A theorem about the asymptotic behaviour of stationary functions of some inf-convolution equations

Philippe Jacquet

UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
BP105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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A THEOREM ABOUT THE ASYMPTOTIC BEHAVIOUR OF STATIONARY FUNCTIONS OF SOME INF-CONVOLUTION EQUATIONS

Philippe JACQUET

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Abstract. Let $f(x)$ and $g(x)$ be two positive real valued functions defined on the positive real numbers x, such that $f(0) = g(0) = 0$. Let $h(x)$ be another positive real valued function, we note $f \star h$ the function which gives to $x \geq 0$ the value of the lower bound of function $h(y) + f(x - y)$ for $y \in [0, x]$. We suppose that $h(x)$ is a stationary function of the (f, g)-convolution, namely $h(x) = f \star h(x) + g(x)$ for all $x \geq 0$. We show that under very general conditions, the asymptotic behavior of $h(x)$ when $x \to \infty$, only depends on the asymptotic behavior of $f(x)$ and $g(x)$. We precisely describe this behavior when $f(x)$ and $g(x)$ are asymptotically polynomial.

Un théorème sur le comportement asymptotique des fonctions stationnaires de quelques équations en inf-convolution

Résumé. Soient $f(x)$ et $g(x)$ deux fonctions à valeurs positives définies pour x réels positifs telles que $f(0) = g(0) = 0$. Soit $h(x)$ une autre fonction à valeurs positive, on note $f \star h$ la fonction qui à $x \geq 0$ associe la valeur de la borne inférieure de la fonction $h(y) + f(x - y)$ quand $y \in [0, x]$. On suppose que $h(x)$ est une fonction stationnaire de la (f, g)-convolution, c'est à dire $h(x) = f \star h(x) + g(x)$ pour tout $x \geq 0$. Nous montrons sous des hypothèses très générales que le comportement asymptotique de $h(x)$ quand $x \to \infty$ ne dépend que des comportements asymptotiques de $f(x)$ et $g(x)$. On précise ce comportement quand $f(x)$ et $g(x)$ sont asymptotiquement polynomiales.

1 Introduction

Let $f(x)$ and $g(x)$ be positive real valued functions defined on the positive real numbers x. We assume $f(0) = g(0) = 0$. We suppose that functions $f(x)$ and $g(x)$ are *locally* bounded. A function is locally bounded if it is bounded on every compact. Note that upper semicontinuity implies local boundness. We define the sequence of positive real valued functions $h_n(x)$ as

$$h_n(x) = \min_{x_1+\cdots+x_n=x} \left\{ \sum_{i=1}^{i=n} f(x_i) + g(x_1 + \cdots + x_i) \right\} . \tag{1}$$

It is equivalent to define $h_n(x)$ using iterated *inf-convolutions*, namely

$$h_{n+1}(x) = \min_{y\in[0,x]} \{h_n(y) + f(x-y)\} + g(x) , \tag{2}$$

with $h_0(x) = f(x) + g(x)$. If $h(x)$ is a positive real valued function we note by $f \star h(x)$ the function $\min_{y\in[0,x]}\{h_n(y) + f(x-y)\}$. This kind of convolution is called *inf-convolution.* The specific operation which maps function h to function $f\star h+g$ will be called the (f,g)-convolution of function h. Therefore the sequence h_n is the result of iterated (f,g)-convolutions over initial function h_0. Formulation (2) reveals that our problem finds numerous applications in dynamic programming. But the formulation (1) refers to the statement of the problem arising in the asymptotic evaluation of the tail distribution of waiting times in LaPalice queueings. There are also applications in task scheduling problems.

Our purpose is to determine the asymptotic behavior of the limiting function $h(x) = \lim_{n\to\infty} h_n(x)$. Note that the sequence $h_n(x)$ is decreasing with respect to subscript n and $h(x) \geq g(x)$.

Theorem 1 *If $f(x) \sim Ax^\alpha$ and $g(x) \sim Bx^\beta$ when $x \to +\infty$ with $\alpha > 1$ and $\beta > 0$, we have the asymptotic equivalence $h(x) \sim Cx^\gamma$ when $x \to \infty$ with $\gamma = \beta + 1 - \beta/\alpha$ and $C = \frac{(A\alpha)^{1/\alpha}}{\gamma}(B\frac{(\alpha-1)}{\alpha})^{(\alpha-1)/\alpha}$.*
Note that $\beta < \gamma < \alpha$ and $\gamma > 1$. The theorem is in fact a kind of consequence of the following more general theorem.

Theorem 2 *Let $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ be four locally bounded positive functions with value zero at $x = 0$. Let $h'(x)$ (respectively $h''(x)$) be stationary function of the (f,g)-convolution (respectively the (f',g')-convolution), namely $h(x) = f \star h(x) + g(x)$. We assume $f(x)$ and $g(x)$ both tending to infinity when $x \to \infty$. If $f(x) \sim f'(x)$ and $g'(x) \sim g(x)$ when $x \to \infty$, then $h'(x) \sim h''(x)$, when $x \to \infty$.*

2 Proof of the theorems

In order to prove the main theorem we will refer to some simple lemmas. As a notational convention we set $f' \geq f''$ when $f'(x)$ and $f''(x)$ are functions as meaning $\forall x$: $f'(x) \geq f''(x)$. The following one is obvious.

Lemma 4 *Let $f'(x)$ and $f''(x)$, $g'(x)$ and $g''(x)$, $h'_0(x)$ and $h''_0(x)$ be positive functions defined on the positive real numbers, let us consider the sequence $h'_n(x)$ (respectively $h''_n(x)$) defined with the iterated (f', g')-convolution with initial function $h'_0(x)$ (respectively with the (f'', g'')-convolutions with initial function $h''_0(x)$). If $f' \geq f''$, or $g' \geq g''$, or $h'_0 \geq h''_0$, then $h'_n \geq h''_n$ for all integer $n \geq 1$.*

It is obvious that functions $h_n(x)$ are locally bounded, therefore function $h(x)$ is upper semicontinuous as a limit of a decreasing sequence of locally bounded functions. Since $h(x)$ and $f(x)$ are not continuous function $h(y) + f(x - y)$ does not necessary reaches its minimum when y varies in $[0, x]$. Therefore we need the following tool.

Let $h'(x)$ be an arbitrarily chosen positive real valued function. Since function $h'(x)$ is *a priori* a general function $h'(y) + f(x - y)$ does not necessarily reach its minimum on $[0, x]$. Let ε be a real strictly positive number, we note $d(x, \varepsilon)$ *one* of the value of y which approaches the minimum of $h(y) + f(x - y)$ closer than ε: $h(d(x, \varepsilon)) + f(x - d(x, \varepsilon)) \leq f \star h(x) + \varepsilon$. The strangeness of this definition comes from the fact that we do not force the choice of $d(x, \varepsilon)$ among certainly numerous possibilities. Strictly speaking we maybe have to refer to the *axiom of choice* in order to have proper definition. But it is unuseful for the restrictive application of our analysis. For convenience of notation we define $d^n(x, \varepsilon) = d(d^{n-1}(x, \varepsilon), \varepsilon 2^{-n})$ for for all integer n greater than 1, we set $d^0(x, \varepsilon) = x$.

Lemma 3 *Function $h(x)$ is a stationary function of the (f, g)-convolution:*

$$h(x) = f \star h(x) + g(x) \ , \tag{3}$$

Proof. This lemma seems obvious, but in fact it is not, since we only know that convergence $h_n(x) \to h(x)$ is simple. Since $h_n(x)$ is a decreasing sequence we have $f \star h \leq f \star h_n$. Therefore for all $x \geq 0$ $f \star h(x) + g(x) \leq h_{n+1}(x)$ and, at the limit

$$f \star h(x) + g(x) \leq h(x) \ . \tag{4}$$

Let function $d(x, \varepsilon)$ defined for function $h(x)$. We have

$$f \star h(x) \geq h(d(x, \varepsilon)) + f(x - d(x, \varepsilon)) - \varepsilon \ .$$

Therefore since obviously $h_n(d(x,\varepsilon))+f(x-d(x,\varepsilon)) \geq \min\{h_n(y)+f(x-y)\}$,

$$f \star h(x) + g(x) \geq h_{n+1}(x) - \varepsilon + h(d(x,\varepsilon)) - h_n(d(x,\varepsilon)) \ ,$$

and at the limit, since $\lim h_n(d(x,\varepsilon)) = h(d(x,\varepsilon))$,

$$f \star h(x) + g(x) \geq h(x) - \varepsilon \ , \tag{5}$$

Since ε can be made as small as possible, this last inequality, together with (4), completes the proof of the lemma. ∎

Lemma 5 *Every function $h'(x)$ stationary function of the (f,g)-convolution satisfies $f+g \geq h' \geq g$.*

Corollary 6 *Every function $h'(x)$ which is stationary function of the (f,g)-convolution is locally bounded*

Lemma 7 *Let $h'(x)$ be a stationary function of the (f,g)-convolution. Let $D > 0$ and $x \geq 0$ the number of integers n such that $d^n(x,\varepsilon) \geq D$ is less than $(h'(x)+\varepsilon)\max_{y \geq D}\{g(x)^{-1}\}$.*

Proof. The sequence $d^n(x,\varepsilon)$ obviously decreases, since $d(x,\varepsilon) \leq x$. Using identity (3), we reach inequality

$$h(x) + \varepsilon \geq h(d(x,\varepsilon)) + f(x - d(x,\varepsilon)) + g(x) \tag{6}$$

that can be iterated again from right hand side, where expression $h(d(x,\varepsilon))$ can be found. Therefore, for each integer n we have

$$h(x) + (1-2^{-n})\varepsilon \geq h(d^n(x,\varepsilon)) + \sum_{i=1}^{i=n} f(d^i(x,\varepsilon) - d^{i-1}(x,\varepsilon)) + g(d^{i-1}(x,\varepsilon)) \ . \tag{7}$$

From the above inequality it is clear that $\sum_i g(d^i(x,\varepsilon)) \leq h'(x) + \varepsilon$. ∎

Lemma 8 *Let $h'(x)$ be a stationary function of the (f,g)-convolution, we have $h \geq h'$.*

Proof. Applying lemma 3 with $h'_0(x) = h'(x)$ we have through lemma 5 $h_0 \geq h'_0$. Therefore, for all integer n, $h_n \geq h'_n$. Since $h'_n = h'$ because $h'(x)$ is stationary function, we get for all n, $h_n \geq h'$ and at the limit $h \geq h'$. ∎

Lemma 9 *Let $D > 0$ such that* $\min_{y \geq D}\{g(y)\} > 0$. *Let $h'(x)$ be a stationary function of the (f,g)-convolution, therefore $h(x) - h(x)$ is bounded.*

Proof. Let $h(x) = h'(x) + \delta(x)$ with $\delta(x) \geq 0$, by lemma 8. Function $\delta(x)$ is locally bounded by corollary 6. Let us consider mapping $d(x, \varepsilon)$ defined for $h'(x)$. Since $h(x)$ is a stationary function we have

$$h'(x) + \delta(x) = \min_{y \in [0,x]} \{h'(y) + \delta(y) + f(x-y)\} + g(x) \ . \tag{8}$$

Therefore

$$h'(x) + \delta(x) \leq h'(d(x,\varepsilon)) + \delta(d(x,\varepsilon)) + f(x - d(x,\varepsilon)) + g(x) \ . \tag{9}$$

Using the definition of $d(x, \varepsilon)$ and the fact that $h'(x)$ is a stationary function of the (f, g)-convolution we finally get

$$\delta(x) \leq \delta(d(x,\varepsilon)) + \varepsilon \ . \tag{10}$$

Iterating (10) using the $d^n(x, \varepsilon)$'s we obtain $\delta(x) \leq \delta(d^n(x,\varepsilon)) + (1 - 2^{-n})\varepsilon$. Since $g(x) \to \infty$ when $x \to \infty$ let $D > 0$ such that $\min_{y \geq D}\{g(y)\} > 0$. By lemma 7, sequence $d^n(x, \varepsilon)$ gets below threshold D in a finite number of steps and therefore for all x: $\delta(x) \leq \max_{y \leq D}\{\delta(y)\} + \varepsilon$. ∎

Corollary 10 *Let $g(x) \to \infty$ when $x \to \infty$, all stationary functions of the (f, g)-convolution are equivalent when $x \to \infty$.*

Lemma 11 *Let $f(x)$ and $g(x)$ be continuous at $x = 0$. Furthermore, let $g(x)$ be an increasing function with $x = 0$ as only root (solution of the equation $g(x) = 0$). $h(x)$ is the only stationary function of the (f, g)-convolution such that $h(0) = 0$.*

Proof. By lemma 7, the sequence $d^n(x, \varepsilon)$ obviously converges to zero. Therefore iterating (10) we obtain $\delta(x) \leq \delta(d^n(x,\varepsilon)) + (1 - 2^{-n})\varepsilon$. Since $\delta(x)$ is continuous at $x = 0$ (because both function $h'(x)$ and $h(x)$ are between g(x) and $f(x) + g(x)$), we have $\lim_n \delta(d^n(x, \varepsilon)) = 0$ and we terminate the proof of the lemma because ε can be made as small as possible. ∎

Now we are ready to prove the following main theorem.

Lemma 12 *Let $g(x) \to \infty$ when $x \to \infty$. Let $g'(x)$ be a function following the same statement than $g(x)$ with additional $g'(x) \sim g(x)$ when $x \to \infty$. Let $h(x)$ and $h'(x)$ be functions being respectively stationary functions of the (f, g)-convolution and of the (f, g')-convolution. We have $h'(x) \sim h(x)$ when $x \to \infty$.*

Proof. If $g(x) \to \infty$ then $h(x) \to \infty$. It does not hurt the generality of the proposition to suppose $g' \geq g$. Indeed, if it is not the case, we can define functions $\bar{g}(x) = \sup\{g(x), g'(x)\}$ and $\tilde{g}(x) = \inf\{g(x), g'(x)\}$. We have $\bar{g} \geq g, g' \geq \tilde{g}$ and $\bar{g}(x) \sim \tilde{g}(x)$ when $x \to \infty$. Let $\bar{h}(x)$ and $\tilde{h}(x)$ be respectively the stationary function of the $(f, \bar{g})$ and $(f, \tilde{g})$-convolutions, by lemma 3 $\bar{h} \geq, h, h' \geq \tilde{h}$ and if we prove $\bar{h}(x) \sim \tilde{h}(x)$ when $x \to \infty$, then we will prove $h(x) \sim h'(x)$.

Let $g'(x) = g(x) + r(x)$ with $r(x) \geq 0$, therefore $h' \geq h$ and let $h'(x) = h(x) + \delta(x)$ with $\delta(x) \geq 0$. We will refer to mappings $d(x, \varepsilon)$ and $d^n(x, \varepsilon)$, with arbitrary $\varepsilon > 0$, as defined as for function $h(x)$ in the (f, g)-convolution. We have

$$h(x) + \delta(x) = \min_{y \in [0,x]} \{h(y) + \delta(y) + f(x-y)\} + g(x) + r(x) . \tag{11}$$

Since $d(x, \varepsilon) \in [0, x]$ we readily have

$$h(x) + \delta(x) \leq h(d(x,\varepsilon)) + \delta(d(x,\varepsilon)) + f(x - d(x,\varepsilon)) + g(x) + r(x) . \tag{12}$$

Using the definition of $d(x, \varepsilon)$ and the fact that $h(x)$ is stationary function of the (f, g)-convolution, we get

$$\delta(x) \leq \delta(d(x,\varepsilon)) + r(x) + \varepsilon . \tag{13}$$

Therefore for all integer $n \geq 1$ we have

$$\delta(x) \leq (1 - 2^{-n})\varepsilon + \delta(d^n(x,\varepsilon)) + \sum_{i=0}^{i=n-1} r(d^i(x,\varepsilon)) . \tag{14}$$

Let D be a real number such that $\forall x \geq D$: $r(x)/g(x) \leq \varepsilon$ and such that $\min_{y \geq D}\{g(y)\} > 0$ that is a direct consequence of $g(x) \sim g'(x)$ and $g(x) \to \infty$ when $x \to \infty$. Since $h(x)$ and $h'(x)$ are locally bounded, there exists $\Delta > 0$ such that $\sup_{x \leq D}\{h'(x)\} \leq \Delta$. For all $x \geq D$ let $N(x)$ be the first integer n such that $d^n(x, \varepsilon) \leq D$. We have in every case, when $x \geq D$

$$\delta(x) \leq \Delta + \sum_{i=0}^{i=N(x)-1} r(d^i(x,\varepsilon)) + \varepsilon , \tag{15}$$

that we can rewrite and extend to

$$\delta(x) \leq \Delta + \varepsilon + \varepsilon \sum_{i=0}^{i=N(x)-1} g(d^i(x,\varepsilon)) + f(d^{i+1}(x,\varepsilon) - d^i(x,\varepsilon)) . \tag{16}$$

Using equation (7) we know that

$$\sum_{i=1}^{i=n} f(d^i(x,\varepsilon) - d^{i-1}(x,\varepsilon)) + g(d^i(x,\varepsilon)) \leq h(x) - h(d^n(x,\varepsilon)) + \varepsilon .$$

Therefore for all $x \geq D$

$$\delta(x) \leq \varepsilon h(x) + \Delta + \varepsilon + \varepsilon^2 \tag{17}$$

which proves that $h'(x) \sim h(x)$ since ε can be made as small as possible and $h(x) \to \infty$ when $x \to \infty$. ∎

The following theorem finds similar proof.

Lemma 13 *Let $f(x) \to \infty$ and $g(x) \to \infty$ when $x \to \infty$. Let $f'(x)$ be a function following the same statement than $f(x)$ with additional condition $f'(x) \sim f(x)$ when $x \to \infty$. Let $h(x)$ and $h'(x)$ be function be respectively stationary functions of the (f,g)-convolution and of the (f',g)-convolution. We have $h'(x) \sim h(x)$ when $x \to \infty$.*

Proof. We follow the same proof as for lemma 12, except that we now set $f'(x) = f(x) + s(x)$ with $s(x) \geq 0$. We obtain companion inequality of (13):

$$\delta(x) \leq \delta(d(x,\varepsilon)) + s(x - d(x,\varepsilon)) + \varepsilon \ , \tag{18}$$

and we obtain

$$\delta(x) \leq \varepsilon + (d^n(x,\varepsilon)) + \sum_{i=0}^{i=n-1} s(d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)) \ . \tag{19}$$

Let E such that $\forall x \geq E$: $s(x)/f(x) \leq \varepsilon$ and let Ω be $\max_{x \leq E}\{f'(x)\}$. Let D be such that $\forall x \geq D$: $g(x) \geq \Omega\varepsilon^{-1}$ and let $N(x)$ and Δ having the same formal definitions as in the proof of lemma 12. For $x > D$ we now have companion inequality of (14):

$$\delta(x) \leq \Delta + \varepsilon + \sum_{i=0}^{N(x)-1} s\left(d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)\right) \ . \tag{20}$$

The term $d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)$ is either greater or smaller than E. Let $I(x)$ be the set of integer i smaller than $N(x)$ such that $d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon) \leq E$. Let $J(x)$ be the complementary set of integers i smaller than $N(x)$ such that $d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon) > E$. We can split summation in inequation (20) and obtain

$$\sum_{i=0}^{N(x)-1} s\left(d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)\right) = \sum_{i \in I(x)} + \sum_{i \in J(x)} \ . \tag{21}$$

It is clear that

$$\sum_{i\in J(x)} s\left(d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)\right) \leq$$

$$\varepsilon \sum_{i=0}^{i=N(x)-1} g(d^i(x,\varepsilon)) + f(d^{i+1}(x,\varepsilon) - d^i(x,\varepsilon)) \; .$$

Using arguement as in proof of lemma 12 we get

$$\sum_{i\in J(x)} s\left(d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)\right) \leq \varepsilon h(x) + \varepsilon^2 \; . \tag{22}$$

About the second part of the split summation it is also clear that

$$\sum_{i\in I(x)} s\left(d^i(x,\varepsilon) - d^{i+1}(x,\varepsilon)\right) \leq N(x)\Omega \; .$$

Calling to lemma 7 and the exact definition of D, it appears that $N(x) \leq (h(x)+\varepsilon)\varepsilon\Omega^{-1}$. Therefore we readily obtain

$$\delta(x) \leq \Delta + \varepsilon + 2(\varepsilon h(x) + \varepsilon^2) \; . \tag{23}$$

which allows us to conclude, since ε can be made as small as possible. ∎

In order to conclude about theorem 1 we have to enter a technical lemma. The difficult point about these (f,g)-convolution is that it is generally very difficult to derive function $h(x)$ from function $f(x)$ and $g(x)$. Conversely, given function $f(x)$ and $h(x)$, it is very easy to derive function $g(x)$ such that function $h(x)$ is stationary function of the (f,g)-convolution.
Indeed

$$g(x) = h(x) - f \star h(x) \; . \tag{24}$$

Lemma 14 *Let $f(x) = Ax^\alpha$ and $h(x) = Cx^\gamma$ such that $\alpha > \gamma > 1$, function $g(x)$ defined by (24) satisfies $g(x) \sim Bx^\beta$, such that $\beta > 0$ and $\gamma = \beta + 1 - \beta/\alpha$ and $C = \frac{(A\alpha)^{1/\alpha}}{\gamma}(B\frac{(\alpha-1)}{\alpha})^{(\alpha-1)/\alpha}$.*

Proof. Since the proof is technically classic and far from the techniques developed in this section, we defer it to appendix.

3 One eccentric case

One eccentric case consists in considering $f(x)$ as an *improper* function. We say that function $f(x)$ is an improper function when there exists $D > 0$ such that $\forall x \geq D$: $f(x) = +\infty$, and $f(x)$ is finite for all $x < D$. We call the real number D the *edge* of function $f(x)$. We suppose $f(0) = 0$ and additional conditions such as local boundness on right open interval $[0, D[$, can be assumed in order to match conditions listed in the first paragraph of the introduction. We consider function $g(x)$ as a proper function. It is easy to parallelize analysis of section 2. Without difficulty, lemmas, theorems remain true up to lemma 12. One needs to be more careful about lemma 13. First we have to consider one special class of improper functions, namely the class of bounded improper functions. An improper function $f(x)$ with edge D is said to be bounded if $f(x)$ is uniformly bounded on $[0, D[$. Little reflection allow us to derive the following theorem.

Theorem 15 *Let $f(x)$ and $f'(x)$ be two bounded improper functions with the same edge. Let $g(x)$ such that $g(x \to \infty$ when $x \to \infty$. Let $h(x)$ and $h'(x)$ be function be respectively stationary function of the (f, g)-convolution and of the (f', g)-convolution. We have $h'(x) \sim h(x)$ when $x \to \infty$.*

In the same order of idea we have the companion lemma of lemma 14.

Lemma 16 *Let $f(x) = 0$ if $x < D$ and $f(x) = +\infty$ when $x \geq D$. Let $g(x)$ be an increasing upper semicontinuous function (always with $g(0) = 0$). Let $h(x)$ be the stationary function of the (f, g)-convolution, we have*

$$h(x) = g(x) + g(x - D) + \cdots + g(x - nD) \ , \tag{25}$$

with n the greatest integer such that $x - nD \geq 0$.
Proof It is clear that $h(x) = g(x)$ when $x < D$ by lemma 5. It is also clear that $h(x)$ is an increasing upper semicontinuous function. Therefore $h(x)$ is right semicontinuous and

$$\min_{y \in [0,x]} \{h(y) + f(x - y)\} = h(x - D) \ . \tag{26}$$

Therefore $h(x) = g(x) + h(x - D)$. ∎
If we apply the previous lemma to $g(x) = Bx^\beta$ the following corollary holds.

Corollary 17 *Let $f(x)$ be a bounded improper function with edge D, and let $g(x)$ such that $g(x) \sim Bx^\beta$, then $h(x) \sim \frac{B}{(\beta+1)D} x^{\beta+1}$.*

The next theorem comes very easily

Theorem 18 *Let $f(x)$ be simply improper function with edge D, and let $g(x)$ such that $g(x) \sim Bx^{\beta}$, therefore $h(x) \sim \frac{B}{(\beta+1)D}x^{\beta+1}$.*

Proof. From lemma 16 it is clear that there exists function $h'(x) \sim B/Dx^{\beta+1}$ such that $h \geq h'$. It suffices to take $h'(x)$ as the stationary function of the (f', g)-convolution with $f'(x)$, a bounded improper function with edge D such that $f' \leq f$. If we take $f''(x)$ as the improper function of edge $D/(1+\varepsilon)$ identical to $f(x)$ when $x < D/(1+\varepsilon)$, it is clear that $f''(x)$ is a bounded improper function (because $f(x)$ is locally bounded) which dominates $f(x)$. Let $h''(x)$ be the stationary function of the (f', g)-convolution, we have $h'' \geq h$ and $h''(x) \sim \frac{(1+\varepsilon)B}{(\beta+1)D}x^{\beta+1}$. ∎

Reference

[1] P. JACQUET "Tail distributions in LaPalice queueings", in preparation.

APPENDIX

Proof of lemma 14. Our purpose is to develop an asymptotic expansion of $\phi(x) = \min_{y\in[0,x]}\{Cy^\gamma + A(x-y)^\alpha\}$. We can rewrite

$$\phi(x) = \min_{t\in[0,1]}\{Cx^\gamma(1-t)^\gamma + Ax^\alpha t^\alpha\} \tag{27}$$

The minimum is reached at t_0 which satisfies the zero gradient condition, namely

$$\gamma C x^\gamma (1-t_0)^{\gamma-1} = \alpha A x^\alpha t^{\alpha-1} \ . \tag{28}$$

We have from this expression the following identity

$$t_0 = \left(\frac{\gamma C}{\alpha A} x^{\gamma-\alpha}(1-t_0)^\gamma\right)^{1/(\alpha-1)} , \tag{29}$$

and the obvious inequality

$$t_0 \le t_0^* = \left(\frac{\gamma C}{\alpha A} x^{\gamma-\alpha}\right)^{1/(\alpha-1)} . \tag{30}$$

From (29) we obtain

$$\phi(x) = Cx^\gamma(1-t_0)^\gamma + Ax^\alpha\left(\frac{\gamma C}{\alpha A} x^{\gamma-\alpha}(1-t_0)^\gamma\right)^{\alpha/(\alpha-1)} ,$$

therefore, using (30)

$$\phi(x) \ge Cx^\gamma(1-t_0^*)^\gamma + Ax^\alpha\left(\frac{\gamma C}{\alpha A} x^{\gamma-\alpha}(1-t_0^*)^\gamma\right)^{\alpha/(\alpha-1)} . \tag{31}$$

Easy algebra, using elementary tools as $(1-a)^\mu \ge 1-\mu a$ when $\mu \ge 0$, leads to the estimate

$$\phi(x) \ge Cx^\gamma - Bx^\beta - \frac{\alpha}{\alpha-1} A \left(\frac{\gamma C}{\alpha A}\right)^{\frac{\alpha+1}{\alpha-1}} x^{\beta-\frac{(\alpha-\gamma)}{\alpha-1}} . \tag{32}$$

An upper bound is given by

$$\phi(x) \le Cx^\gamma(1-t_0^*)^\gamma + Ax^\alpha(t_0^*)^\alpha \tag{33}$$

which gives the following estimate, arguing with $(1-a)^\mu \le 1-\mu a + (\mu-1)a^2/2$,

$$\phi(x) \le Cx^\gamma - Bx^\beta + \frac{\gamma-1}{2} C \left(\frac{\gamma C}{\alpha A}\right)^{\frac{2}{\alpha-1}} x^{\beta-\frac{(\alpha-\gamma)}{\alpha-1}} . \tag{34}$$

Inequalities (32) and (34) are sufficient to conclude that $g(x) = h(x) - \phi(x) \sim Bx^\beta$ when $x \to \infty$. ∎

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