coset $x + C$. We shall denote by $b^{(0)}, b^{(1)}, ..., b^{(\gamma)}$ the distinct rows $B(x)$. It particular, $b^{(0)}$ is the weight distribution of C itself. (By definition, the parameter γ is the combinatorial number of C.) Let us define the symmetric relations $R_0, R_1, ..., R_\gamma$ over the coset space F^n/C as follows:

$$R_u = \{(x + C, y + C) : B(x - y) = b^{(u)}\}. \tag{32}$$

Note that R_0 is the identity relation, and that the relations R_u form a partition of the Cartesian square of the coset space. We first recall a result of [5], in a form that is appropriate to our purpose.

Theorem 4.1 [5]. *The distinct rows of the distance distribution matrix of C are linearly independent (or, equivalently, the combinatorial number γ of C is equal to its external distance t) if and only if the restriction of the Hamming scheme to the orthogonal code $C^\perp$ is an association scheme. In this case, the $t + 1$ relations $R_0, R_1, ..., R_t$ endow the coset space F^n/C with the structure of an association scheme, which is the dual of the scheme carried by $C^\perp$.*

Recall that the external distance t of a linear code C is the number of nonzero weights of the orthogonal code $C^{\perp}$. From Theorems 3.2 and 4.1 we immediately deduce the caracterization alluded to above; the result is the following.

Theorem 4.2 *A linear code C with external distance t admits a t-partition design if and only if the restriction of the Hamming scheme to $C^{\perp}$ is an association scheme.*

The association scheme $\{R_u : 0 \leq u \leq t\}$ carried by the space F^n/C (under the assumption $\bar{r} = t$) is called the coset scheme of C. As shown by Montpetit [11], the character matrix of this scheme (having as its (i,j) entry the ith eigenvalue of the jth relation matrix, for $0 \leq i,j \leq t$) can be determined from the associate matrix M of the unique t-partition design admitted by C. For $i = 0, 1, ..., t$, let $z_i = (1, z_{i1}, ..., z_{it})$ be the normalized left eigenvector of M corresponding to the eigenvalues $\lambda_i = P_1(w_i)$. Thus, we have

$$z_i M = P_1(w_i) z_i. \tag{33}$$

Theorem 4.3 [11]. *Let C be a linear code satisfying $\bar{r} = t$. The rows of the character matrix of the coset scheme of C are the normalized left eigenvectors $z_0, z_1, ..., z_t$ of the associate matrix M of the t-partition design admitted by C.*

Remark : The last results, Theorems 4.1–4.3, can be extended to the case where C is an additive code, i.e., a subgroup of the direct product of n Abelian groups of order q (for any value of q).

Let us conclude with two examples. Recall the basic inequalities $\rho \leq t \leq \gamma \leq \bar{r}$ (see Proposition 3.1), together with the striking fact that $t = \gamma$ implies $\gamma = \bar{r}$ (by Theorem 3.2). Our first example, with $\rho = t = 6$ and $\gamma = \bar{r} = 7$, shows that the converse statement is not true in general, even when $\rho = t$. This examples stands in contrast with the Goethals–van Tilborg theorem [7], as stated in [1], which tells us that $\rho = t = 2$ implies $\bar{r} = 2$.

To prove the existence of a unique r-partition design with $r = \gamma$ admitted by the considered code we call upon the argument used for proving Theorem 3.2. There will here be a unique solution to (23) although $\gamma + 1$ is larger than the rank $t + 1$ of the linear system because the variables $m_{uv}(x)$ are constrained to be nonnegative integers and because some of them must vanish.

Let C be the first order Reed-Muller code of length $n = 16$ (with $q = 2$). As shown in p. 418 of [9], the distance distribution matrix B of C has eight distinct rows (so, $\gamma = 7$), and its covering radius ρ equals 6. Furthermore, as the orthogonal code $C^{\perp}$ (the extended Hamming code) has six nonzero weights (4, 6, 8, 10, 12, 16), the external distance of C is also equal to 6. It can be proved that the combinatorial partition of $\mathbf{F}^{16}$ relative to the code C is a partition design. In view of Theorem 3.1, this means that the combinatorial number $\bar{r}$ of C is given by $\bar{r} = \gamma = 7$. Thus, C admits a unique 7-partition design $\{E_0, E_1, ..., E_7\}$, where a class E_u contains all points $x \in \mathbf{F}^{16}$ for which the coset $x + C$ has a given weight distribution $b^{(u)}$, for $u = 0, 1, ..., 7$. (This example is a linear type partition design, as described in Proposition 4.2). The associate matrix M is

$$M = \begin{bmatrix} 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 15 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 14 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 12 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 16 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 16 & 0 \end{bmatrix}. \qquad (34)$$

The result $\bar{r} = 7$, together with the explicit matrix (34), can be obtained by the following argument. First, we compute the combinatorial matrix A from the distance distribution matrix B given in [9], by use of Proposition 2.2. For a point $x \in E_u$, define the integer $m_{uv}(x)$ as the number of points $y \in E_v$ satisfying $d(x, y) = 1$ (for $u, v = 0, 1, ..., 7$). We have to show that $m_{uv}(x)$ is independent of x, and is equal to the (u, v) entry in (34). From the basic properties of C it can be seen that $m_{uv}(x)$ vanishes (for all $x \in E_u$) for each coordinate position (u, v), with $u < v$, in which the corresponding entry of (34) is zero. By symmetry, the same conclusion holds for the positions (u,v) such that $u > v$.

Now consider the system of linear equations (19), for all $j \geq 1$, in the unknowns m_{uv} (with $0 \leq u,v \leq r = 7$). It can be verified that this system admits a unique solution, given in (34), under the following two constraints: m_{uv} is a nonnegative integer, and $m_{uv} = 0$ implies $m_{vu} = 0$. By definition, the numbers $m_{uv}(x)$ satisfy the system (19), and they obey the constraints for the reason explained above. Therefore, we have $m_{uv}(x) = m_{uv}$ for all $x \in E_u$. (Alternatively, the fact that the combinatorial partition relative to C is a partition design can be proved by standard group theoretic arguments, taking into account the result in chapter 14, section 3 of [9].)

In our second example we use a construction method introduced in [2] and based on Proposition 4.2. For a positive integer k, consider a subgroup G of the general linear group $GL_k(\mathsf{F})$, together with a subset Ω of F^k that is preserved by G. Assume that any two distinct vectors in Ω are linearly independent, and that Ω is a spanning set for F^k. Let $\Delta_0 = \{0\}, \Delta_1, ..., \Delta_r$ denote the orbits of G on the space F^k, and let H be a $k \times n$ matrix, with $n = |\Omega|$, having the elements of Ω as its columns (in any order). Define C as the linear code, of length n and codimension k, having H as a parity check matrix. It readily follows from Proposition 4.2 that C admits the linear type r-partition design $\{E_0, E_1, ..., E_r\}$, where E_u contains the points $x \in \mathsf{F}^n$ satisfying $Hx \in \Delta_u$.

Let us now assume that F is the field with two elements ($q = 2$). Define G as the subgroup of $GL_k(\mathsf{F})$ containing all permutation matrices (of order k), and define Ω as the set of vectors $h \in \mathsf{F}^k$ satisfying $\|h\| = p$, where p is a given integer, with $1 \leq p \leq k-1$ and $p \equiv 1 \pmod 2$. It is obvious that the orbits of G are the weight classes $\Delta_0, \Delta_1, ..., \Delta_k$, where Δ_u contains the vectors h in F^k satisfying $\|h\| = u$. Let us give the values of the entries of the associate matrix M (see Proposition 4.2). For $u, v = 0, 1, ..., k$, set $\ell = (u - v + p)/2$. Then we have

$$m_{uv} = \binom{u}{\ell}\binom{k-u}{v-u+\ell}, \tag{35}$$

if ℓ is an integer (and $p \geq |u - v|$), and $m_{uv} = 0$ otherwise.

As a specific example we consider the case $k = 11$ and $p = 3$, which yields $n = 165$. It can be verified that the covering radius and the external distance of the code C under discussion are $\rho = 5$ and $t = 9$. Furthermore, it turns out that the cosets of C have exactly 12 distinct weight distributions. (In other words, two cosets that are not equivalent under the group G have unequal weight distributions.) Thus, the regularity number $\bar{r}$ coincides with the combinatorial number γ; we have $\bar{r} = \gamma = k = 11$.

Note

The present paper is a revised version of INRIA report ♯ 626, february 1987, with the same title, which was submitted to Discrete Mathematics in 1986.

The fact that the publication of this paper has been delayed is due to the extremely long refeering procedure of Discrete Mathematics.

A first application of the results in INRIA report ♯ 626 was published by B. COURTEAU and A. MONTPETIT in Discrete Mathematics : "*Dual distance of completely regular codes*", received november 1986, appeared DM 88 (1991).

Some of the results were further generalized by André MONTPETIT to distance regular graphs in "*Codes dans les graphes réguliers*" Rapport N° 43, Université de Sherbrooke, Canada 1988 and "*Codes et partitions cohérentes dans les graphes réguliers*" Annales des sciences mathématiques du Québec, Vol. 14, N° 2, 1990 (received september 1988). See also "*Coherent Partitions and Codes*" Eurocode 1990, to appear in Springer Verlag Lecture Notes in Computer Science. The particular case of Distance-Regular Graphs is dealt with in A.E. BROUWER, A.M. COHEN and A. NEUMAIER, "*Distance-regular graphs*", Ergebnisse der Mathematik 3.18, Springer, Heidelberg (1989).

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