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A RELATIONSHIP BETWEEN STABILIZED FINITE ELEMENT METHODS AND THE GALERKIN METHOD WITH BUBBLE FUNCTIONS

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Mai 1991



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**A Relationship Between Stabilized Finite Element Methods
and the Galerkin Method with Bubble Functions**

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A Relationship between Stabilized Finite Element Methods and the Galerkin Method with Bubble Functions

Abstract :

A relation between stabilized finite element methods and the Galerkin method employing interpolations with bubble functions is established for the advective-diffusive model and for the linearized compressible Navier-Stokes equations. The bubble functions are shown to help in stabilizing the advective operator without recourse to upwinding or any other numerical strategy. In particular, for the advective-diffusive model, the Galerkin method employing piecewise linears with bubble functions is shown to be equivalent to the Streamline-Upwind/Petrov-Galerkin (SUPG) method in the diffusive limit.

Une Relation entre les Méthodes d'Eléments Finis

Stabilisées et la Méthode de Galerkin avec Fonctions Bulles

Résumé :

On établit, pour l'équation de diffusion-convection et pour les équations de Navier-Stokes compressibles linéarisées, un lien entre les méthodes d'éléments finis stabilisées et la méthode de Galerkin associée à des approximations avec fonctions bulles. On montre que les fonctions bulles aident à stabiliser l'opérateur de convection sans utiliser ni décentrage, ni autres artifices numériques. En particulier, pour l'équation modèle de diffusion-convection, on démontre que la méthode de Galerkin utilisant des fonctions linéaires par morceaux plus des fonctions bulles est équivalente à la méthode SUPG (Streamline-Upwind/Petrov-Galerkin) quand le terme de diffusion est prédominant.

1. The Advective-Diffusive Model

Let us first consider the (homogeneous-Dirichlet) steady advective-diffusive problem which consists in finding the scalar field $u = u(\mathbf{x})$, such that

$$\mathbf{a} \cdot \nabla u - \nabla \cdot (\kappa \nabla u) = f \quad \text{in } \Omega, \quad (1)$$

$$u = 0 \quad \text{on } \Gamma \quad (2)$$

where the given data are: the velocity field $\mathbf{a}(\mathbf{x})$, the diffusivity $\kappa = \kappa(\mathbf{x}) > 0$ and the source function $f(\mathbf{x})$. Herein $\Omega \subset \mathbb{R}^2$, is a bounded domain with a polygonal boundary Γ .

A standard variational formulation of (1) can be written as: Find $u \in H_0^1(\Omega)$ such that

$$(\mathbf{a} \cdot \nabla u, v) + (\kappa \nabla u, \nabla v) = (f, v) \quad v \in H_0^1(\Omega). \quad (3)$$

To introduce finite element approximations of (3), let us consider a partition \mathcal{C}_h of $\bar{\Omega}$ into elements consisting of triangles in the usual way (i.e., no overlapping is allowed between any two elements of the partition, the union of all element domains K reproduces $\bar{\Omega}$, etc.). Quasiuniformity is *not* assumed. As usual, we denote by $P_1(K)$ the set of all linear functions defined on K .

The simplest conforming Galerkin method to approximate (3) searches a solution in $V_1 \subset H_0^1(\Omega)$ where

$$V_1 = \{v_1 \in H_0^1(\Omega) \mid v_1|_K \in P_1(K), K \in \mathcal{C}_h\}. \quad (4)$$

More precisely, this method searches $u_1 \in V_1$ such that

$$(\mathbf{a} \cdot \nabla u_1, v_1) + (\kappa \nabla u_1, \nabla v_1) = (f, v_1) \quad v_1 \in V_1. \quad (5)$$

For uniform meshes, this method can be written as a finite-difference stencil which reproduces a centered difference approximation of the advective operator. It is well-known in the finite difference literature that for advective-dominated flows, i.e., when the mesh-Peclet number, defined by $Pe = |\mathbf{a}|h/(2\kappa)$, is large, spurious oscillations will frequently be detected in the solution of (5). A cure to this drawback widespread in the numerical literature is that some “artificial” viscosity has to be added to the formulation. In the finite element method this is achieved through the use of a Petrov-Galerkin formulation [6,9] or by a Stabilization Method such as SUPG of Hughes and Brooks [5,10] or Streamline Diffusion of Johnson *et al.* [12,13]. Heretofore, there has been a consensus that the Galerkin method could not be used to stabilize the advective operator. The reasoning is simple: Take $V_h \subset H_0^1(\Omega)$ to be a typical finite element space consisting of piecewise continuous polynomials or other sophisticated approximation. Then, as before, the method consists of finding $u_h \in V_h$ such that

$$B(u_h, v_h) = (\mathbf{a} \cdot \nabla u_h, v_h) + (\kappa \nabla u_h, \nabla v_h) = (f, v_h) \quad v_h \in V_h. \quad (6)$$

The stability of this method can be roughly examined by taking $v_h = u_h$ which yields

$$B(u_h, u_h) = (\mathbf{a} \cdot \nabla u_h, u_h) + \|\kappa^{1/2} \nabla u_h\|_0^2.$$

For a given velocity vector field with $\nabla \cdot \mathbf{a} = 0$, note that by integration by parts and using the homogeneous boundary condition (2) it follows that

$$(\mathbf{a} \cdot \nabla u_h, u_h) = 0.$$

Therefore, apparently, the stability of (6) relies, in this case, entirely on $\|\kappa^{1/2} \nabla u_h\|_0^2$ which can be “small” for very “small” κ ! From this argument, it

seems hopeless that the Galerkin method can produce stable solutions for small κ . We will see below, that this is not entirely true. For the moment let us review some stabilized methods for this equation:

i) Classical Upwind:

$$B_U(u_h, v_h) = (\mathbf{a} \cdot \nabla u_h, v_h) + ((\kappa + h)\nabla u_h, \nabla v_h) = (f, v_h) \quad v_h \in V_h \quad (7)$$

ii) Streamline Upwind Petrov-Galerkin:

$$\begin{aligned} B_{SUPG}(u_h, v_h) &= (\mathbf{a} \cdot \nabla u_h, v_h) + (\kappa \nabla u_h, \nabla v_h) \\ &\quad + \sum_{K \in \mathcal{C}_h} (\mathbf{a} \cdot \nabla u_h - \nabla \cdot (\kappa \nabla u_h), \tau \mathbf{a} \cdot \nabla v_h)_K \\ &= (f, v_h) + \sum_{K \in \mathcal{C}_h} (f, \tau \mathbf{a} \cdot \nabla v_h)_K, \end{aligned} \quad (8)$$

where

$$\tau(\mathbf{x}, \text{Pe}_K(\mathbf{x})) = \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p} \xi(\text{Pe}_K(\mathbf{x})) \quad (9)$$

$$\text{Pe}_K(\mathbf{x}) = \frac{m_k |\mathbf{a}(\mathbf{x})|_p h_K}{2\kappa(\mathbf{x})} \quad (10)$$

$$\xi(\text{Pe}_K(\mathbf{x})) = \begin{cases} \text{Pe}_K(\mathbf{x}) & , 0 \leq \text{Pe}_K(\mathbf{x}) < 1 \\ 1 & , \text{Pe}_K(\mathbf{x}) \geq 1 \end{cases} \quad (11)$$

$$|\mathbf{a}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^N |a_i(\mathbf{x})|^p \right)^{1/p} & , 1 \leq p < \infty \\ \max_{i=1, N} |a_i(\mathbf{x})| & , p = \infty \end{cases} \quad (12)$$

$$m_k = \min \left\{ \frac{1}{3}, 2\tilde{C}_k \right\} \quad (13)$$

$$\tilde{C}_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\Delta v\|_{0,K}^2 \leq \|\nabla v\|_0^2 \quad v \in V_h \quad (14)$$

(see Hughes and Brooks [10] for the original version of SUPG and [8] and references therein for other constructions of τ and further elaborations).

Contrary to the Galerkin method, these stabilized methods have improved stability characteristics even for “small” κ (or high mesh-Peclet number). Indeed, for the classical upwind method

$$B_U(u_h, u_h) = \|(\kappa + h)^{1/2} \nabla u_h\|_0^2$$

which is clearly more coercive than $B(u_h, u_h)$ for small κ . However the classical upwind method has the following major drawbacks: i) it is too dissipative in the “crosswind direction”; ii) it is not a consistent method, i.e., the exact solution does not satisfy (7) producing *consistency errors* that have to be estimated. This drawback limits the use of (7) to linear elements in the absence of sources, since a high order accurate solution is not possible, in general, by employing high order interpolations for (7).

For the SUPG method additional stability is obtained in the streamline direction, based on a residual formulation. This method combines improved stability with high order accuracy (i.e., high order polynomials yield better convergence rates for sufficiently smooth solutions). Indeed, stability of SUPG follows by:

$$B_{SUPG}(u_h, u_h) = \|\kappa^{1/2} \nabla u_h\|_0^2 + \|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 - \sum_{K \in \mathcal{C}_h} (\nabla \cdot (\kappa \nabla u_h), \tau \mathbf{a} \cdot \nabla u_h)_K \quad (15)$$

where by the design of τ , for $Pe_K \geq 1$

$$\tau = \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p} \frac{1}{Pe_K(\mathbf{x})} \frac{m_K |\mathbf{a}(\mathbf{x})|_p h_K}{2\kappa(\mathbf{x})} \leq \frac{m_K h_K^2}{4\kappa(\mathbf{x})}$$

and therefore the third term in (15) can be bounded from below as follows (as-

suming constant κ)

$$\begin{aligned}
-\sum_{K \in \mathcal{C}_h} (\nabla \cdot (\kappa \nabla u_h), \tau \mathbf{a} \cdot \nabla u_h)_K &= -\sum_{K \in \mathcal{C}_h} (\tau^{1/2} \nabla \cdot (\kappa \nabla u_h), \tau^{1/2} \mathbf{a} \cdot \nabla u_h)_K \\
&\geq -\frac{1}{2} \|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 - \frac{1}{2} \sum_{K \in \mathcal{C}_h} \|\tau^{1/2} \kappa \Delta u_h\|_{0,K}^2 \\
&\geq -\frac{1}{2} \|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 - \frac{1}{2} \sum_{K \in \mathcal{C}_h} \frac{m_K h_K^2}{4} \kappa \|\Delta u_h\|_{0,K}^2 \quad (16) \\
&\geq -\frac{1}{2} \|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 - \frac{1}{2} \frac{m_K}{4C_K} \kappa \|\nabla u_h\|_0^2 \\
&\geq -\frac{1}{2} \|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 - \frac{\kappa}{4} \|\nabla u_h\|_0^2
\end{aligned}$$

and therefore combining with (15) yields

$$B_{SUPG}(u_h, u_h) \geq C(\|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 + \kappa \|\nabla u_h\|_0^2). \quad (17)$$

Then an error estimate may be established in the “energy” norm

$$(\|\tau^{1/2} \mathbf{a} \cdot \nabla u_h\|_0^2 + \kappa \|\nabla u_h\|_0^2)^{1/2}.$$

Further discussion and analysis can be found in [8,12,13] and references therein.

We are now ready to come back to the central point of this work. Consider the Galerkin method given by (6) with $V_h = V_1^b$ where

$$V_1^b = \{v \in H_0^1(\Omega) \mid v|_K \in P_1(K) \oplus B(K), K \in \mathcal{C}_h\}, \quad (18)$$

and $B(K)$ denotes standard bubble functions defined on K with the following properties

$$\int_K \phi \, d\Omega = C_{1K} h_K^2, \quad \phi|_K \in B(K), \quad K \in \mathcal{C}_h \quad (19)$$

$$\|\nabla \phi\|_{0,K} = C_{2K}, \quad \phi|_K \in B(K), \quad K \in \mathcal{C}_h \quad (20)$$

where C_{1K} and C_{2K} are independent of the element parameter h_K . These equations hold for the cubic bubble, for example.

Our result can be stated as:

THEOREM: *Assuming the given data $(\mathbf{a}, \kappa$ and $f)$ to be piecewise constant, then the Galerkin method employing $V_h = V_1^b$ is equivalent to the SUPG method with $V_h = V_1$ for diffusive-dominated flows and appropriately defined element parameters h_K .*

Proof: Let us write the solution u_h of the Galerkin method with $V_h = V_1^b$ as

$$u_h(x) = u_1(x) + \sum_{K \in \mathcal{C}_h} \phi(x) u_b \quad (21)$$

where $u_1(x) \in V_1$, u_b is the bubble nodal value and the bubble function $\phi|_K \in B(K) \subset H_0^1(K)$ satisfies

$$\phi(x) = 0 \quad \text{on } \partial K, \quad K \in \mathcal{C}_h. \quad (22)$$

First consider the ‘‘bubble-equation’’ obtained by taking $v = \phi|_K$ on K and $v = 0$ elsewhere. Note that $v \in H_0^1(K) \subset H_0^1(\Omega)$. Then in (6)

$$(\mathbf{a} \cdot \nabla u_1, \phi)_K + (\mathbf{a} \cdot \nabla(\phi u_b), \phi)_K + (\kappa \nabla u_1, \nabla \phi)_K + (\kappa \nabla(\phi u_b), \nabla \phi)_K = (f, \phi)_K. \quad (23)$$

However, under the hypothesis of piecewise constant data, by integration-by-parts the second and third terms above are zero. Indeed,

$$(\mathbf{a} \cdot \nabla \phi, \phi)_K = -(\phi, \mathbf{a} \cdot \nabla \phi)_K - (\phi^2, \nabla \cdot \mathbf{a})_K + (\mathbf{a} \cdot \mathbf{n}, \phi^2)_{\partial K} \quad (24)$$

$$(\kappa \nabla u_1, \nabla \phi)_K = -(\nabla \cdot (\kappa \nabla u_1), \phi)_K + (\kappa \nabla u_1 \cdot \mathbf{n}, \phi)_{\partial K} \quad (25)$$

and for piecewise-constant \mathbf{a} and κ , $\nabla \cdot \mathbf{a} = 0$ on K , and $\nabla \cdot (\kappa \nabla u_1) = \kappa \Delta u_1 = 0$ on K ; and since $\phi = 0$ on ∂K then the assertion follows. Therefore the bubble equation (23) simplifies to:

$$u_b(\kappa \nabla \phi, \nabla \phi)_K = (f - \mathbf{a} \cdot \nabla u_1, \phi)_K. \quad (26)$$

Since f and \mathbf{a} are piecewise-constants by hypothesis and since ∇u_1 is also a piecewise-constant function, then we may pull $f - \mathbf{a} \cdot \nabla u_1$ out of the right-hand-side integral and write the bubble nodal value as

$$u_b = \frac{-1}{\kappa_{|K} \|\nabla \phi\|_{0,K}^2} (\mathbf{a} \cdot \nabla u_1 - f)|_K \int_K \phi \, d\Omega. \quad (27)$$

Let us now consider $v = v_1 \in V_1$. Then in (6):

$$(\mathbf{a} \cdot \nabla u_1, v_1) + \sum_{K \in \mathcal{C}_h} (\mathbf{a} \cdot \nabla(\phi u_b), v_1)_K + (\kappa \nabla u_1, \nabla v_1) = (f, v_1) \quad (28)$$

(We have used the fact that $(\kappa \nabla \phi, \nabla v_1)_K = 0$, $K \in \mathcal{C}_h$, as in (25)).

Note that, extracting the second term from (28), we obtain the Galerkin method employing linear elements, cf. eq. (5). Let us now examine the second term of (28). For each element K we have:

$$\begin{aligned} (\mathbf{a} \cdot \nabla(\phi u_b), v_1)_K &= -u_b(\phi, \mathbf{a} \cdot \nabla v_1)_K \\ &= + \frac{1}{\kappa_{|K} \|\nabla \phi\|_{0,K}^2} (\mathbf{a} \cdot \nabla u_1 - f)|_K (\mathbf{a} \cdot \nabla v_1)|_K \left(\int_K \phi \, d\Omega \right)^2. \end{aligned}$$

Using the properties of the standard bubble function (eq.(19)) we can rewrite

$$\int_K \phi \, d\Omega = C_{3K} \text{Measure}(K)$$

and pull back the piecewise-constant values to an integral on the element interior, i.e.,

$$(\mathbf{a} \cdot \nabla(\phi u_b), v_1)_K = \frac{C_{1K} C_{3K}}{C_{2K}^2} \frac{h_K^2}{\kappa_{|K}} (\mathbf{a} \cdot \nabla u_1 - f, \mathbf{a} \cdot \nabla v_1)_K. \quad (29)$$

Therefore combining this result with (28) yields the SUPG method using P1 approximation with a coefficient τ of $O(h_K^2/\kappa|K|)$, which is the order of the parameter in the SUPG for diffusive-dominated flows. ■

Consider the numerical test discussed, e.g., in [8,10] for a moderately advection-dominated flow skew to the mesh. Taking a regular mesh with $h = 0.05$, and given data $\kappa = 0.005$ and $|\mathbf{a}|_2 = 1$ we have by (10) for a linear element ($k = 1$), $Pe = \frac{1}{3} \times 1 \times 0.05 / (2 \times 0.005) = 5/3 > 1$. Elevation plots of u are shown in Figure 1 for the Galerkin method using linear elements and linear elements with bubbles. Note that similarly to the SUPG method, the Galerkin method with bubbles does not present spurious oscillations in smooth regions. For the linear element case, oscillations propagate even in smooth regions.

REMARKS:

1. The THEOREM shows that the Galerkin method employing $V_h = V_1^b$ has the improved stability characteristics of “artificial” diffusion in the streamline direction. This is a rather surprising result at first sight, since the bubble function, that induces this feature, is a perfectly symmetric test and trial function and does not have any “trick” built-in *a priori*. It comes out of its condensation that the linear support of the V_1^b -space has similar convergence properties of the SUPG method for low mesh-Peclet numbers. From what we know of the SUPG method, there is no reason to compute the bubble nodal value afterwards, since the vertex nodal values produce a desirable numerical solution. We should note that the relationship between bubbles and SUPG was first noted in G. Rogé’s thesis [17] (see also [4]) for a one-dimensional model in absence of source. For one dimension the effect of artificial diffusion could be

viewed as classical upwinding (no preferential direction) or as (degenerated) streamline upwinding. By going to the multidimensional case, it becomes clear that the bubbles do have the streamline diffusion properties of the SUPG method.

2. Once the result is understood, the next question is what to do when $\kappa \rightarrow 0$. In other words, when the mesh-Peclet number is large, will the Galerkin method with bubbles perform satisfactorily? In the proof of the THEOREM the condensation procedure is shown in (26), which indicates that there will be a division by $(\kappa \nabla \phi, \nabla \phi)_K$. One possibility to overcome the undesirable effect when $\kappa \rightarrow 0$ is to simply replace $(\kappa \nabla \phi, \nabla \phi)_K$ by $C_{4K} |\mathbf{a}|_p h_K$ and proceed as before. Clearly in this case consistency is not affected, since the bubble term now becomes

$$(\mathbf{a} \cdot \nabla(\phi u_b), v_1)_K = \frac{C_{1K} C_{3K}}{C_{4K}} \frac{h_K}{|\mathbf{a}|_p} (\mathbf{a} \cdot \nabla u_1 - f, \mathbf{a} \cdot \nabla v_1)_K. \quad (30)$$

In other words, SUPG can be completely recovered by this procedure with a parameter $O(h_K/|\mathbf{a}|_p)$ when $\kappa \rightarrow 0$.

3. The solution advocated in Remark 2 can be found alternatively by redefining the bubble functions. Consider, for example, a strip crossing a given triangle in the mesh, through its barycenter. The strip is constructed to have the width of $\kappa/|\mathbf{a}|_p$. Its length is approximately equal to the element diameter h_K . Inside the strip we let the bubble value go linearly from its value along the border of the strip (approximately one near the barycenter) to zero in the mid-line parallel to the length of the strip. This construction implies that the area of the strip is $Ch_K \kappa/|\mathbf{a}|_p$ and the Euclidean norm of the gradient of the

bubble is approximately $|\mathbf{a}|_p/\kappa$. Therefore

$$\begin{aligned} (\kappa \nabla \phi, \nabla \phi)_K &\cong C \kappa \left(\frac{|\mathbf{a}|_p}{\kappa} \right)^2 \frac{h_K \kappa}{|\mathbf{a}|_p} \\ &= C |\mathbf{a}|_p h_K \end{aligned} \quad (31)$$

and we achieved the same goal as in the previous remark, since the integrals $\int_K \phi \, d\Omega$, $K \in \mathcal{C}_h$ are still $O(h_K^2)$ by this construction.

4. Let us consider in this remark a constant diffusivity κ . By multiplying equation (28) by κ and letting $\kappa \rightarrow 0$, we obtain the least-squares method for hyperbolic equations. If we wish to compute with this method in this limit ($\kappa \rightarrow 0$), then in the implementation divisions by small numbers should be avoided in the condensation of the bubbles. One way to carry this out is to set $\kappa = 1$ in eq. (27) and multiply the first, third and fourth terms of (28) by κ .
5. Pierre [14,15] and Bank and Welfert [2] have established, for the Stokes problem, the relation between a Galerkin method with bubble functions (the so-called Mini-element of Arnold, Brezzi and Fortin [1]) and the stabilized methods of Brezzi and Pitkäranta [3] and Hughes, Franca and Balestra [11]. The present work points out that bubbles help not only to fulfill compatibility between spaces as in Stokes, but also to stabilize the advective operator. This is more apparent in the next section.

2. Linearized Compressible Navier-Stokes Equations

Good numerical results were obtained at INRIA and Dassault Aviation (see [4,17]) for the simulation of the compressible Navier-Stokes equations at low Reynolds number, employing the Galerkin method with density and temperature approximated by a piecewise linear function and velocity with a piecewise linear

plus a bubble function. The first attempts to explain this good behaviour were associated with the Stokes-like equations obtained in the limit as Reynolds number goes to zero [7,16]. However, we have seen in the previous section that, in fact, the bubble function also adds stability to the advective operator. For systems it is not clear, *a priori*, which variables to consider with bubble functions. We will restrict our attention to a simplified set of linearized steady compressible Navier-Stokes equations, to shed some light on the improvement of stability obtained in the aforementioned works.

Let us then consider the problem of finding the density ρ , the velocity \mathbf{u} and the temperature T such that

$$\mathbf{u}_o \cdot \nabla \rho + \rho_o \nabla \cdot \mathbf{u} = 0 \quad (32)$$

$$(\nabla \mathbf{u}) \mathbf{u}_o + (\gamma - 1) \left(\frac{T_o}{\rho_o} \nabla \rho + \nabla T \right) - \frac{1}{\text{Re} \rho_o} \left[\Delta \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right] = \mathbf{f} \quad (33)$$

$$\mathbf{u}_o \cdot \nabla T + (\gamma - 1) T_o \nabla \cdot \mathbf{u} - \frac{\gamma}{\text{Re} \rho_o \text{Pr}} \Delta T = 0 \quad (34)$$

where ρ_o , \mathbf{u}_o and T_o are given piecewise constant fields, γ is the ratio between specific heats. Re is the Reynolds number and Pr is the Prandtl number. The dissipation due to viscous terms have been neglected in equation (34) for keeping simple the arguments that follow. For external flows we let Γ_∞ be the far-field boundary of the domain and Γ_B the boundary of a rigid body immersed in the flow. Denoting by

$$\Gamma_\infty^- = \{x \in \Gamma_\infty \mid \mathbf{u}_\infty \cdot \mathbf{n} < 0\}$$

$$\Gamma_\infty^+ = \Gamma / (\Gamma_\infty^- \cup \Gamma_B)$$

where \mathbf{u}_∞ is the free stream velocity and \mathbf{n} is the outward unit vector to Γ . To (32)-(34) we append the following set of boundary conditions:

$$\rho = \rho_p \quad \text{on } \Gamma_\infty^- \quad (35)$$

$$\mathbf{u} = \mathbf{u}_p \quad \text{on } \Gamma_B \cup \Gamma_\infty^- \quad (36)$$

$$T = T_p \quad \text{on } \Gamma_B \cup \Gamma_\infty^- \quad (37)$$

and

$$(\nabla \mathbf{u} + \frac{1}{3}(\nabla \cdot \mathbf{u})\mathbf{I})\mathbf{n} = \mathbf{0} \quad \text{on } \Gamma_\infty^+ \quad (38)$$

$$\nabla T \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_\infty^+ \quad (39)$$

where ρ_p , \mathbf{u}_p and T_p are prescribed functions at the boundaries and \mathbf{I} is the identity tensor.

Let us consider the following finite element spaces:

$$U_p = \{\varphi \in H^1(\Omega) \mid \varphi|_K \in P_1(K), K \in \mathcal{C}_h, \varphi = \rho_p \text{ on } \Gamma_\infty^-\} \quad (40)$$

$$\mathbf{V}_p^b = \{\mathbf{v} \in (H^1(\Omega))^N \mid \mathbf{v}|_K \in (P_1(K) \oplus B(K))^N, K \in \mathcal{C}_h, \mathbf{v} = \mathbf{u}_p \text{ on } \Gamma_B \cup \Gamma_\infty^-\} \quad (41)$$

$$W_p = \{\theta \in H^1(\Omega) \mid \theta|_K \in P_1(K), K \in \mathcal{C}_h, \theta = T_p \text{ on } \Gamma_B \cup \Gamma_\infty^-\} \quad (42)$$

and U_o , \mathbf{V}_o^b , W_o which are the spaces obtained by setting to zero the boundary conditions in (40)-(42). The Galerkin method we wish to study consists in searching for $\{\rho_h, \mathbf{u}_h, T_h\} \in U_p \times \mathbf{V}_p^b \times W_p$ such that

$$(\mathbf{u}_o \cdot \nabla \rho_h, \varphi) + (\rho_o \nabla \cdot \mathbf{u}_h, \varphi) = 0 \quad \varphi \in U_o \quad (43)$$

$$\begin{aligned} & ((\nabla \mathbf{u}_h)\mathbf{u}_o, \mathbf{v}) + (\gamma - 1)\left(\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h, \mathbf{v}\right) \\ & + \frac{1}{\text{Re} \rho_o} \left[(\nabla \mathbf{u}_h, \nabla \mathbf{v}) + \frac{1}{3}(\nabla \cdot \mathbf{u}_h, \nabla \cdot \mathbf{v}) \right] = (\mathbf{f}, \mathbf{v}) \quad \mathbf{v} \in \mathbf{V}_o^b \end{aligned} \quad (44)$$

$$(\mathbf{u}_o \cdot \nabla T_h, \theta) + (\gamma - 1)(T_o \nabla \cdot \mathbf{u}_h, \theta) + \frac{\gamma}{\text{Re} \rho_o \text{Pr}} (\nabla T_h, \nabla \theta) = 0 \quad \theta \in W_o. \quad (45)$$

Similarly to the proof of the THEOREM in the last section, we wish to show that the Galerkin method defined in equations (43) to (45) is equivalent to a stabilization procedure for the linearized Navier-Stokes equations. To this end the first step is to “statically condensate” the bubble from the momentum equation. Let us decompose the trial function for the velocity as

$$\mathbf{u}_h(x) = \mathbf{u}_1(x) + \sum_{K \in \mathcal{C}_h} \phi(x) \mathbf{u}_b. \quad (46)$$

Next let us take $\mathbf{v} = \phi \mathbf{e}_i$, on K , for $i = 1, \dots, N$, and zero elsewhere, where \mathbf{e}_i is the unity cartesian vector with components $(\mathbf{e}_i)_j = \delta_{ij}$ for $j = 1, \dots, N$. For this choice of weighting functions and replacing (46) in (44) we get:

$$\begin{aligned} & ((\nabla \mathbf{u}_1) \mathbf{u}_o, \phi \mathbf{e}_i)_K + (\gamma - 1) \left(\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h, \phi \mathbf{e}_i \right)_K + \frac{1}{\text{Re} \rho_o} \left[(\nabla(\phi \mathbf{u}_b), \nabla(\phi \mathbf{e}_i))_K \right. \\ & \left. + \frac{1}{3} (\nabla \cdot (c \mathbf{u}_b), \nabla \cdot (\phi \mathbf{e}_i))_K \right] = (\mathbf{f}, \phi \mathbf{e}_i)_K \quad i = 1, \dots, N. \end{aligned} \quad (47)$$

In obtaining (47), we have used the facts that

$$(\nabla(\phi \mathbf{u}_b) \mathbf{u}_o, \phi \mathbf{e}_i)_K = 0 \quad (48)$$

$$(\nabla \mathbf{u}_1, \nabla(\phi \mathbf{e}_i))_K = 0 \quad (49)$$

$$(\nabla \cdot \mathbf{u}_1, \nabla \cdot (\phi \mathbf{e}_i))_K = 0 \quad (50)$$

by integration-by-parts. Also note that

$$\begin{aligned} & (\nabla(\phi \mathbf{u}_b), \nabla(\phi \mathbf{e}_i))_K + \frac{1}{3} (\nabla \cdot (\phi \mathbf{u}_b), \nabla \cdot (\phi \mathbf{e}_i))_K \\ & = \mathbf{u}_b \cdot \left[\int_K (\nabla \phi \cdot \nabla \phi \mathbf{e}_i + \frac{1}{3} (\nabla \phi \otimes \nabla \phi) \mathbf{e}_i) d\Omega \right] \\ & = \left[\int_K (\nabla \phi \cdot \nabla \phi \mathbf{I} + \frac{1}{3} \nabla \phi \otimes \nabla \phi) d\Omega \right] \mathbf{u}_b \cdot \mathbf{e}_i. \end{aligned} \quad (51)$$

Therefore (47) becomes

$$\frac{1}{\text{Re}\rho_o} \mathbf{A} \mathbf{u}_b \cdot \mathbf{e}_i = -((\nabla \mathbf{u}_1) \mathbf{u}_o + (\gamma - 1) \left(\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h \right) - \mathbf{f}, \phi \mathbf{e}_i)_K \quad (52)$$

where

$$\mathbf{A} = \int_K (\nabla \phi \cdot \nabla \phi \mathbf{I} + \frac{1}{3} \nabla \phi \otimes \nabla \phi) d\Omega. \quad (53)$$

Or, using the piecewise constant data assumption

$$\mathbf{u}_b = -\text{Re}\rho_o \mathbf{A}^{-1} ((\nabla \mathbf{u}_1) \mathbf{u}_o + (\gamma - 1) \left(\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h \right) - \mathbf{f})|_K \int_K \phi d\Omega \quad (54)$$

Once we have computed the bubble nodal values, let us examine the effect of the decomposition (46) in each equilibrium equation. Starting with (43) we obtain

$$(\mathbf{u}_o \cdot \nabla \rho_h, \varphi) + (\rho_o \nabla \cdot \mathbf{u}_1, \varphi) + \sum_{K \in \mathcal{C}_h} (\rho_o \nabla \cdot (\phi \mathbf{u}_b), \varphi)_K = 0 \quad \varphi \in U_o. \quad (55)$$

The first two terms of equation (55) are just the ones obtained by employing the Galerkin method with equal-order linear elements. The third term at each element can be written as:

$$\begin{aligned} (\rho_o \nabla \cdot (\phi \mathbf{u}_b), \varphi)_K &= -\mathbf{u}_b \cdot (\rho_o \phi \cdot \nabla \varphi)_K \\ &= -\mathbf{u}_b \cdot (\rho_o \nabla \varphi)|_K \int_K \phi d\Omega. \end{aligned} \quad (56)$$

Substituting (46) into (44) with $\mathbf{v} = \mathbf{v}_1 \in \mathbf{V}_o^1$ leads us to:

$$\begin{aligned} ((\nabla \mathbf{u}_1) \mathbf{u}_o, \mathbf{v}_1) + \sum_{K \in \mathcal{C}_h} (\nabla (\phi \mathbf{u}_b) \mathbf{u}_o, \mathbf{v}_1)_K + (\gamma - 1) \left(\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h, \mathbf{v}_1 \right) \\ + \frac{1}{\text{Re}\rho_o} \left[(\nabla \mathbf{u}_1, \nabla \mathbf{v}_1) + \frac{1}{3} (\nabla \cdot \mathbf{u}_1, \nabla \cdot \mathbf{v}_1) \right] = (\mathbf{f}, \mathbf{v}_1) \quad \mathbf{v}_1 \in \mathbf{V}_o^1 \end{aligned} \quad (57)$$

(We have used $(\nabla(\phi \mathbf{u}_b), \nabla \mathbf{v}_1)_K = 0$ and $(\nabla \cdot (\phi \mathbf{u}_b), \nabla \cdot \mathbf{v}_1)_K = 0$.) Except for the second term, all terms are present in the equal-order linear Galerkin method. The second term in (57) can be written at each element as:

$$\begin{aligned} (\nabla(\phi \mathbf{u}_b) \mathbf{u}_o, \mathbf{v}_1)_K &= -\mathbf{u}_b \cdot (\phi, (\nabla \mathbf{v}_1) \mathbf{u}_o)_K \\ &= -\mathbf{u}_b \cdot ((\nabla \mathbf{v}_1) \mathbf{u}_o)|_K \int_K \phi \, d\Omega. \end{aligned} \quad (58)$$

Next, by substituting (46) into (45) yields to:

$$\begin{aligned} (\mathbf{u}_o \cdot \nabla T_h, \theta) + (\gamma - 1)(T_o \nabla \cdot \mathbf{u}_1, \theta) + (\gamma - 1) \sum_{K \in \mathcal{C}_h} (T_o \nabla \cdot (\phi \mathbf{u}_b), \theta)_K \\ + \frac{\gamma}{\text{Re} \rho_o \text{Pr}} (\nabla T_h, \nabla \theta) = 0 \quad \theta \in W_o. \end{aligned} \quad (59)$$

Again, all terms are in the Galerkin method with equal-order linear elements, except for the third term, that can be written at each element level as:

$$\begin{aligned} (\gamma - 1)(T_o \nabla \cdot (\phi \mathbf{u}_b), \theta)_K &= -(\gamma - 1) \mathbf{u}_b \cdot (T_o \phi, \nabla \theta)_K \\ &= -(\gamma - 1) \mathbf{u}_b \cdot (T_o \nabla \theta)|_K \int_K \phi \, d\Omega. \end{aligned} \quad (60)$$

Therefore, examining (55) to (60) we have shown that the Galerkin method given by equations (43) to (45) is equivalent to the equal-order linear Galerkin method plus the following perturbation terms (PT), rewritten from (56), (58) and (60) as:

$$PT = - \sum_{K \in \mathcal{C}_h} \int_K \phi \, d\Omega \, \mathbf{u}_b \cdot (\rho_o \nabla \varphi + (\nabla \mathbf{v}_1) \mathbf{u}_o + (\gamma - 1) T_o \nabla \theta)|_K, \quad (61)$$

which combined with (54) gives us

$$\begin{aligned} PT &= \sum_{K \in \mathcal{C}_h} \left(\int_K \phi \, d\Omega \right)^2 \text{Re} \rho_o \mathbf{A}^{-1} ((\nabla \mathbf{u}_1) \mathbf{u}_o + (\gamma - 1) \left(\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h \right) - \mathbf{f})|_K \\ &\quad \cdot (\rho_o \nabla \varphi + (\nabla \mathbf{v}_1) \mathbf{u}_o + (\gamma - 1) T_o \nabla \theta)|_K, \end{aligned} \quad (62)$$

and by (19)

$$\begin{aligned}
PT &= \sum_{K \in \mathcal{C}_h} C_{1K} C_{3K} h_K^2 \text{Re} \rho_o (\mathbf{A}^{-1} ((\nabla \mathbf{u}_1) \mathbf{u}_o + (\gamma - 1) (\frac{T_o}{\rho_o} \nabla \rho_h + \nabla T_h) \\
&\quad - \mathbf{f}), \rho_o \nabla \varphi + (\nabla \mathbf{v}_1) \mathbf{u}_o + (\gamma - 1) T_o \nabla \theta)_K \\
&= \sum_{K \in \mathcal{C}_h} C_{1K} C_{3K} h_K^2 \text{Re} \rho_o \begin{pmatrix} \nabla \rho_h \\ (\nabla \mathbf{u}_1) \mathbf{u}_o \\ \nabla T_h \end{pmatrix}^T \\
&\quad \begin{pmatrix} (\gamma - 1) T_o \mathbf{A}^{-1} & (\gamma - 1) \frac{T_o}{\rho_o} \mathbf{A}^{-1} & (\gamma - 1)^2 \frac{T_o^2}{\rho_o} \mathbf{A}^{-1} \\ \rho_o \mathbf{A}^{-1} & \mathbf{A}^{-1} & (\gamma - 1) T_o \mathbf{A}^{-1} \\ (\gamma - 1) \rho_o \mathbf{A}^{-1} & (\gamma - 1) \mathbf{A}^{-1} & (\gamma - 1)^2 T_o \mathbf{A}^{-1} \end{pmatrix} \begin{pmatrix} \nabla \varphi \\ (\nabla \mathbf{v}_1) \mathbf{u}_o \\ \nabla \theta \end{pmatrix} \\
&\quad - \sum_{K \in \mathcal{C}_h} C_{1K} C_{3K} h_K^2 \text{Re} \rho_o (\mathbf{A}^{-1} \mathbf{f}, \rho_o \nabla \varphi + (\nabla \mathbf{v}_1) \mathbf{u}_o + (\gamma - 1) T_o \nabla \theta)_K.
\end{aligned} \tag{63}$$

Therefore the perturbation terms that emanate by condensation are in fact adding a least-squares type form of the momentum equation, with a coefficient $O(h_K^2 \cdot \text{Re} \rho_o)$, and the source is being corrected by the same operator.

Further examining (63) we note that the matrix

$$\mathbf{C} = \begin{pmatrix} (\gamma - 1) T_o \mathbf{A}^{-1} & (\gamma - 1) \frac{T_o}{\rho_o} \mathbf{A}^{-1} & (\gamma - 1)^2 \frac{T_o^2}{\rho_o} \mathbf{A}^{-1} \\ \rho_o \mathbf{A}^{-1} & \mathbf{A}^{-1} & (\gamma - 1) T_o \mathbf{A}^{-1} \\ (\gamma - 1) \rho_o \mathbf{A}^{-1} & (\gamma - 1) \mathbf{A}^{-1} & (\gamma - 1)^2 T_o \mathbf{A}^{-1} \end{pmatrix}$$

is positive-semidefinite, and therefore the perturbation terms are indeed adding a numerical dissipation. Note also that adding the viscous and thermal dissipation due to the Galerkin terms with linear elements, gives us positive contributions to the block matrices \mathbf{C}_{22} and \mathbf{C}_{33} rendering \mathbf{C} positive definite.

Summing up, we have shown that the method given by eqs. (43)-(45) is

equivalent to the equal-order linear Galerkin method plus a least-squares form of the momentum equation. ■

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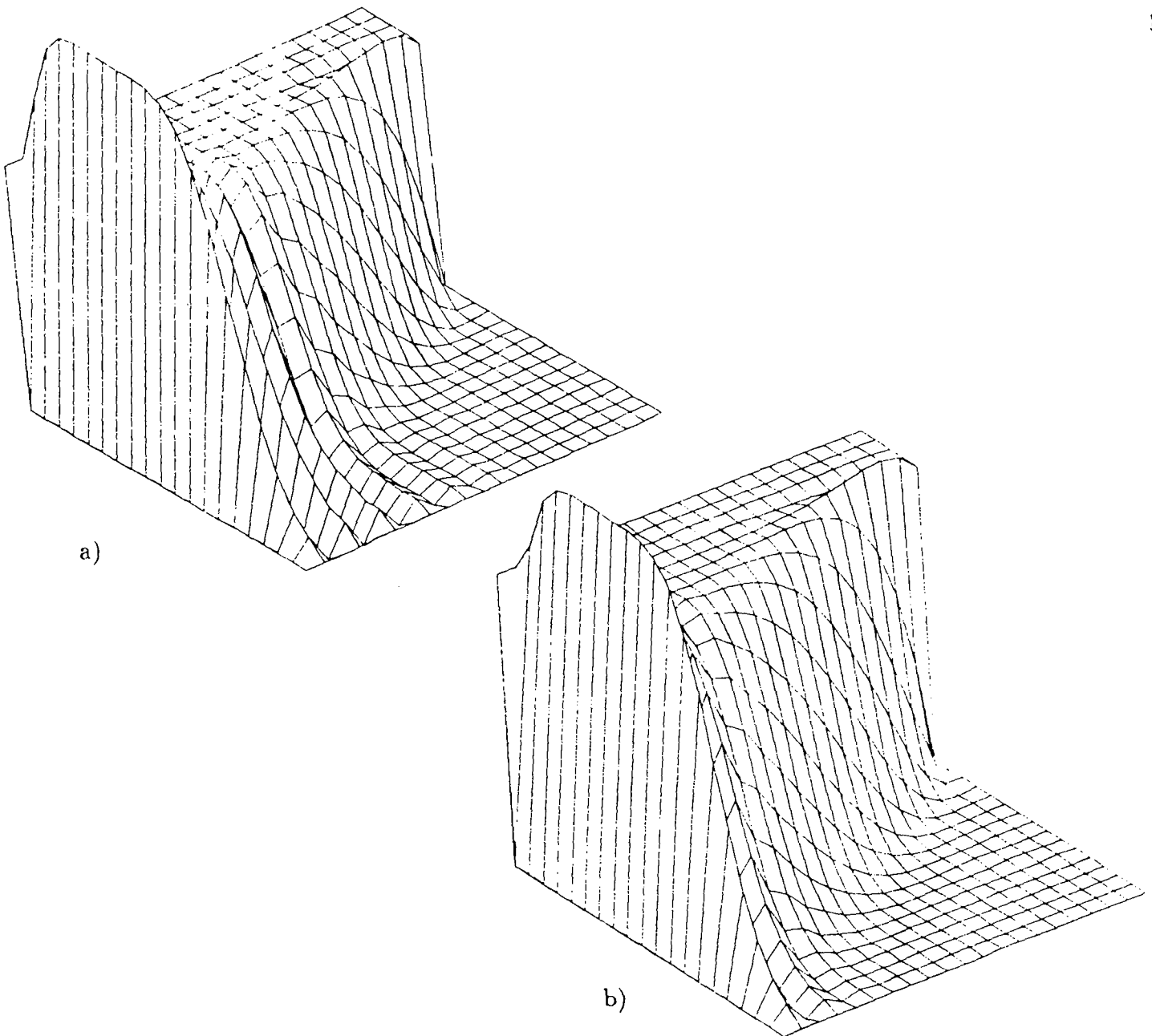


Figure 1: Advection skew to the mesh: The Galerkin method employing: a) Linear elements; b) Linear elements with bubble functions.

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