Optimality of a threshold policy in the M/M/1 queue with repeated vacations

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Abstract

Consider an M/M/1 queueing system with server vacations where the server is turned off as soon as the queue gets empty. We assume that the vacation durations form a sequence of i.i.d. random variables with exponential distribution. At the end of a vacation period, the server may either be turned on if the queue is non empty or take another vacation. The following costs are incurred: a holding cost of h per unit of time and per customer in the system and a fixed cost of γ each time the server is turned on. We show that there exists a threshold policy that minimizes the long-run average cost criterion. The approach we use was first proposed in [3] and enables us to determine explicitly the optimal threshold and the optimal long-run average cost in terms of the model parameters.

Keywords:

Queues with vacations; control of queues; Markov decision processes; dynamic programming.

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OPTIMALITE D'UNE POLITIQUE A SEUIL DANS UNE FILE D'ATTENTE M/M/1 AVEC VACANCES DU SERVEUR

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Résumé

Nous considérons une file d'attente M/M/1 où le serveur prend des vacances dès que la file se vide. On suppose que les durées successives des vacances du serveur forment une suite de variables aléatoires indépendantes et équidistribuées suivant une loi exponentielle. A la fin d'une période de vacances, le serveur peut soit être réactivé si la file est non vide, soit prendre de nouvelles vacances. Les coûts suivants sont introduits: un coût h par unité de temps et par client dans la file et un coût fixe γ à chaque réactivation du serveur. Nous montrons qu'il existe une politique à seuil qui minimise un critère ergodique de type coût moyen. L'approche suivie nous permet de déterminer explicitement la valeur optimale du seuil ainsi que la valeur optimale du critère de coût.

Mots-Clés: Théorie des files d'attente; contrôle des files d'attente; processus de décision markoviens; programmation dynamique.

1 Introduction

Queueing systems with vacations of the server have already received much attention in the literature and a comprehensive discussion can be found in the survey papers by Doshi [5] and Teghem [19]. These models are commonly used for modeling and tuning various systems ranging from manufacturing systems to communication and computer systems (cf. Doshi [5]). Two types of vacation models can be identified: models with repeated vacations of the server and models with a removable server. In the former case the length of a vacation period is driven by an external process (e.g., the vacation lengths are i.i.d. random variables, see Gelenbe and Mitrani [7], Gelenbe and Iasnogorodski [8], Kella [10,11], Levy and Yechiali [13]), while in the latter case the length of a vacation period is driven by the arrival process (see Heyman and Sobel [9], pp. 336-337, Yadin and Naor [21]). In both cases, the server may be turned off at any service completion epoch of a customer and it may be turned on only at an arrival epoch of a customer in the case of a removable server, or only at the end of a vacation period in the case of repeated vacations. Of particular interest is the stationary policy that turns the server on only when the number of customers in the queue is equal to or larger than a given value, and turns the server off when the queue is empty. This policy will be referred to as the *threshold policy*.

A natural objective is to seek for an optimal vacation policy that optimizes a given cost function among certain classes of policies. For the M/G/1 queue with a removable server Heyman and Sobel [9] and Talman [17] have shown that a fairly general average cost criterion is minimized by a threshold policy among the class of all policies. For the more difficult problem of controlling the service process in a queue with repeated vacations, only a few optimization results have been reported so far. Recently, Kella [10] has computed the best threshold policy for an M/G/1 queue with repeated vacations, and Lee and Srinivasan [12] have carried out the same analysis in the case of batch arrivals. In the case where the decision to take another vacation is based on a random outcome depending on the number of consecutive vacations already taken, Kella [11] has shown that a control policy of a limit type minimizes a long-run average cost criterion.

Our contribution is to establish the optimality of a threshold policy over the class of all policies, including policies that may depend on the history of the system (i.e., number of customers at any time and previous decisions made).

More precisely, consider an M/M/1 queue under the exhaustive service discipline and with repeated vacations, where the lengths of the vacation periods are i.i.d. random variables with an exponential distribution. Assume that a holding cost $h \geq 0$ is incurred per unit of time and per customer and that a fixed cost $\gamma \geq 0$ is incurred when the server is turned on (cf. Remark 2.1). We show in this paper that there exists a threshold policy that minimizes the long-run average cost criterion.

The approach we use closely follows that proposed by Blanc, de Waal, Nain and Towsley [3] and enables us, in particular, to explicitly compute both the optimal threshold and the optimal long-run average cost in terms of the model parameters.

The paper is organized as follows. In Section 2, the problem is formulated as a Markov decision problem. A related discounted cost problem is then introduced and solved via dynamic programming and the technique proposed in [3] (Sections 3 and 4). This finally enables us to derive the optimal policy for the long-run average cost criterion (Section 5).

In a companion paper [1] we show through a completely different analysis that the result of this paper actually extends to the M/G/1 queue with (arbitrary) repeated vacations. However, the optimal threshold cannot be determined explicitly in this case and the numerical procedure proposed by Kella [10] has to be used instead. Extensions of our results to nonexhaustive service disciplines (gated-type disciplines, semi-exhaustive discipline, etc., see Takagi [18]) are the subject of ongoing research.

2 The Model

All of the random variables considered in this paper are defined on some fixed probability triple $(\Omega, \mathcal{F}, \mathrm{P})$. $\mathbb{N}$ will denote the set of nonnegative integers. Define $\mathbb{N}^* := \mathbb{N} - \{0\}$.

We consider an $M/M/1$ queueing system, where the server may take a sequence of vacations whose lengths are i.i.d. exponential random variables with mean $1/\nu$, $\nu > 0$. We assume that the vacation process is independent of the arrival and service processes. Let $\lambda > 0$ be the arrival rate and let $\mu > 0$ be the service rate.

The vacations are taken according to the following scheme: the server serves the queue exhaustively and then takes a vacation; at the end of a vacation, the server may either be turned on if the queue is non empty or take another vacation. The server is never turned on when the queue is empty at the end of a vacation period. The following costs are incurred (cf. Remark 2.1): a cost $h \geq 0$ per unit of time and per customer in the queue and a fixed cost $\gamma \geq 0$ each time the server is turned on (restarting cost). Our objective is to find a control policy of the service process that minimizes the long-run average cost (to be made more precise).

To do so, we place our problem in the framework of Markov Decision Processes (MDP's) theory (Lippman [14], Ross [15]). Let us first define the decision epochs $\{t_n\}_1^\infty$. For $n \geq 1$, let:

- a_n, $0 \leq a_1 < a_2 < \cdots$, be the arrival time of the n-th customer;
- d_n, $0 \leq d_1 < d_2 < \ldots$, be the departure time of the n-th customer;
- $\mathrm{V} := \{v_n\}_1^\infty$ be a Poisson process with intensity $\nu > 0$. This process represents the *virtual* vacation process.

Then, $\{t_n\}_1^\infty := \{a_n\}_1^\infty \cup \{d_n\}_1^\infty \cup \{v_n\}_1^\infty$ with $0 \leq t_1 < t_2 < \ldots$. We assume that the first decision takes place at time 0 (i.e., $t_1 = 0$). By convention, we also assume that the n-th decision epoch takes place just after the n-th event (arrival, departure or virtual vacance completion) has occured.

In order to define the state of the system let us introduce

- $X(t)$ to be the number of customers in the queue at time $t \geq 0$, including the one in service, if any. We assume that the sample paths of $\{X(t),\, t \geq 0\}$ are right-continuous;
- $Y(t)$ to be the state of the server at time $t \geq 0$, where $Y(t) = 1$ (resp. $Y(t) = 0$) if the server is on (resp. off). We assume that the sample paths of $\{Y(t),\, t \geq 0\}$ are left-continuous.

The state of the system at time t_n is represented by the random vector $(X_n, Y_n, Z_n) \in \mathbb{N} \times \{0,1\}^2$, where $X_n := X(t_n)$, $Y_n := Y(t_n)$, and $Z_n := \mathbf{1}(t_n \in \mathrm{V})$ for all $n \geq 1$.

An admissible policy can be recursively defined as follows. For $n \geq 1$, let $\mathbb{H}_n := (X_j,\, 1 \leq j \leq n;\, u_j,\, 1 \leq j \leq n-1)$ be the history of the system at t_n, where $u_j \in \{0,1\}$ is the decision made at t_j, with the convention that $u_j = 1$ (resp. $u_j = 0$) if the decision is to turn the server on (resp. to take a vacation). An admissible control policy U is a sequence $\{U_n\}_1^\infty$ of $[0,1]$-valued random variables such that U_n is measurable with respect to the σ-field $\mathcal{F}_n := \sigma(\mathbb{H}_n)$ with the interpretation that $\mathrm{P}\left(u_n = 1 \,|\, \mathcal{F}_n\right) = U_n$ for $n \geq 1$. A control policy is said to be *stationary* if U_n only depends on the value of X_n and if it is non randomized, i.e., $U_n \in \{0,1\}$, $n \geq 1$.

Under the foregoing assumptions, it is easily seen that $\mathbf{R} := \{(X_n, Y_n, Z_n),\, n \geq 1\}$ is an MDP with state space $\mathrm{S} := \mathbb{N} \times \{0,1\}^2 - \{(0,1,1),(0,0,0)\}$.

Let $A_{x,y,z}$ be the action space when the system is in state (x,y,z). Define:

$$A_{x,1,1} = \{1\}, \text{ for } x \in \mathbb{N}; \tag{2.1}$$

$$A_{x,1,0} = \begin{cases} \{1\}, & \text{for } x \in \mathbb{N}^*; \\ \{0\}, & \text{for } x = 0; \end{cases}$$

$$A_{x,0,1} = \begin{cases} \{0,1\}, & \text{for } x \in \mathbb{N}^*; \\ \{0\}, & \text{for } x = 0; \end{cases}$$

$$A_{x,0,0} = \{0\}, \text{ for } x \in \mathbb{N}^*. \tag{2.2}$$

Note that the action spaces always contain one element except in the case when the decision epoch takes place at the end of a vacation period and that the queue is non empty, which agrees with the initial formulation of the problem.

Last, let us introduce $\mathrm{Q}\left(\cdot,\cdot,\cdot \,|\, x,y,z;u\right)$, the one-step probabiLity transition of the MDP $\mathbf{R}$ given that the current state is (x,y,z) and that action $u \in A_{x,y,z}$ is chosen. We have (with $\beta := \lambda+\mu+\nu$):

$$\begin{aligned} \mathrm{Q}\left(x-1,1,0 \,|\, x,1,1;1\right) &= \mu/\beta, \text{ for } x \in \mathbb{N}^*; \\ \mathrm{Q}\left(x+1,1,0 \,|\, x,1,1;1\right) &= \lambda/\beta, \text{ for } x \in \mathbb{N}^*; \\ \mathrm{Q}\left(x,1,1 \,|\, x,1,1;1\right) &= \nu/\beta, \text{ for } x \in \mathbb{N}^*; \end{aligned}$$

$$\begin{aligned} \mathrm{Q}\left(x-1,1,0 \,|\, x,1,0;1\right) &= \mu/\beta, \text{ for } x \in \mathbb{N}^*; \\ \mathrm{Q}\left(x+1,1,0 \,|\, x,1,0;1\right) &= \lambda/\beta, \text{ for } x \in \mathbb{N}^*; \end{aligned}$$

$$
\begin{aligned}
Q(x,1,1\,|\,x,1,0;1) &= \nu/\beta, \text{ for } x \in \mathbb{N}^*;\\
Q(0,0,1\,|\,0,1,0;0) &= \nu/(\lambda+\nu);\\
Q(1,0,0\,|\,0,1,0;0) &= \lambda/(\lambda+\nu);
\end{aligned}
$$

$$
\begin{aligned}
Q(x-1,1,0\,|\,x,0,1;1) &= \mu/\beta, \text{ for } x \in \mathbb{N}^*;\\
Q(x+1,1,0\,|\,x,0,1;1) &= \lambda/\beta, \text{ for } x \in \mathbb{N}^*;\\
Q(x,1,1\,|\,x,0,1;1) &= \nu/\beta, \text{ for } x \in \mathbb{N}^*;\\
Q(x+1,0,0\,|\,x,0,1;0) &= \lambda/(\lambda+\nu), \text{ for } x \in \mathbb{N};\\
Q(x,0,1\,|\,x,0,1;0) &= \nu/(\lambda+\nu), \text{ for } x \in \mathbb{N};
\end{aligned}
$$

$$
\begin{aligned}
Q(x+1,0,0\,|\,x,0,0;0) &= \lambda/(\lambda+\nu), \text{ for } x \in \mathbb{N}^*;\\
Q(x,0,1\,|\,x,0,0;0) &= \nu/(\lambda+\nu), \text{ for } x \in \mathbb{N}^*.
\end{aligned}
$$

We are now in position to formulate our optimization problem (called problem **P1**). We want to find a policy $U \in \mathcal{U}$ that minimizes for all $(x,y,z) \in \mathbf{S}$ the long-run average cost criterion

$$V^U(x,y,z) := \mathrm{limsup}_{T\uparrow\infty} V_T^U(x,y,z), \tag{2.3}$$

where

$$V_T^U(x,y,z) := \mathrm{E}_U\left[T^{-1}\int_0^T hX(t)\,dt + \gamma(1-Y(t-))\,dY(t)\,|\,(X_1,Y_1,Z_1)=(x,y,z)\right]. \tag{2.4}$$

(Observe that $X(0)=X_1$ and $Y(0)=Y_1$ since we have assumed that $t_1=0$.)

Solving directly for problem **P1** turns out to be a difficult task. A standard method for solving problem **P1** is to look at a *discounted version* of that problem, and then to show that the structure of the optimal discounted policy (e.g., threshold policy) is captured by the optimal long-run average policy (cf. Ross [15], Sennott [16]) by letting the discount factor go to 0.

So, our first objective is to minimize over $\mathcal{U}$ the α-discounted cost function ($\alpha > 0$)

$$V_\alpha^U(x,y,z) := \mathrm{E}_U\left[\sum_{n\geq 1} e^{-\alpha t_n}\, c(X_n,Y_n,Z_n;u_n)\,|(X_1,Y_1,Z_1)=(x,y,z)\right], \tag{2.5}$$

for all (x,y,z) in $\mathbf{S}$, $U \in \mathcal{U}$, where

$$c(x,y,z;u) := \frac{h}{\nu}\,x\,z + \gamma\,(1-y)\,z\,\mathbf{1}(u=1). \tag{2.6}$$

Let

$$V_\alpha(x,y,z) := \inf_{U\in\mathcal{U}} V_\alpha^U(x,y,z). \tag{2.7}$$

Our second objective is to minimize over $\mathcal{U}$ the long-run average cost function (called problem **P2**)

$$W^U(x,y,z) := \limsup_{T\uparrow\infty} W_T^U(x,y,z), \tag{2.8}$$

for all $(x,y,z) \in \mathrm{S}$, $U \in \mathcal{U}$, where

$$W_T^U(x,y,z) := \mathrm{E}_U \left[T^{-1} \sum_{0\le t_n \le T} c(X_n, Y_n, Z_n; u_n) \,|\, (X_1, Y_1, Z_1) = (x,y,z) \right]. \tag{2.9}$$

The link between both long-run average cost functions (2.3) and (2.8) is established in the following proposition:

Proposition 2.1 *For all $T > 0$, $(x,y,z) \in \mathrm{S}$, $U \in \mathcal{U}$, $V_T^U(x,y,z) = W_T^U(x,y,z)$.*

Proof. Define $N(t) := \sup\{n \ge 0 \,:\, v_n \le t\}$ with $v_0 := 0$. We have, cf. (2.6),

$$\begin{aligned} W_T^U(x,y,z) &= \mathrm{E}_U \left[T^{-1} \int_0^T hX(t)\, dN(t)/\nu + \gamma(1 - Y(t-))\, dY(t) \,|(X_1, Y_1, Z_1) = (x,y,z) \right], \\ &= \mathrm{E}_U \left[T^{-1} \int_0^T hX(t-)\, dN(t)/\nu + \gamma(1 - Y(t-))\, dY(t) \,|(X_1, Y_1, Z_1) = (x,y,z) \right], \end{aligned}$$

where the above equality follows from the fact that with probability one $\{X(t),\, t > 0\}$ and $\mathbf{V}$ have no common jumps. The result now follows from Brémaud [2], formula 2.3, p. 24 (with $C_s := X(s-)\,\mathbf{1}(s \le T)$, $N_s := N(s)$, $\lambda_s := \nu$) ■

In particular, Proposition 2.1 shows that both problems **P1** and **P2** are equivalent.

Remark 2.1 Additional costs/rewards could be considered. In particular, the system could receive a reward for each unit of time the server is on vacation, and a constant cost could be incurred each time the server is turned off. However, and as observed by Kella [10], p. 116, these extensions can easily be captured by our model.

3 The Dynamic Programming Equation

For each function $f : \mathrm{S} \to \mathbb{R}$, set

$$\| f \| := \sup_{\substack{(x,y,z)\in \mathrm{S} \\ x \ne 0}} |f(x,y,z)|\, x^{-1}, \tag{3.1}$$

and define B to be the Banach space (with norm given in (3.1)) of all such f for which $\| f \| < \infty$.

Theorem 3.1 *There exists an optimal stationary policy for the α-discounted problem. In addition, V_α is the unique solution in B to the DP equation*

$$V_\alpha(x,y,z) = \min_{u \in A_{x,y,z}} \left\{ c(x,y,z;u) + \frac{\theta_{x,y,z}(u)}{\alpha + \theta_{x,y,z}(u)} \sum_{(x',y',z') \in \mathrm{S}} V_\alpha(x',y',z')\, Q(x',y',z' \,|\, x,y,z;u) \right\}, \tag{3.2}$$

for all $(x,y,z) \in \mathrm{S}$, where $\theta_{x,y,z}(u)$ is the transition rate out of state (x,y,z) given that action $u \in A_{x,y,z}$ is chosen.

Furthermore, the stationary control which selects an action minimizing the right-hand side of (3.2) for all $(x,y,z) \in \mathrm{S}$ is optimal.

Proof. First note that the costs (2.6) are unbounded. However, one can easily establish that Assumptions 2 and 3 in Lippman [14] are satisfied. Therefore, the proof follows from Theorem 1 in [14]. ■

From the model description it is easily seen that the transition rates $\theta_{x,y,z}(u)$ read:

$$\begin{aligned}
\theta_{x,1,1}(1) &= \beta, \text{ for } x \in \mathbb{N}; \\
\theta_{x,1,0}(1) &= \beta, \text{ for } x \in \mathbb{N}^*; \\
\theta_{0,1,0}(0) &= \lambda + \nu; \\
\theta_{x,0,1}(1) &= \beta, \text{ for } x \in \mathbb{N}^*; \\
\theta_{x,0,1}(0) &= \lambda + \nu, \text{ for } x \in \mathbb{N}; \\
\theta_{x,0,0}(0) &= \lambda + \nu, \text{ for } x \in \mathbb{N}^*.
\end{aligned}$$

From Theorem 3.1 we easily obtain that

$$\begin{aligned}
(\alpha+\beta)\, V_\alpha(x,1,1) &= x\,(\alpha+\beta) + \mu\, V_\alpha(x-1,1,0) + \lambda\, V_\alpha(x+1,1,0) \\
&+ \nu\, V_\alpha(x,1,1), \text{ for } x \in \mathbb{N}^*;
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
(\alpha+\beta)\, V_\alpha(x,1,0) &= \mu\, V_\alpha(x-1,1,0) + \lambda\, V_\alpha(x+1,1,0) \\
&+ \nu\, V_\alpha(x,1,1), \text{ for } x \in \mathbb{N}^*;
\end{aligned} \tag{3.4}$$

$$(\alpha+\lambda+\nu)\, V_\alpha(0,1,0) = \lambda\, V_\alpha(1,0,0) + \nu\, V_\alpha(0,0,1); \tag{3.5}$$

$$(\alpha+\lambda+\nu)\, V_\alpha(0,0,1) = \lambda\, V_\alpha(1,0,0) + \nu\, V_\alpha(0,0,1); \tag{3.6}$$

$$V_\alpha(x,0,1) = \min\left\{ x + \frac{\lambda}{\alpha+\lambda+\nu}\, V_\alpha(x+1,0,0) + \frac{\nu}{\alpha+\lambda+\nu}\, V_\alpha(x,0,1); \right.$$

$$x + \gamma + \frac{\mu}{\alpha+\beta} V_\alpha(x-1,1,0) + \frac{\lambda}{\alpha+\beta} V_\alpha(x+1,1,0)$$

$$\left. + \frac{\nu}{\alpha+\beta} V_\alpha(x,1,1) \right\}, \quad \text{for } x \in \mathbb{N}^*; \tag{3.7}$$

$$(\alpha+\lambda+\nu) V_\alpha(x,0,0) \;=\; \lambda V_\alpha(x+1,0,0) + \nu V_\alpha(x,0,1), \quad \text{for } x \in \mathbb{N}^*. \tag{3.8}$$

Combining (3.5) and (3.6) gives us

$$V_\alpha(0,1,0) = V_\alpha(0,0,1) = \frac{\lambda}{\alpha+\lambda} V_\alpha(1,0,0). \tag{3.9}$$

Using (3.4) and (3.8) we see that (3.7) can be rewritten as

$$V_\alpha(x,0,1) = x + \min\left\{V_\alpha(x,0,0); \gamma + V_\alpha(x,1,0)\right\}, \quad \text{for } x \in \mathbb{N}^*. \tag{3.10}$$

Introducing (3.10) into (3.8) yields for $x \in \mathbb{N}^*$

$$(\alpha+\lambda+\nu) V_\alpha(x,0,0) = \nu x + \lambda V_\alpha(x+1,0,0) + \nu \min\left\{V_\alpha(x,0,0); V_\alpha(x,1,0)+\gamma\right\}. \tag{3.11}$$

On the other hand, we observe from (3.3) and (3.4) that

$$V_\alpha(x,1,1) = x + V_\alpha(x,1,0), \quad \text{for } x \in \mathbb{N}^*. \tag{3.12}$$

Introducing (3.12) into (3.4) yields for $x \in \mathbb{N}^*$

$$(\alpha+\lambda+\mu) V_\alpha(x,1,0) = \nu x + \lambda V_\alpha(x+1,1,0) + \mu\, V_\alpha(x-1,1,0). \tag{3.13}$$

Relations (3.9), (3.11), (3.12) and (3.13) contain all of the information given by the DP equation (3.2).

A glance at relation (3.13) indicates that it defines a difference equation of which the solution is given in the next lemma.

Lemma 3.1 *For any $\alpha > 0$,*

$$V_\alpha(x,1,0) = \left(V_\alpha(1,0,0)\left(\frac{\lambda}{\alpha+\lambda}\right) + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right)\right)\beta_{\alpha,1}^x + \frac{\nu}{\alpha}x + \nu\left(\frac{\lambda-\mu}{\alpha^2}\right), \tag{3.14}$$

for all $x \in \mathbb{N}$, where $\beta_{\alpha,1}$, $0 < \beta_{\alpha,1} < 1$, is the smallest root of the polynomial (in z) $\lambda z^2 - (\alpha + \lambda + \mu) z + \mu$.

Proof. Fix $\alpha > 0$. The general solution of the difference equation (3.13) is

$$V_\alpha(x,1,0) = a\,\beta_{\alpha,1}^x + b\,\beta_{\alpha,2}^x + c\,x + d, \tag{3.15}$$

for $x \in \mathbb{N}$, where $\beta_{\alpha,1}$ and $\beta_{\alpha,2}$ are the roots of the polynomial (in z) $\lambda\,z^2 - (\alpha+\lambda+\mu)\,z + \mu$, with $0 < \beta_{\alpha,1} < 1 < \beta_{\alpha,2}$.

By remembering that $V_\alpha(x,1,0)/x$ is uniformly bounded in $\mathbb{N}^*$, we see from (3.15) that necessarily $b = 0$ since $\beta_{\alpha,2} > 1$. The remaining coefficients a, c and d are easily identified by introducing (3.15) into (3.13) and by using (3.9). ∎

4 The Discounted Cost Problem

From now on we shall assume that $\lambda < \mu$ (condition of ergodicity). For any $\alpha > 0$, let $U_\alpha(x,y,z)$ be the optimal stationary control when in state $(x,y,z) \in \mathrm{S}$. As already observed, we only need to focus on states $(x,y,z) \in \mathrm{S}$ such that $x \neq 0$, $y = 0$ and $z = 1$. With a slight abuse of notation, define $U_\alpha(x) := U_\alpha(x,0,1)$ for $x \neq 0$, with the interpretation that $U_\alpha(x) = 1$ (resp. 0) if the decision is to turn the server on (resp. to take another vacation).

As already mentioned in the introduction, we follow the method proposed by Blanc, de Waal, Nain and Towsley [3] for determining the optimal α-discounted policy. For fixed $\alpha > 0$, the procedure goes as follows: (1) we first assume that the optimal policy is a threshold policy with threshold $L \geq 1$ (this policy will be called U_L), (2) we construct a function that would be a solution of the DP equation if policy U_L were indeed optimal, (3) we show the existence of an integer L such that the function we have constructed in (2) satisfies the DP equation.

The first two points of the above scheme yield the following proposition:

Proposition 4.1 *Fix $\alpha > 0$. Assume that there exists a finite integer $L \in \mathbb{N}^*$ and a family of L numbers $\{Y_L(x)\}_{x=1}^L$ that satisfy* [1]

$$\begin{aligned}(\alpha+\lambda)Y_L(x) &= \nu x + \lambda\,Y_L(x+1), \quad \textit{for } x = 1,2,\ldots,L-1; && (4.1)\\ Y_L(L) &= C_\alpha\left(Y_L(1)\left(\frac{\lambda}{\alpha+\lambda}\right) + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right)\right)\beta_{\alpha,1}^L + \frac{\nu}{\alpha}\,L + K_\alpha, && (4.2)\end{aligned}$$

and such that

$$\begin{aligned}Y_L(1) &\geq 0; && (4.3)\\ Y_L(x) &< Z_L(x) + \gamma, \quad \textit{for } x = 1,2,\ldots,L-1; && (4.4)\\ Y_L(L) &\geq Z_L(L) + \gamma, && (4.5)\end{aligned}$$

[1] Clearly, all numbers $L, Y_L(1),\ldots,Y_L(L)$ depend on α. This dependence has been skipped for sake of simplicity.

where

$$Z_L(x) := \left(Y_L(1)\left(\frac{\lambda}{\alpha+\lambda}\right) + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right)\right)\beta_{\alpha,1}^x + \frac{\nu}{\alpha}x + \nu\left(\frac{\lambda-\mu}{\alpha^2}\right); \tag{4.6}$$

$$C_\alpha := \frac{\nu}{\alpha+\nu+\lambda(1-\beta_{\alpha,1})}; \tag{4.7}$$

$$K_\alpha := \left(\frac{\nu}{\alpha+\nu}\right)\left(\frac{\lambda}{\alpha} + \nu\left(\frac{\lambda-\mu}{\alpha^2}\right) + \gamma\right). \tag{4.8}$$

Then, $U_\alpha(x) = \mathbf{1}(x \geq L)$ *for* $x \in \mathbb{N}^*$.

Proof. Fix $\alpha > 0$ and let $\{Y_L(x)\}_{x=1}^L$ be a family of numbers that satisfy (4.1)-(4.5).

Define:

$$\tilde{Y}_L(x) := \begin{cases} Y_L(x), & \text{for } x = 1,2,\ldots,L; \\ C_\alpha\left(Y_L(1)\left(\frac{\lambda}{\alpha+\lambda}\right) + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right)\right)\beta_{\alpha,1}^x + \frac{\nu}{\alpha}x + K_\alpha, & \text{for } x \geq L+1. \end{cases} \tag{4.9}$$

Let us first comment the above definition. As explained in (1) above, $\tilde{Y}_L(x)$ is obtained by assuming that the threshold policy U_L is optimal (i.e., $V_\alpha(x,0,0) = \tilde{Y}_L(x)$). This clearly motivates the definition of $\tilde{Y}_L(x)$ for $x = 1,2,\ldots,L$ (see (3.11) and (4.1)). The expression of $\tilde{Y}_L(x)$ for $x \geq L+1$ is obtained by introducing (3.14) into (3.11) (still assuming that U_L is optimal), which gives for $x \geq L$,

$$(\alpha+\lambda+\nu)\,V_\alpha(x,0,0) = \nu x + \lambda\,V_\alpha(x+1,0,0) + \nu a_\alpha\,\beta_{\alpha,1}^x + \frac{\nu^2}{\alpha}x + \frac{\nu^2}{\alpha^2}(\lambda-\mu) + \nu\gamma, \tag{4.10}$$

where a_α is the coefficient of $\beta_{\alpha,1}^x$ in (3.14). Again, we have a difference equation, of which the general solution is given by

$$V_\alpha(x,0,0) = K_0\,\beta_{\alpha,1}^x + K_1\left(\frac{\alpha+\lambda+\nu}{\lambda}\right)^x + K_2\,x + K_3, \quad \text{for } x \geq L. \tag{4.11}$$

Since $V_\alpha(x,0,0)/x$ must be uniformly bounded in x, we deduce from (4.11) that necessarily $K_1 = 0$. The remaining constants K_0, K_2 and K_3 are easily identified by pluging (4.11) into (4.10), which yields

$$V_\alpha(x,0,0) = a_\alpha\,C_\alpha\,\beta_{\alpha,1}^x + \frac{\nu}{\alpha}x + K_\alpha, \quad \text{for } x \geq L, \tag{4.12}$$

where C_α and K_α are defined in (4.7) and (4.8), respectively, which therefore explains the definition of $\tilde{Y}_L(x)$ for $x \geq L$.

Let us now show that $\tilde{Y}_L(x) \geq Z_L(x) + \gamma$ for $x \geq L$, where $Z_L(x)$ is defined in (4.6).

For $x \geq L$, we have from (4.6) and (4.9)

$$\tilde{Y}_L(x) - Z_L(x) - \gamma = (C_\alpha - 1)\Big(Y_L(1)\left(\frac{\lambda}{\alpha+\lambda}\right) + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right)\Big)\beta_{\alpha,1}^x + K_\alpha + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right) - \gamma. \quad (4.13)$$

We know from (4.5) and the definition of $\tilde{Y}_L(L)$ that the right-hand side of (4.13) is non negative for $x = L$. Since the right-hand side of (4.13) is a nondecreasing function of x (because $\lambda < \mu$, $Y_L(1) \geq 0$, $0 < \beta_{\alpha,1} < 1$, and $C_\alpha < 1$), we therefore deduce from (4.13) that

$$\tilde{Y}_L(x) \geq Z_L(x) + \gamma, \quad \text{for } x \geq L. \quad (4.14)$$

Using this result, (3.14), (4.1), (4.6), and (4.9), it is easily seen that $V_\alpha(x,0,0) = \tilde{Y}_L(x)$ and $V_\alpha(x,1,0) = Z_L(x)$) satisfy (3.11) and (3.13).

Consequently, we have found a solution to the DP equation that belongs to B (because $\tilde{Y}_L(x)/x$ and $Z_L(x)/x$ are uniformly bounded in $\mathbb{N}^*$). Since such a solution is unique from Theorem 3.1, we deduce that necessarily (cf. also (3.9), (3.12)),

$$\begin{aligned} V_\alpha(x,0,0) &= \tilde{Y}_L(x), \quad \text{for } x \in \mathbb{N}^*; & (4.15)\\ V_\alpha(0,1,0) &= (\lambda/(\alpha+\lambda))\,Y_L(1); & (4.16)\\ V_\alpha(x,1,0) &= Z_L(x), \quad \text{for } x \in \mathbb{N}^*; & (4.17)\\ V_\alpha(x,1,1) &= x + Z_L(x), \quad \text{for } x \in \mathbb{N}^*. & (4.18) \end{aligned}$$

Hence, cf. (4.4), (4.15), (4.17),

$$V_\alpha(x,0,0) < V_\alpha(x,1,0) + \gamma, \quad (4.19)$$

for $x = 1, 2, \ldots, L-1$, and, cf. (4.14), (4.15), (4.17),

$$V_\alpha(x,0,0) \geq V_\alpha(x,1,0) + \gamma, \quad (4.20)$$

for all $x \geq L$, or equivalently, $U_\alpha(x) = 1(x \geq L)$ for all $\in \mathbb{N}^*$, which concludes the proof. ■

We are now left with proving the existence of the integer L of Proposition 4.1. To do so, let us first introduce further notation. Let $\rho := \lambda/\mu$, $a := \lambda/\nu$ and define x_0 as the unique zero in $[0,\infty)$ of the polynomial (in w) $w^2 + (2a+1)\,w - 2a\gamma\,(1-\rho)$, that is

$$x_0 := \frac{-(2a+1) + \sqrt{(2a+1)^2 + 8a\gamma\,(1-\rho)}}{2}. \quad (4.21)$$

Last, let

l_0 be the smallest integer larger or equal to x_0 such that $l_0 \geq 1$. (4.22)

The symbol $O(\alpha)$ (resp. $O(1)$) will denote a function such that $\lim_{\alpha \downarrow 0} O(\alpha) = 0$ (resp. $= K$, $|K| < \infty$).

The following result holds:

Proposition 4.2 *Assume that $x_0 < l_0$. Then, there exists $\alpha_0 > 0$, such that for all $\alpha \in (0, \alpha_0)$, there exists a family of l_0 numbers $\{Y_{\alpha,l_0}(x)\}_{x=1}^{l_0}$ that satisfy (4.1)-(4.5).*

Proof. We first proceed with the solution $\{Y_{\alpha,L}(x)\}_{x=1}^{L}$ of the system of equations (4.1)-(4.2) when $L \geq 1$ and $\alpha > 0$ are both fixed.

From (4.1), we obtain that

$$Y_{\alpha,L}(x) = \left(\frac{\alpha+\lambda}{\lambda}\right)^{x-1} Y_{\alpha,L}(1) - \frac{\lambda\nu}{\alpha^2}\left[\left(\frac{\alpha+\lambda}{\lambda}\right)^{x} - 1\right] + \frac{\nu}{\alpha}\,x, \tag{4.23}$$

for $x = 2, \ldots, L$.

Combining (4.23) for $x = L$ and (4.2) yields $Y_{\alpha,L}(1)$, from which we deduce with (4.23) that

$$\begin{aligned} Y_{\alpha,L}(x) &= \left[\frac{\frac{\lambda\nu}{\alpha^2}\left(\left(\frac{\alpha+\lambda}{\lambda}\right)^{L} - 1\right) + \nu\left(\frac{\mu-\lambda}{\alpha^2}\right)\beta_{\alpha,1}^{L}\,C_\alpha + K_\alpha}{\left(\frac{\alpha+\lambda}{\lambda}\right)^{L} - \beta_{\alpha,1}^{L}\,C_\alpha}\right]\left(\frac{\alpha+\lambda}{\lambda}\right)^{x} \\ &+ \frac{\lambda\nu}{\alpha^2}\left[1 - \left(\frac{\alpha+\lambda}{\lambda}\right)^{x}\right] + \frac{\nu}{\alpha}\,x, \end{aligned} \tag{4.24}$$

for $x = 1, 2, \ldots, L$.

Finally, using (4.24) and the definition of $Z_{\alpha,L}(x)$ gives

$$Y_{\alpha,L}(x) - Z_{\alpha,L}(x) = \left[\frac{\frac{\nu\mu}{\alpha^2}\left(\beta_{\alpha,1}^{L}\,C_\alpha - \frac{\nu}{\alpha+\nu}\right) + \frac{\gamma\,\nu}{\alpha+\nu}}{\left(\frac{\alpha+\lambda}{\lambda}\right)^{L} - \beta_{\alpha,1}^{L}\,C_\alpha}\right]\left[\left(\frac{\alpha+\lambda}{\lambda}\right)^{x} - \beta_{\alpha,1}^{x}\right] + \frac{\nu\mu}{\alpha^2}\left(1 - \beta_{\alpha,1}^{x}\right), \tag{4.25}$$

for $x = 1, 2, \ldots, L$.

The first step is to prove the existence of $\alpha_1 > 0$ such that $Y_{\alpha,L}(1) > 0$ for $\alpha \in (0, \alpha_1)$. This property follows from the fact that (cf. Appendix A)

$$Y_{\alpha,L}(1) = \frac{A}{\alpha} + O(1), \tag{4.26}$$

where $A > 0$ does not depend on α.

The next step consists in evaluating the difference $Y_{\alpha,L}(x) - Z_{\alpha,L}(x)$ when α gets close to 0. After tedious algebra (cf. Appendix A), we obtain that

$$Y_{\alpha,L}(x) - Z_{\alpha,L}(x) = \left(\frac{L^2 + 2a\,L - x\,(L+a) + 2a^2 + 2a\,\gamma\,(1-\rho) + a}{L+a}\right)\left(\frac{x}{2a\,(1-\rho)}\right) + O(\alpha), \tag{4.27}$$

for $x = 1, 2, \ldots, L$.

On the other hand, it is seen from (4.27) that

$$Y_{\alpha,L}(L) - Z_{\alpha,L}(L) - \gamma = \frac{P(L)}{2(L+a)(1-\rho)} + O(\alpha), \tag{4.28}$$

where

$$P(w) := w^2 + (2a+1)\,w - 2a\,\gamma\,(1-\rho). \tag{4.29}$$

Recall the definitions of x_0 and l_0. Since $x_0 < l_0$ by assumption, we see that

$$P(L) \;<\; 0, \text{ for } L = 1, 2, \ldots, l_0 - 1; \tag{4.30}$$

$$P(l_0) \;>\; 0, \tag{4.31}$$

and so, cf. (4.28),

$$Y_{\alpha,L}(L) \;<\; Z_{\alpha,L}(L) + \gamma, \text{ for } L = 1, 2, \ldots, l_0 - 1; \tag{4.32}$$

$$Y_{\alpha,l_0}(l_0) \;>\; Z_{\alpha,l_0}(l_0) + \gamma, \tag{4.33}$$

for $\alpha \in (0, \alpha_2)$.

It remains to prove that for α small enough,

$$Y_{\alpha,l_0}(x) < Z_{\alpha,l_0}(x) + \gamma, \text{ for } x = 1, 2, \ldots, l_0 - 1. \tag{4.34}$$

To do so, rewrite $Y_{\alpha,L}(x) - Z_{\alpha,L}(x)$ as, cf. (4.27),

$$Y_{\alpha,L}(x) - Z_{\alpha,L}(x) = Y_{\alpha,L-1}(x) - Z_{\alpha,L-1}(x) + \left(\frac{x}{2a\,(1-\rho)}\right)(Q(L) - Q(L-1)) + O(\alpha), \tag{4.35}$$

for $x = 1, 2, \ldots, L-1$, where

$$Q(w) := \frac{w^2 + 2a\,w + 2a^2 + 2a\,\gamma\,(1-\rho) + a}{w+a}. \tag{4.36}$$

Since

$$\frac{\partial Q(w)}{\partial w} = \frac{P(w) - (w+a)}{(w+a)^2} < 0 \text{ for } 0 \le w \le x_0,$$

we get that $Q(L) - Q(L-1) < 0$ for $L = 1, 2, \ldots, l_0 - 1$. Further, it is shown in Appendix B that $Q(l_0) - Q(l_0 - 1) < 0$ when $x_0 < l_0$. Consequently,

$$Q(L) - Q(L-1) < 0, \text{ for } l = 1, 2, \ldots l_0, \tag{4.37}$$

which implies from (4.35) that for α small enough,

$$Y_{\alpha,L}(x) - Z_{\alpha,L}(x) < Y_{\alpha,L-1}(x) - Z_{\alpha,L-1}(x), \tag{4.38}$$

for $x = 1, 2, \ldots, L-1$, $L = 1, 2, \ldots, l_0$.

Combining (4.32) and (4.38), it is easily seen that there exists $\alpha_3 > 0$, such that for $\alpha \in (0, \alpha_3)$,

$$Y_{\alpha, l_0}(x) - Z_{\alpha, l_0}(x) < \gamma, \tag{4.39}$$

for $x = 1, 2, \ldots, l_0 - 1$. The proof is concluded by letting $\alpha_0 := \min(\alpha_1, \alpha_2, \alpha_3)$. ■

Propositions 4.1 and 4.2 yield the following

Proposition 4.3 *Assume that $x_0 < l_0$. Then, there exists $\alpha_0 > 0$ such that $U_\alpha(x) = \mathbf{1}(x \geq l_0)$ for $x \in \mathbb{N}^*$.*

Remark 4.1 If

$$0 \leq \gamma < \frac{a+1}{a(1-\rho)}, \tag{4.40}$$

then for α small enough $U_\alpha(x) = 1$. This result follows from (4.28) by noting that the condition (4.40) is equivalent to $P(1) > 0$.

5 The Long-Run Average Cost Problem

In this section we shall discuss the long-run average cost problem (2.8) and we shall establish the optimality of a threshold policy.

Since $V_\alpha^U(x, y, z)$ is well defined for all $(x, y, z) \in \mathrm{S}$, $U \in \mathcal{U}$ (see (2.5)), we know from a Tauberian theorem (Widder [20], pp. 181-182) that

$$\text{limsup}_{\alpha \downarrow 0}\, \alpha V_\alpha^U(x, y, z) \leq \text{limsup}_{T \uparrow \infty} W_T^U(x, y, z), \tag{5.1}$$

for all $(x, y, z) \in \mathrm{S}$, $U \in \mathcal{U}$. Further, if $\lim_{T \uparrow \infty} W_T^U(x, y, z)$ exists then $\lim_{\alpha \downarrow 0} \alpha V_\alpha^U(x, y, z)$ exists as well, and

$$W^U(x, y, z) = \lim_{\alpha \downarrow 0} \alpha V_\alpha^U(x, y, z), \tag{5.2}$$

for all $(x, y, z) \in \mathrm{S}$, $U \in \mathcal{U}$.

Fix $x \in \mathbb{N}^*$ and let $U \in \mathcal{U}$ be an arbitrary policy. Two cases need be distinguished:

Case 1: $x_0 < l_0$.

From Proposition 4.3 we have

$$V_\alpha^{U_{l_0}}(x, 0, 1) \leq V_\alpha^U(x, 0, 1), \tag{5.3}$$

for all $\alpha \in (0, \alpha_0)$. Hence, cf. (5.1),

$$\limsup_{\alpha \downarrow 0} \alpha\, V_\alpha^{U_{l_0}}(x,0,1) \leq W^U(x,0,1). \tag{5.4}$$

Under the threshold policy U_{l_0} and the condition $\rho < 1$, it is readily seen that $\lim_{T\uparrow\infty} W_T^{U_{l_0}}(x,0,1)$ exists and is independent of the initial condition, which implies from (5.2) and (5.4) that

$$W^{U_{l_0}}(x,0,1) \leq W^U(x,0,1). \tag{5.5}$$

Case 2: $x_0 = l_0$.

From now on, $V_\alpha^U(x,0,1)$, $W^U(x,0,1)$, x_0 and l_0 are considered as functions of the parameter γ, and denoted as $V_{\alpha,\gamma}^U(x,0,1)$, $W_\gamma^U(x,0,1)$, x_0^γ and l_0^γ, respectively.

Let λ, μ, ν and $\gamma = \gamma_0$ be such that $x_0^{\gamma_0} = l_0^{\gamma_0}$. Since the mapping $\gamma \to x_0^\gamma$ is strictly increasing and continuous in $[0,\infty)$, cf. (4.21), we see that there exists $H > 0$ such that

$$l_0^{\gamma_0} - 1 < x_0^{\gamma_0 - h} < l_0^{\gamma_0}, \tag{5.6}$$

for $0 < h < H$. Therefore, Proposition 4.3 applies to the parameters λ, μ, ν and $\gamma_0 - h$ for $0 < h < H$, which implies that for $h \in (0, H)$,

$$V_{\alpha,\gamma_0-h}^{U_{l_0^{\gamma_0}}}(x,0,1) \leq V_{\alpha,\gamma_0-h}^U(x,0,1) \leq V_{\alpha,\gamma_0}^U(x,0,1), \tag{5.7}$$

for $\alpha \in (0, \alpha_0(h))$, where the second inequality follows from the fact that the mapping $\gamma \to V_{\alpha,\gamma}^U(x,0,1)$ is nondecreasing in $[0,\infty)$ for $\alpha > 0$. Similarly to Case 1 above, we derive from (5.7) that

$$W_{\gamma_0-h}^{U_{l_0^{\gamma_0}}}(x,0,1) \leq W_{\gamma_0}^U(x,0,1), \tag{5.8}$$

for $0 < h < H$, which in turn yields

$$W_{\gamma_0}^{U_{l_0^{\gamma_0}}}(x,0,1) \leq W_{\gamma_0}^U(x,0,1), \tag{5.9}$$

since the mapping $\gamma \to W_\gamma^{U_{l_0^{\gamma_0}}}(x,0,1)$ is continuous (in fact linear) in $[0,\infty)$ (use (2.8) and (2.9), together with the property that $\lim_{T\uparrow\infty} W_T^U(x,0,1)$ exists and is finite when a threshold is employed).

This shows the optimality of a threshold policy.

Let us now compute the optimal long-run average cost. Assume first that $x_0 < l_0$. Because $\lim_{\alpha\downarrow 0} \alpha V_\alpha(x,0,1)$ does not depend on x, this limit can be obtained (in particular) from (4.26) and (A.14). Therefore, the optimal long-run average cost $W^{U_{l_0}}$ is given by

$$W^{U_{l_0}} = \left(\frac{(l_0-1)\,l_0}{2a} + \left(\frac{\rho}{a(1-\rho)} + 1\right)(l_0+a) + \gamma(1-\rho)\right)\left(\frac{\lambda}{l_0+a}\right). \tag{5.10}$$

By using now the continuity of the mapping $\gamma \to W_\gamma^{U_{l_0}^\gamma}(x,y,z)$ (cf. Appendix C) we deduce that (5.10) also gives the optimal long-run average cost when $x_0 = l_0$.

The results of this paper are collected in the following proposition (we relax the assumption that $h/\nu = 1$):

Proposition 5.1 *Assume that $\lambda < \mu$. Then, there exists a threshold policy that solves the long-run average cost problem (2.8). The long-run average cost corresponding to the optimal policy is given by*

$$W^{U_{l_0}} = \left(\frac{l_0\,(l_0-1)}{2(l_0+a)} + \frac{\rho}{(1-\rho)} + a\right) h + \gamma\,\frac{\lambda\,(1-\rho)}{l_0+a}, \tag{5.11}$$

where the optimal threshold l_0 is given in (4.22) once γ is substituted for γ/h in (4.21).

Remark 5.1 It may be checked from (5.11) that the policy U_{l_0+1} is also optimal whenever $x_0 = l_0$ (i.e., $W^{U_{l_0}} = W^{U_{l_0+1}}$). Note that this property can directly be obtained by considering $\gamma + h$ instead of $\gamma - h$ in the proof of Proposition 5.1.

Remark 5.2 From (2.9) and the definition of the cost c (see Section 2) one immediately deduces from (5.11) that $(L+a)/(\lambda\,(1-\rho))$ is the average return time to an empty queue when the threshold policy U_L is used, for any $L \geq 1$. Similarly, one observes that $L\,(L-1)/(2\,(L+a)) + \rho/(1-\rho) + a$ is the expected queue length under policy U_L, for any $L \geq 1$.

A Appendix

In this appendix we establish formulas (4.26) and (4.27).

To do so, the following power series expansions are needed:

$$\frac{\nu}{\alpha+\nu} = 1 - \frac{\alpha}{\nu} + \frac{\alpha^2}{\nu^2} + o\left(\alpha^2\right); \tag{A.1}$$

$$\beta_{\alpha,1}^x = 1 + \frac{x\alpha}{\lambda-\mu} + \left(\frac{(x-1)x}{2(\lambda-\mu)^2} - \frac{x\mu}{(\lambda-\mu)^3}\right)\alpha^2 + o\left(\alpha^2\right); \tag{A.2}$$

$$\left(\frac{\alpha+\lambda}{\lambda}\right)^x = 1 + \frac{x}{\lambda}\alpha + \frac{(x-1)x}{2\,\lambda^2}\alpha^2 + o\left(\alpha^2\right); \tag{A.3}$$

$$C_\alpha = 1 + \frac{\mu}{(\lambda-\mu)\nu}\alpha + \left(\frac{(\lambda\mu - \lambda\nu - \mu^2)\,\mu}{\nu^2\,(\lambda-\mu)^3}\right)\alpha^2 + o\left(\alpha^2\right), \tag{A.4}$$

where $\lim_{\alpha\downarrow 0} o(\alpha^i)/\alpha^i = 0$ for any $i \in \mathbb{N}^*$.

From (A.1)-(A.4), we readily obtain that

$$\frac{\nu\mu\,(1-\beta^x_{\alpha,1})}{\alpha^2} = \frac{\nu\lambda\,a_1}{\alpha} - \frac{\nu\mu\,x}{(\lambda-\mu)^2}\left(\frac{x-1}{2} - \frac{\mu}{\lambda-\mu}\right) + O(\alpha); \tag{A.5}$$

$$\left(\frac{\alpha+\lambda}{\lambda}\right)^x - \beta^x_{\alpha,1} = a_1\,\alpha + a_2\,\alpha^2 + o\left(\alpha^2\right); \tag{A.6}$$

$$\frac{\nu\mu}{\alpha^2}\left(\beta^L_{\alpha,1}\,C_\alpha - \frac{\nu}{\alpha+\nu}\right) + \frac{\nu\,\gamma}{\alpha+\nu} = -\frac{\nu\lambda\,b_1}{\alpha} - \nu\mu\left(b_2 + \frac{1}{\nu^2} - \frac{(L-1)L}{2\lambda^2}\right) + \gamma + O(\alpha); \tag{A.7}$$

$$\left(\frac{\alpha+\lambda}{\lambda}\right)^L - \beta^L_{\alpha,1}\,C_\alpha = b_1\,\alpha + b_2\,\alpha^2 + o\left(\alpha^2\right); \tag{A.8}$$

$$\frac{\left(\frac{\alpha+\lambda}{\lambda}\right)^x - \beta^x_{\alpha,1}}{\left(\frac{\alpha+\lambda}{\lambda}\right)^L - \beta^L_{\alpha,1}\,C_\alpha} = \frac{a_1}{b_1} + \left(\frac{a_2}{b_1} - \frac{a_1\,b_2}{b_1^2}\right)\alpha + o(\alpha), \tag{A.9}$$

with

$$a_1 := \frac{\mu\,x}{\lambda\,(\mu-\lambda)}; \tag{A.10}$$

$$a_2 := -\left(\frac{x-1}{2\,(\lambda-\mu)^2} - \frac{\mu}{(\lambda-\mu)^3} - \frac{x-1}{2\,\lambda^2}\right)x; \tag{A.11}$$

$$b_1 := \left(\frac{\mu}{\mu-\lambda}\right)\left(\frac{L}{\lambda} + \frac{1}{\nu}\right); \tag{A.12}$$

$$b_2 := -\left(\frac{(L-1)L}{2\,(\lambda-\mu)^2} - \frac{L\mu}{(\lambda-\mu)^3} + \frac{\mu(\lambda\mu-\lambda\nu-\mu^2)}{\nu^2\,(\lambda-\mu)^3} + \frac{L\mu}{\nu\,(\lambda-\mu)^2} - \frac{(L-1)L}{2\lambda^2}\right). \tag{A.13}$$

Using (4.25), (A.5)-(A.13) we obtain after straightforward algebra

$$\begin{aligned} Y_{\alpha,L}(x) - Z_{\alpha,L}(x) &= \left(\frac{\nu\,x}{L+\frac{\lambda}{\nu}}\right)\left(\frac{(L-1)\,L\mu}{2\lambda\,(\mu-\lambda)} + \frac{\mu\,L}{\nu\,(\mu-\lambda)} + \frac{\mu\,\lambda}{\nu^2\,(\mu-\lambda)} + \frac{\gamma}{\nu}\right) \\ &- \frac{\nu\mu\,(x-1)\,x}{2\lambda\,(\mu-\lambda)} + O(\alpha), \\ &= \left(\frac{x}{L+a}\right)\left(\frac{(L-1)\,L}{2a\,(1-\rho)} + \frac{L}{1-\rho} + \frac{a}{1-\rho} + \gamma\right) - \frac{(x-1)\,x}{2a\,(1-\rho} + O(\alpha), \\ &= \left(\frac{L^2 + 2a\,L - x\,(L+a) + 2a^2 + 2a\,\gamma\,(1-\rho) + a}{L+a}\right)\left(\frac{x}{2a\,(1-\rho)}\right) + O(\alpha). \end{aligned}$$

On the other hand, using the series expansions (cf. (A.1)-(A.4))

$$\frac{1}{\left(\frac{\alpha+\lambda}{\lambda}\right)^L - \beta^L_{\alpha,1}\,C_\alpha} = \frac{1}{b_1\,\alpha} - \frac{b_2}{b_1^2} + O(\alpha);$$

$$\beta^L_{\alpha,1}\, C_\alpha - \frac{\nu}{\alpha+\nu} \;=\; -\frac{\lambda b_1\, \alpha}{\mu} - \left(b_2 + \frac{1}{\nu^2} - \frac{(L-1)\,L}{2\lambda^2}\right)\alpha^2 + o(\alpha^2),$$

and (4.24), we readily obtain that

$$Y_{\alpha,L}(1) = \left(\frac{(L-1)\,L}{2a} + \left(\frac{\rho}{a\,(1-\rho)} + 1\right)(L+a) + \gamma\,(1-\rho)\right)\left(\frac{\lambda}{L+a}\right)\alpha^{-1} + O(1). \qquad \text{(A.14)}$$

B Appendix

We show that

$$Q(l_0) - Q(l_0 - 1) < 0, \qquad \text{(B.1)}$$

when $x_0 < l_0$.

From the definition of $P(w)$ and $Q(w)$ (cf. (4.29) and (4.36), respectively), it is easily seen that (B.1) holds if and only if

$$P(l_0) < 2\,(l_0 + a). \qquad \text{(B.2)}$$

Let us prove (B.2). We have:

$$\begin{aligned} P(l_0) &= P(l_0) - P(x_0), \\ &= (l_0 - x_0)(l_0 + x_0 + 2a + 1), \\ &= \Delta\,(2l_0 - \Delta + 2a + 1) := f(\Delta), \end{aligned}$$

where $\Delta := l_0 - x_0$ with $0 < \Delta < 1$.

The proof is now completed by observing that the mapping $\Delta \to f(\Delta)$ is nondecreasing in $[0,1]$ (since $l_0 \geq 1$) and by noting that $f(1) = 2\,(l_0 + a)$.

C Appendix

We show in this appendix that the mapping $\gamma \to W_\gamma^{U_{l_0}^{\gamma}}(x,y,z)$ is continuous in $[0,\infty)$. Fix $(x,y,z) \in$ S.

Since $\rho < 1$, it is readily seen that $\lim_{T\uparrow\infty} W_T^{U_L}(x,y,z)$ exists (in particular) for any threshold policy U_L with finite threshold L. By combining this result together with (2.6) and (2.8), we readily conclude that the mapping $\gamma \to W_\gamma^{U_L}(x,y,z)$ is nondecreasing (property (P1)) and continuous (property (P2)) in $[0,\infty)$ for $L < \infty$.

Let $0 \leq \gamma_1 < \gamma_2 < \infty$. Then,

$$W_{\gamma_1}^{U_{l_0}^{\gamma_1}}(x,y,z) \leq W_{\gamma_1}^{U_{l_0}^{\gamma_2}}(x,y,z) \leq W_{\gamma_2}^{U_{l_0}^{\gamma_2}}(x,y,z) \leq W_{\gamma_2}^{U_{l_0}^{\gamma_1}}(x,y,z), \qquad \text{(C.1)}$$

where we have used (P1) to establish the second inequality.

Using now (P2) and (C.1) we obtain that

$$\lim_{\gamma_2 \to \gamma_1} W_{\gamma_2}^{U_2}(x,y,z) = W_{\gamma_1}^{U_1}(x,y,z),$$

which concludes the proof.

References

[1] E. Altman and P. Nain, "Optimal control of an M/G/1 queue with repeated vacations." Submitted to the *IEEE Transactions on Automatic Control.*

[2] P. Brémaud, *Point Processes and Queues.* Springer Verlag. New York, 1981.

[3] J. P. C. Blanc, P. de Waal, P. Nain, and D. Towsley, "A new device for the synthesis problem of optimal control of admission to an M/M/c queue," INRIA Report no. 1310, Oct. 1990. Submitted to the *IEEE Transactions on Automatic Control.*

[4] K. L. Chung, *Markov Chains with Stationary Transition Probabilities.* 2nd. Ed., Springer Verlag. New York, 1967.

[5] B. Doshi, "Queueing systems with vacations - A survey," *Queueing Systems*, vol. 1, pp. 29-66, 1986.

[6] B. Doshi, "Conditional and unconditional distributions for M/G/1 type queues with server vacations," *Queueing Systems*, vol. 7, pp. 229-252, 1990.

[7] E. Gelenbe and I. Mitrani, *Analysis and Synthesis of Computer Systems.* Academic Press. London, 1980.

[8] E. Gelenbe and R. Iasnogorodski, "A queue with server of walking type (autonomous service)," *Ann. Inst. Henry Poincaré*, vol. XVI, no. 1, pp. 63-73, 1980.

[9] D. P. Heyman and M. J. Sobel, *Stochastic Models in Operations Research*, Volume II, Stochastic Optimization. McGraw-Hill Book Company. New York, 1984.

[10] O. Kella, "The threshold policy in the M/G/1 queue with server vacations," *Naval Research Quaterly*, vol. 33, pp. 111-123, 1989.

[11] O. Kella, "Optimal control of the vacation scheme in an M/G/1 queue," *Operations Research*, vol. 38, no. 4, pp. 724-728, 1990.

[12] H-S Lee and M. M. Srinivasan, "Control policies for the $M^x/G/1$ queueing system," *Management Science*, vol. 35, no. 6, pp. 708-721, 1989.

[13] Y. Levy and U. Yechiali, "Utilization of idle time in an M/G/1 queueing system," *Management Science*, vol. 22, no. 2, pp. 202-211, 1975.

[14] S. A. Lippman, "On dynamic programming with unbounded rewards," *Management Science*, vol. 21, no. 11, pp. 1225-233, 1975.

[15] S. M. Ross, *Applied Probability Models with Optimization Applications*. Holden-Day. San Francisco, 1970.

[16] L. I. Sennott, "Average cost semi-Markov decision processes and the control of queueing system," *Probability in the Engineering and the Informational Sciences*, vol. 3, pp. 247-272, 1989.

[17] A. J. J. Talman, "A simple proof of the optimality of the best N-policy in the M/G/1 queueing control problem with removable server," *Statistica Neerlandica*, vol. 33, pp. 143-150, 1979.

[18] H. Takagi, *Analysis of Polling Systems*. MIT Press. 1986.

[19] J. Teghem Jr., "Control of the service process in a queueing system," *European Journal of Operations Research*, vol. 23, pp. 141-158, 1986.

[20] D. V. Widder, *The Laplace Transform*. Princeton University Press. Princeton, 1941.

[21] M. Yadin and P. Naor, "Queueing systems with a removable service station," *Operations Research Quaterly*, vol. 14, no. 4, pp. 393-405, 1963.

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