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Fonctions de Lyapounov pour les réseaux de Jackson

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Résumé

Nous proposons une construction explicite de fonctions de Lyapounov pour les réseaux de Jackson markoviens. On obtient directement deux corollaires : d'abord une preuve des conditions nécessaires et suffisantes d'ergodicité, sans utiliser la fameuse forme produit ; ensuite, une convergence exponentielle vers la distribution d'équilibre. Nous considérons aussi de petites perturbations des probabilités de transition (conduisant donc à des réseaux qui ne sont pas de Jackson) et prouvons que la distribution stationnaire est une fonction analytique de ces perturbations.

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Abstract

We construct explicitly Lyapounov functions for markovian Jackson networks. Two direct corollaries are obtained : first a proof of the necessary and sufficient conditions for ergodicity, without using the famous Jackson's product form ; secondly, an exponential convergence rate to the stationary distribution. We also consider small perturbations of the transition probabilities (yielding thus non Jackson networks) and prove that the corresponding stationary distribution is an analytic function of these perturbations.

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INTRODUCTION

Jackson networks are now classical models for communication networks. Jackson [1] obtained the famous product form for their stationary probabilities. Sufficient ergodicity conditions follow from this product form. Proof of necessity of these conditions was obtained by many authors [2, 3, 4]. From the general theory of countable Markov chains, it follows that the n-step transition probabilities converge to stationary probabilities when $n \to \infty$. It was not much known about the rate of this convergence. The only result we know is the proof of exponential convergence under some smallness assumption in [5].

Here we consider Markovian Jackson networks, i.e. with Poisson arrivals and exponential service times. In this case, they are equivalent to a class of random walks in $\mathbf{Z}_+^N$, where N is the number of nodes in a network. The main result of this paper is that we explicitely construct Lyapounov functions. They are either almost linear in the terminology of [6], or just piecewise linear.

Using the results of [7], we get the corollary : exponential convergence for any ergodic Jackson network. Also we define a class of networks which are not of Jackson's type, but are small perturbations of Jackson networks. The examples are small dependence between nodes, simultaneous arrivals etc. We prove that stationary probabilities (which cannot be given by a product-form) depend analytically on the parameters.

Let us note that we never use Jackson's product form in the proofs.

1 Ergodicity conditions for Jackson networks

Here we recall some well-known facts and prove a useful geometric lemma. We consider an open Jackson network with N nodes, let $\xi^i(t)$ be the length of the queue at the i-th node at time t. We restrict ourselves here to the simplest assumptions : independent Poisson inputs with parameter $\lambda_i > 0$ for any node i, exponential service times with parameters $\mu_i > 0$. After a customer being served at the i-th node, he is immediately transferred with probability p_{ij} to the end of the queue at node j, $j = 1, ..., N$ and, with the probability

$$p_{i0} = 1 - \sum_{j=1}^{n} p_{ij} \ ,$$

he leaves the network. It is convenient (but not necessary) to assume that $p_{ii} = 0$, for all i.
In other words, we consider a continuous time random walk $\tilde{L}$ on $\mathbf{Z}_+^N$ with transition intensities $\lambda_{\alpha\beta}$, from the state $\alpha = (\alpha^1, ..., \alpha^N)$ to the state $\beta = (\beta^1, ..., \beta^N)$,

$$\lambda_{\alpha\beta} = \begin{cases} \mu_{0i} \doteq \lambda_i \ , & \text{if } \beta - \alpha = e_i, \\ \mu_{i0} \doteq \mu_i \ p_{i0} \ , & \text{if } \beta - \alpha = -e_i \\ \mu_{ij} \doteq \mu_i \ p_{ij} \ , & \text{if } \beta - \alpha = -e_i + e_j, \\ & 1 \leq i, j \leq N \ . \end{cases} \tag{1.1}$$

Here e_i is the vector $(0, ..., 0, 1, 0, ..., 0)$, with i-th coordinate equal to 1. It is convenient to denote the zero vector by e_0. We recall now Jackson's equations. Assuming a stationary regime, we denote by ν_j "the mean number of customers" coming to the node j from the outside and from the other nodes during a unit time interval. Using the law of large numbers, Jackson wrote the following system of equations (we call it Jackson's system)

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \nu_i \ p_{ij} \ \ , \ \ j = 1, ..., N \ . \tag{1.2}$$

Let us note that these equations can be solved by the iteration scheme

$$\nu_j = \lambda_j + \sum_{k=1}^{\infty} \sum_{i=1}^{N} \lambda_i \ p_{ij}^{(k)} \ , \tag{1.3}$$

where

$$\| \ p_{ij}^{(k)} \ \| = \mathbf{P}^k \ , \ \mathbf{P} \doteq \| \ p_{ij} \ \|_{i,j=0,1,\dots,N}$$

and we put $p_{0i} \equiv 0, \ i \neq 0, \ p_{00} = 1$.
The series in the right-hand side of (1.3) converges if

$$p_{ij}^{(k)} \leq C(1-\epsilon)^k \tag{1.4}$$

for some $\epsilon > 0, \ C > 0$. For this it is necessary and sufficient to assume that

(A) *Starting from any state we reach 0 with positive probability (a.s.) in the Markov chain with $N+1$ states $0, 1, ..., N$, defined by the stochastic matrix* $\mathbf{P}$.
So we can rewrite (1.3) as follows

$$\nu_j = \lambda_j + \sum_{i=1}^{N} \lambda_i \, m_{ij}^0$$

where m_{ij}^0 is the mean number of hitting j starting from i in this finite-state Markov chain.
So we see that the solution of (1.2) is unique.

Theorem 1.1 *(Jackson). The network is ergodic iff*

$$\nu_j < \mu_j \;\; for \; all \;\; j = 1, ..., N \, .$$

Below we give a new proof of this theorem in the geometrical setting which will be useful in the following sections.
Later on we use different results for discrete time Markov chains from [7]. We note that all of them could be easily rewritten for continuous time case. But we avoid this rewritting by introducing the following discrete time random walk L in $\mathbf{Z}_+^N$. We choose its transition probabilities as

$$p_{\alpha\beta} = w_\alpha \, \lambda_{\alpha\beta} \tag{1.5}$$

for some

$$0 < w_\alpha \leq (\sum_\beta \lambda_{\alpha\beta})^{-1}$$

E.g. if we choose

$$w_\alpha = (\sum_\beta \lambda_{\alpha\beta})^{-1} \, ,$$

we get the imbedded chain. It is more convenient to choose

$$w_\alpha \equiv w \leq \min_\alpha \, (\sum_\beta \lambda_{\alpha\beta})^{-1} \tag{1.6}$$

Stationary probabilities $\tilde{\pi}_\alpha$ of $\tilde{L}_\alpha$ and those π_α of L are connected as

$$\pi_\alpha = w\tilde{\pi}_\alpha \tag{1.7}$$

and so $\tilde{L}$ is ergodic iff L is ergodic,

We want to recall now some definitions from [7].

We consider a discrete time homogeneous Markov chain L which is assumed to be irreducible and aperiodic unless otherwise stated. The set of states is $\mathbf{Z}_+^N = \{(z_1, ..., z_N) : z_i \geq 0 \ are \ integers\}$, $p_{\alpha\beta}^k$ be k-step transition probabilities on L, $M^k(\alpha) = (M_1^k(\alpha), ..., M_N^k(\alpha))$ be the vector of the mean jump from the point α in k steps ; $p_{\alpha\beta}^1 = p_{\alpha\beta}$, $M^1(\alpha) = M(\alpha)$.

For any $\bigwedge \subseteq \{1, 2, ..., N\}$, we define the face $B^\wedge$ of $R_+^N = \{(r_1, ..., r_N) : r_i \geq 0 \ real\}$ by $B^\wedge = \{(r_1, ..., r_N) : r_i > 0, \ i \in \bigwedge; \ r_i = 0, \ i \bar{\in} \bigwedge\}$.

It is sufficient for us here to consider r.w. with the following conditions *boundedness of jumps* :

$$p_{\alpha\beta} = 0 \ , \ for \ \ \| \alpha - \beta \| > 1 \ ,$$

where

$$\| \alpha \| = \max_i |\alpha_i| \ \ , \ \ \alpha = (\alpha_1, ..., \alpha_N) \ ,$$

which is a stronger condition than in [7], the *homogeneity condition* we use is also stronger : for any $\bigwedge$ and for any $a \in B^\wedge \cap \mathbf{Z}_+^N$,

$$p_{\alpha\beta} = p_{\alpha+a,\beta+a} \ ,$$

for all $\alpha \in B^\wedge \cap \mathbf{Z}_+^N$, $\beta \in \mathbf{Z}_+^N$.

We define the *first* vector field on $\mathbf{R}_+^N$ to be constant on any $B^\wedge$ and equal to

$$M_\Lambda \equiv M(\alpha) \ , \ \alpha \in \wedge \ .$$

For the Markov chain L we have the crucial property

$$M_\Lambda = f_0 + \sum_{i \in \wedge} f_i \ . \tag{1.8}$$

where

$$f_i = w \sum_{j=0}^{N} \mu_{ij}(-e_i + e_j) \,. \tag{1.9}$$

So f_i is the contribution of the transition from i-th node (including the virtual 0-node).
It is clear that the 2^N mean jump vectors M_Λ are the vertices of the parallelepiped which we denote by $\prod$. Its initial point can be taken f_0 and the edges drawn from this point are $f_1, ..., f_N$. This parallelepiped can be degenerate if the vectors $f_1, ..., f_N$ are linearly dependent. We shall use below the following combinatorial criterion of ergodicity equivalent to Jackson's one.

Lemma 1.2 *Jackson's network is ergodic iff M is not degenerate and the point $0 \in R^N$ is one of its internal points. Moreover, if the origin does not belong to M then this chain is transient.*

Proof : Let us consider the following system of equations, w.r.t. $\epsilon_1, ..., \epsilon_N$,

$$f_0 + \epsilon_1 \, f_1 + ... + \epsilon_N \, f_N = 0 \tag{1.10}$$

Note that $\prod$ is not degenerate iff this system is not degenerate. In this case the system (1.10) has a unique solution and 0 is an internal point of $\prod$ iff $0 < \epsilon_i < 1$ for $i = 1, ..., N$.
Inserting (1.1.), (1.9) into (1.10) we get

$$0 = f_0 + \sum_{j=1}^{N} \epsilon_j \, f_j = \sum_{j=1}^{N} \lambda_j \, e_j + \sum_{i=1}^{N} \epsilon_i \sum_{j=0}^{N} \mu_i \, p_{ij}(-e_i + f_j) =$$

$$= \sum_{j=1}^{N} \lambda_j \, e_j + \sum_{i=1}^{N} \epsilon_i \, \mu_i(-e_i + \sum_{j=0}^{N} p_{ij} \, e_j) =$$

$$= \sum_{j=1}^{N} \lambda_j \, e_j - \sum_{j=1}^{N} \epsilon_j \, \mu_j \, e_j + \sum_{i=1}^{N} \epsilon_i \, \mu_i \sum_{j=0}^{N} p_{ij} \, e_j$$

$$= \sum_{j=1}^{N} (\lambda_j - \epsilon_j \, \mu_j + \sum_{i=1}^{N} \epsilon_i \, \mu_i \, p_{ij}) e_j$$

which coincides with (1.2) for $\epsilon_i \equiv \rho_i = \frac{\nu_i}{\mu_i}$.
So when 0 is an internal point of nondegenerate $\prod$ ergodicity follows from Jackson's explicit formulae for stationary probabilities (but in the next sections we prove it without using Jackson's results). Let now 0 lies on the boundary of $\prod$ (this includes the case of degenerate $\prod$ when $\prod$ coincides with its boundary). Then there exists a hyperplane $\mathcal{L}$ of dimension $N-1$ in R^N such that $0 \in \mathcal{L}$ and $\prod$ belongs to the closure of one of the two half-spaces defined by $\mathcal{L}$. But this means that we can construct a linear Lyapounov function f such that

$$\sum p_{\alpha\beta} f(\beta) - f(\alpha) \geq 0 \ , \ f(\alpha) > 0 \text{ for infinite number of } \alpha \in \mathbf{Z}_+^N .$$

Using boundedness of jumps we get nonergodicity by [7], see also Theorem 2.3, [8]. The same is for transience when 0 lies strictly outside $\prod$.

2 Main results

Theorem 2.1 *Let $\tilde{L}$ be a Jackson network such that 0 lies inside $\prod$ and $\tilde{p}_{\alpha\beta}^{(t)}$ its time-t transition probabilities. Then there exist constants $C(\alpha) > 0$ and $\chi > 0$, such that, for any α, β, t,*

$$|\tilde{\pi}_\alpha - \tilde{p}_{\alpha\beta}^{(t)}| < C(\alpha) e^{-\chi t} .$$

Let us now fix some Jackson network with $\lambda_{\alpha\beta}$. In addition to the "jacksonian" jumps of this network, we allow any jumps $\alpha \to \beta$ satisfying boundedness and homogeneity conditions, but the intensities $\nu_{\alpha\beta}$ of these additional jumps are *small.*

Theorem 2.2 *If a fixed Jackson network with intensities $\lambda_{\alpha\beta}$ is such that 0 lies inside $\prod$, then there exists $\nu_0 > 0$ such that, for*

$$\nu_{\alpha\beta} < \nu_0 ,$$

the resulting (non jacksonian) network has stationary probabilities analytically depending on $\nu_{\alpha\beta}$. Moreover this analytic family is a Lyapounov analytic family (see section 4).

Remark 1 *In particular one can expand stationary probabilities π_α as a convergent series in $\nu_{\alpha\beta}$. For such perturbed Markov chains, we also have an exponential convergence to the stationary state.*

Remark 2 *If 0 lies inside $\prod$ ergodicity follows from Theorems 2.1 and 2.2. So we proved ergodicity without using Jackson's product form.*

3 Geometric construction

Let us recall that $\prod$ is a convex hull of the points M_Λ (the ends of the vectors M_Λ with the initial points at 0).
Let a be a fixed point of R^n. Let us put

$$\Gamma = \Gamma^a = \{a + \sum_{i=1}^{n} \beta_i \, f_i : \beta_i \geq 0\}$$

So it is a multidimensional corner (with the vertex a) generated by the vectors f_i. It is convenient to put

$$\Gamma_\Lambda = \{a + \sum_{i \in \Lambda} \beta_i \, f_i : \beta_i \geq 0\} \ , \bigwedge \subset \{1, ..., n\}$$

So

$$\Gamma = \Gamma_{\{1,...,n\}}$$

and we define the surface $\tilde{\Gamma}$ of Γ :

$$\tilde{\Gamma} = \bigcup_{\Lambda \neq \{1,...,n\}} \Gamma_\Lambda$$

Scaling : Let us denote $\alpha\Gamma$, $\alpha\tilde{\Gamma}$, $\alpha\Gamma_\Lambda$, $\alpha \geq 1$, the scaled geometrical objects respectively, with vertex αa.

Lemma 3.1 *Assuming that 0 lies inside $\prod$ we have that*

$$R^n_+ \cap (\alpha\Gamma)$$

is a compact set for any $\alpha \geq 1$

Proof : Let us first note that if it is compact for some a then it is compact for any a. So we can choose a in a convenient way, e.g. to put

$$a = f_0$$

Let us note then that the ray $f_0 + \beta_i\, f_i$, $0 \leq \beta_i < \infty$, intersects the face $x_i = 0$ of R_+^N, as f_i has e_i-component negative and the other positive. From this compactness is readily seen. E.g. we could expand

$$f_0 + \sum_{i=1}^{N} \beta_i\, f_i = \sum_{i=1}^{N} C_i\, e_i \tag{3.1}$$

with

$$C_j = const\ + \sum_{i:i\neq 0,j} \alpha_i\, p_{ij} - \alpha_j\ ,$$

$$\alpha_j = \mu_j\, \beta_j$$

Considering a ray $\alpha_i = tr_i$, $r_i \geq 0$, $t \geq 0$, we see that its intersection with R_+^N is an interval of finite length as

$$\sum_{j=1}^{N} C_j = const\ + t[\sum r_i(1 - p_{i0}) - \sum r_i]$$

and the coefficient is negative. ■

Let us consider some Γ_Λ with $|\Lambda| = N - 1$. This is an hyperplane (of dimension $N-1$) in R^N and it subdivides R^N onto 2 halfspaces $\Gamma_\Lambda^+, \Gamma_\Lambda^-$. We denote Γ_Λ^+ that one which contains Γ.

Lemma 3.2 *Under the conditions of lemma 3.1 let us consider a hyperplane Γ_Λ with $|\Lambda| = N - 1$. Then any vector $M_{\Lambda'}$ with*

$$\Lambda' \not\subset \Lambda \tag{3.2}$$

and with initial point on Γ_Λ, lies in Γ_Λ^+.

Proof : Let us first show that $M_{\{1,\dots,N\}}$ has this property for all $\Gamma_\Lambda, |\Lambda| = N - 1$.

For this let us choose $a = -M_{\{1,\ldots,N\}}$. Let us note that $0 \in \prod$ iff $0 \in (-\prod)$. Then $M_{\{1,\ldots,N\}}$ with initial point a (which lies on all Γ_Λ simultaneously) has 0 as its final point and so belongs to all Γ^+_Λ.
Let us take now e.g. $\bigwedge = \{2, \ldots, N\}$ and any $\bigwedge'$ such that $1 \in \bigwedge'$. Now let us take again $a = -M_{\{1,\ldots,N\}} = -f_0 - \sum_{i\in\Lambda'} f_i - \sum_{i\neq 0, i\bar{\in}\Lambda'} f_i$. Then the point

$$b = a + \sum_{i\neq 0, i\bar{\in}\Lambda'} f_i \ ,$$

belongs to Γ_Λ and $b + M_{\Lambda'} = 0 \in \Gamma^+_\Lambda$. ■

Lemma 3.3 *If a lies strictly inside R^N_+ then Γ_Λ , $\mid \Lambda \mid = N - 1$, has the property :*

$$\Gamma_\Lambda \cap \overline{B^{\Lambda'}} = \emptyset$$

for

$$\Lambda' \subset \Lambda \ .$$

Proof : Let again $\bigwedge = \{2, \ldots, N\}$, then

$$\Gamma_\Lambda = \{a + \sum_{j=2}^{N} \beta_j \ f_j\} \ .$$

But if we expand

$$a + \sum_{j=2}^{N} \beta_j \ f_j = \sum_{i=1}^{N} C_i \ e_i \ ,$$

then C_1 is strictly positive. So it cannot belong to $\overline{B^{\Lambda'}}$ where the first coordinate is zero. ■
Let us consider the following function on R^N_+ :

$$f_x = \alpha \quad if \quad x \in \alpha\, \tilde{\Gamma}^a \tag{3.3}$$

This "piecewise linear" function obtained by scaling is our main Lyapounov function as we show in the section 5.

4 Analytic Lyapounov families

Here we give a compact reformulation of some results in [7] which are necessary to prove the main results. Let us consider a family of Markov chains $\{L^\nu\}$, $\nu \in \mathcal{D}$ which is an interval of the real axis containing 0, with the same state space S. The matrix $P_\nu = (p_{ij}(1,\nu))_{i,j\in S}$ of transition probabilities can be considered as a bounded linear operator in the Banach space $l_1(S)$. Let us assume that P_ν is analytic in ν as a function in $\mathcal{D}$ with values in the Banach algebra of bounded operators in $l_1(S)$. This means that P_ν can be Taylor expanded as

$$P_\nu = \sum_{n=0}^{\infty} P_n \, \nu^n \tag{4.1}$$

where P_n are bounded linear operators with

$$\| P_n \| \le C a^n \tag{4.2}$$

for some C, $a > 0$, i.e. the series is convergent for $|\nu|$ sufficiently small. Under these conditions we say that we have an analytic family of Markov chains.

Definition 1 *We say that this family is an analytic Lyapounov family if in addition also the following conditions are assumed : there exist nonnegative functions f_i, $i \in S$, $\nu \in \mathcal{D}$, on S and positive integer valued functions k_i^ν such that*

(i) $\displaystyle\sup_{i\in S,\nu\in\mathcal{D}} k_i^\nu = b < \infty$;

(ii) the series

$$\sum_{i\in S} exp\,(-b_1\, f_i^\nu) \, ,$$

for any $b_1 > 0$ converges uniformly in $\nu \in \mathcal{D}$;

(iii) there exist $d > 0$ such that

$$p_{ij}(1,\nu) = 0 \text{ } for\ all\ \nu \in \mathcal{D}\ when\ ever\ |f_i^\nu - f_j^\nu| > d \, ,$$

(iv) there exist $k > 0$ and $\delta > 0$ such that for any $i \in S$ and any

$$j \in V_i \stackrel{\text{def}}{=} \{j : \sup_{\nu\in\mathcal{D}} p_{ji}(1,\nu) > 0\} \, ,$$

$$p_{ji}(n,0) > \delta \ ,$$

where $p_{ji}\ (n,\nu)$ are n-step transition functions for the Markov chain L^{ν} ;

(v) for all $\nu \in \mathcal{D}$, $i \in S - B$ where B is a finite subset of S and some $\epsilon > 0$

$$\sum_{j \in S} p_{ij}(k_i^{\nu}, \nu) - f_i^{\nu} < -\epsilon \ .$$

By Foster's criterion the L_{ν} is ergodic for any $\nu \in \mathcal{D}$.
We say that a Markov chain $L = L_0$ is analytic Lyapounov Markov chain if the family $L_{\nu} \equiv L_0$ is analytic Lyapounov.

Theorem 4.1 *If L_{ν} is an analytic Lyapounov family then there exists $\nu_0 > 0$ such that*

1. *there exist C_2, $\delta_2 > 0$ such that*

$$\pi_i(\nu) < C_2 \ exp\ (-\delta_2\ f_i^{\nu}) \tag{4.3}$$

   for all $i \in S$, $\nu \in \mathcal{D}$;

2. *there exist constants σ_2, C_3, $\delta_3 > 0$ such that*

$$\sum_{j \in S} |p_{ij}(n,\nu) - \pi_j\ (\nu)| < C_3\ exp\ (-\delta_3 n) \ ,$$

   for all $\nu \in \mathcal{D}$, $i \in S$, $n > \sigma_2\ f_i^{\nu}$.

3. *stationary probabilities $\pi_i(\nu)$ are analytic in ν for $|\nu| < \nu_0$ for all $i \in S$.*

5 Lyapounov functions

We shall give two constructions. Both are useful for future generalisations.

1^{st} construction
The first one uses smoothing (in fact, the principle of almost linearity [6]), and $k_i \equiv 1$ for the first construction.

Lemma 5.1 *For any $\epsilon > 0$ there exists a smooth convex closed hypersurface (homeomorphic to the boundary $\partial\Pi$ of Π) $\partial\Pi(\epsilon)$ such that for any $x \in \partial\Pi(\epsilon)$*

$$\rho(x, \partial\Pi) < \epsilon ,$$

and, for any $y \in \partial\Pi$,

$$\rho(y, \partial\Pi(\epsilon)) < \epsilon .$$

For a proof it is sufficient to consider the unit cube in R^N and after to use a linear transformation. For the unit cube one can use induction constructing on each step a cylinder smoothed at the ends.

Let us take the intersection of $\partial\Pi(\epsilon)$ with a neighbourhood of the vertex $a = f_0$. We prolongate it by linearity to get a hypersurface $\tilde{\Gamma}(\epsilon)$ (smooth convex) such that the pairs

$$(\tilde{\Gamma} \cap R^N_+,\ \tilde{\Gamma} \cap \partial(R^N_+)) \ and$$

$$(\tilde{\Gamma}(\epsilon) \cap R^N_+,\ \tilde{\Gamma} \cap \partial(R^N_+)) \ are\ homeomorphic$$

and

$$\rho(x, \tilde{\Gamma}(\epsilon)) < \epsilon \ \ for\ any\ x \in \tilde{\Gamma} \cap R^N_+ ,$$

$$\rho(y, \tilde{\Gamma}) < \epsilon \ \ for\ any\ y \in \tilde{\Gamma}(\epsilon) \cap R^N_+ .$$

Then we use (as in [6]) scaling to define a Lyapounov function

$$f_x = \alpha \ , \ \ x \in \alpha\tilde{\Gamma}(\epsilon) \tag{5.1}$$

After this the principle of almost linearity gives us the following result

Theorem 5.2 *With the Lyapounov function (5.1) and with $k_x \equiv 1$ an ergodic Jackson network is an analytic Lyapounov Markov chain.*

2^{nd} construction

In this construction we use the Lyapounov function (3.3) with $k_x \equiv k$ sufficiently large.

Theorem 5.3 *Let us consider Jackson network such that $0 \in \prod$ and choose the function f_x as in (3.3) with the point a lying inside R^N_+. Then for this Lyapounov function Jackson network is an analytic Lyapounov Markov chain.*

Proof : We must only prove that there exist k_i satisfying (i) such that (v) is true. Let us fix a constant d - maximal length of a jump in some metrics ρ on R^N. In our case $d = 1$ in the metrix

$$\rho(x, y) = \max_i |x_i - y_i|$$

where e.g. $x = (x^1, ..., x^N)$

Lemma 5.4 *Let us fix $\epsilon > 0$ sufficiently small. There exists $\rho_0 > 0$ such that for any $x \in \mathbf{Z}_+^N$ with*

$$\rho = \rho(x, \partial R_+^N) > \rho_0 ,$$

for any k such that

$$\rho_0 < kd \equiv k \leq \rho ,$$

we have

$$\sum_y p_{xy}^{(k)} f_y - f_x < -\epsilon \tag{5.2}$$

Proof: For any $\Lambda, |\Lambda| = N-1$, let us consider a new function $f_x^\Lambda = \alpha$, *if* x belongs to the hyperplane R_Λ generated by $\alpha\Gamma_\Lambda$. Let us consider our random walk starting from x :

$$x = \xi_0,\ \xi_1, ..., \xi_{k-1} ,$$

during k steps. Then

$$f_{\xi_0}^\Lambda,\ f_{\xi_1}^\Lambda, ..., f_{\xi_{k-1}}^\Lambda ,$$

is a supermartingale satisfying

$$M(f_{\xi_i}^\Lambda / f_0^\Lambda, ..., f_{i-1}^\Lambda) < -\epsilon',\ \epsilon' > 0 ,$$

as $M_{\{1,...,N\}}$ is directed to the corresponding side of the hyperplane R_Λ. So by lemma 1.1 of [7] we can find constants $C_\Lambda,\ \delta_\Lambda,\ \epsilon_\Lambda > 0$ such that

$$f_y^\Lambda - f_x^\Lambda < -\epsilon_\Lambda k \tag{5.3}$$

with probability $1 - C_\Lambda\, e^{-\delta_\Lambda k}$.

So if we take ρ_0 sufficiently large (5.3) takes place for *all* Λ with probability

$$1 - C\, e^{-\delta k} ,$$

for some constants $C,\ \delta > 0$. Using the boundedness of jumps and the fact that our function f grows linearly with α we immediately have (5.2) for some $\epsilon > 0$. ■

Lemma 5.5 *Again choose $\epsilon > 0$ sufficiently small and $i = 1, ..., n$. Then there exist $\rho_i > 0$ such that for any $x \in \mathbf{Z}_+^N$ with*

$$\rho = \max_{\Lambda : i \notin \Lambda} \rho(x, B^\Lambda) > \rho_i \tag{5.4}$$

for any k such that

$$\rho_i < kd \equiv k \leq \rho$$

we have

$$\sum_y p_{xy}^{(k)} f_y - f_x < -\epsilon \tag{5.5}$$

Proof : We repeat the proof of lemma 5.3, using our geometrical construction : mean jump vectors look at the necessary direction from any point which the random walk, starting from x, can visit eventually during $\rho - 1$ steps.

To prove the theorem let us put

$$\tilde{\rho} = \max_{0 \leq i \leq N} \rho_i \ .$$

Then for any point x outside $(\tilde{\rho}+1)$ neighbor hood of the origin in our special metrics, we put

$$k_x \equiv \tilde{\rho}\ ,$$

and note that, for any such point there exists i, such that for $\Lambda = \{1, ..., N\} - \{i\}$ (5.4) holds. ■

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