An algorithm for the forward kinematics of general 6 D.O.F. parallel manipulators

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Un algorithme pour la cinématique directe des manipulateurs parallèles généraux à 6 degrés de liberté

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Résumé

La cinématique directe des robots parallèles a été étudiée pour les architectures mécaniques dans lesquelles le manipulateur présente, en dehors de la base et du plateau mobile, des faces planes. Dans ce cas on sait que l'on peut se ramener, en général, à la résolution d'un polynôme mono-variable de degré 16. Mais la technique employée pour obtenir ce résultat ne peut se généraliser aux manipulateurs sans faces planes. Aucun résultat sur ce sujet n'existe dans la littérature que ce soit pour obtenir l'ensemble des solutions possibles ou concernant le nombre maximum de solutions. Nous proposons un algorithme permettant de trouver l'ensemble des configurations solution et montrons que le nombre de solutions est majorée par 1320. Enfin nous exhibons une configurations à 12 solutions.

Abstract

Forward kinematics has been studied mainly for parallel manipulators with planar faces. In this case the size of the set of equations from the inverse kinematics can be reduced from 6 to 3 and this last set can be combined into a polynomial in one variable of order 16. But this method cannot be extented to general parallel manipulators without planar faces for which there is no known results. We present here an algorithm for the forward kinematic of general manipulators and we show that the number of solutions is bounded by 1320. We present a configuration with twelve solutions.

1 Introduction

Parallel manipulators present a great interest for many industrial tasks due to their high positionning ability and high nominal load. Many applications has been presented in the past either for flight simulator [13] or as robotic devices [4], [3], for example with force-feedback control [12], [8], [9]. This kind of applications uses both inverse kinematics (which is in general straightforward) but also forward kinematics. The later is known to be a difficult problem from a long time [1]. The problem of the direct kinematics has been addressed for manipulators where the mobile plate is a triangle and each of the three articulation points on the vertices of the mobile plate are shared by two links. In this case Hunt [5] conjectured that there will be at most sixteen solutions and this conjecture has been proved geometrically in [10]. Then it has been noticed that for a fixed set of links lengths each articulation point of the mobile plate lie on a circle which center and radius may be determined through the links lengths. Thus its position is fully defined by one angle. By expressing the position of the three articulation points of the mobile as a function of the 3 angles and writing that the distance between these points are known quantities one get three equations in the sines and cosines of the unknown angles. Nanua [11] has combined these equations to get a 24th order polynomial in one of the unknown. Then it has been shown that the order of this polynomial can be reduced to 16, with only even power, if the lines normal to the circles and going through their centers lie in the same plane and a numerical procedure has enabled to find many configurations with sixteen assembly modes ([10], [2]). In [10] other cases has been considered for the position of the lines: if the lines are no more coplanar but intersect or are parallel we may find a polynomial of order 20 and if the lines are in a general position the order will be 24. In the same paper it has been shown that it is possible to find the direct kinematics polynomial for most of the various manipulators find in the literature (as soon as the mobile plate is a triangle) with this analysis and an extensive study of the number of solutions has been performed. It has been noticed that the number of solutions will be bounded by 16 whatever is the number of d.o.f of the manipulator: for example both new prototypes of parallel manipulator developed at INRIA will have at most 16 assembly modes although one is a 6 d.o.f manipulator and the other only a 3 d.o.f rotational wrist. A very recent work [6] has proposed a method to find a polynomial of order 16 in every cases i.e. a minimal polynomial. Thus this paper seems to conclude the study of forward kinematics for parallel manipulator with a triangular mobile plate.

But this method cannot be applied if the manipulator's mobile plate is not a triangle. In this case the manipulator has no more planar faces. A recent work [7] has addressed the problem of a manipulator with 4 planar faces and two non-planar faces. But basically in this case the resolution consists in finding an equivalent manipulator with only planar faces. Thus, to the best of

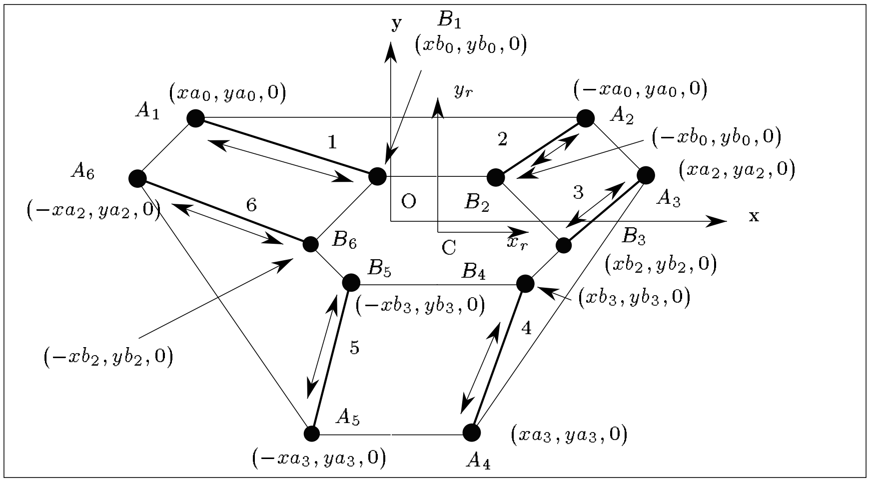

Figure 1: Top view of the considered manipulator

our knowledge, no paper has addressed the problem of the general manipulator either to find a polynomial formulation or an upper bound of the number of assembly modes. We propose here an algorithm to find all the assembly modes which gives also an upper bound of the number of assembly modes. This algorithm reduces the size of the set of equations to solve from 6 to 2 and we present a numerical procedure to solve this reduced set.

We will consider here the case where both plates are hexagons with a symmetry axis (y and y_r in Figure 1).

All the articulation points A_i on the base are coplanar as well as the articulation points B_i on the mobile. The links are numbered from 1 to 6 and we define a reference frame (O,x,y,z) where y is the symmetry axis of the fixed base. We then define a mobile frame (C, x_r, y_r, z_r) for the mobile with y_r the symmetry axis of the mobile plate. The coordinates of the articulations point A_i in the reference frame are (xa_i, ya_i, za_i) and for convenience the axis z of the reference frame is chosen such that $za_i = 0$. In the same manner the coordinates of the articulations points B_i in the mobile frame are $(xb_i, yb_i, 0)$. The position of the mobile plate is defined in the reference frame by the coordinates of point $C(x_0, y_0, z_0)$ and its orientation by three Euler's angles ψ, θ, ϕ with the associated rotation matrix R and ρ_i will denote the length of link i. The subscripts will be omitted each time there cannot be any misunderstanding. A subscript $_r$ will denote that the coordinates are expressed in the mobile frame.

2 Relation between C and R

We will consider here the expression of a links length ρ as a function of the position and orientation of the mobile plate. We have:

$$\underline{AB} = \underline{AO} + \underline{OC} + \underline{CB} = \underline{AO} + \underline{OC} + R\underline{CB_r} \tag{1}$$

Therefore:

$$\begin{aligned}\rho^2 &= \underline{AO}\ \underline{AO}^T + \underline{CB_r}\ \underline{CB_r}^T + 2(\underline{AO}^T + \underline{CB_r}^T R^T)\underline{OC} \\ &\quad + 2\underline{AO}^T R\underline{CB_r} + \underline{OC}\ \underline{OC}^T \end{aligned}\tag{2}$$

The forward kinematic problem is to find the solutions (i.e. $\underline{OC}$ and R) of a system of 6 equations of type (2) for a given set of links lengths. Let us denote d_A the distance between A and O and d_B the distance between B and C. These quantities are constant and are defined by the geometry of the manipulator. Thus equation (2) can be written:

$$\rho^2 = d_A^2 + d_B^2 + 2(\underline{AO}^T + \underline{CB_r}^T R^T)\underline{OC} + 2\underline{AO}^T R\underline{CB_r} + \underline{OC}\ \underline{OC}^T \tag{3}$$

Let us consider now two links. We define:

$$U_{ij} = d_{A_i}^2 + d_{B_i}^2 - d_{A_j}^2 - d_{B_j}^2 \qquad \underline{W_{ij}} = \underline{A_iO}^T - \underline{A_jO}^T$$

$$\underline{T_{ij}} = \underline{CB_{i_r}}^T - \underline{CB_{j_r}}^T \qquad S_{ij} = \underline{A_iO}^T R\underline{CB_{i_r}} - \underline{A_jO}^T R\underline{CB_{j_r}}$$

We notice that $U_{ij}, \underline{W_{ij}}, \underline{T_{ij}}$ are fully determinated by the geometry of the manipulator and that the S_{ij} are only dependent from the orientation of the robot. Let us define ρ_{ij} as:

$$\rho_{ij} = \rho_i^2 - \rho_j^2 \tag{4}$$

Equation (4) can be written as:

$$\rho_{ij} = U_{ij} + 2S_{ij} + 2(\underline{W_{ij}} + \underline{T_{ij}}^T R^T)\underline{OC} \tag{5}$$

This equation is linear in term of the coordinates of C. If we consider the three equations $\rho_{12}, \rho_{45}, \rho_{65}$ we get thus a linear system $\mathcal{S}$ in the three unknowns x_0, y_0, z_0. The resolution of this system enables to find the coordinates of C as a function of ψ, θ, ϕ. Thus we have:

$$\underline{OC} = \underline{H}(\psi, \theta, \phi) \tag{6}$$

Through the geometry of the problem we may notice that if a set (ψ, θ, ϕ) yields to a solution (x_0, y_0, z_0) then the set $(\psi, -\theta, \phi)$ must yield to the solution $(x_0, y_0, -z_0)$. This means that for a given set of links lengths and corresponding angles we must find the symmetrical configuration with respect to the base as a valid configuration.

The system $\mathcal{S}$ has been solved and the values of x_0, y_0, z_0 are given in Appendix 1. The previous remark is satisfied during the resolution. The determinant Δ of the system is:

$$\Delta = 32 \sin\theta(xa_0 xb_3 - xa_3 xb_0)(\sin\psi(yb_3 - yb_2) + \sin\phi(ya_3 - ya_2)) \qquad (7)$$

which will vanished if $\sin\theta = 0$, $\sin\phi = \sin\psi = 0$ or $\sin\psi(yb_3 - yb_2) = -\sin\phi(ya_3 - ya_2)$. In a first step we will suppose that none of these conditions is fulfilled.

3 The ϕ-curve

The system $\mathcal{S}$ being solved we may report the values of x_0, y_0, z_0 back in the two equations ρ_{24}, ρ_{36}. An interesting remark during this operation is that these two equations can be written as:

$$u_1 \cos\theta + u_2 = 0 \qquad\qquad v_1 \cos\theta + v_2 = 0 \qquad (8)$$

where u_1, u_2, v_1, v_2 contain only terms in sine and cosine of ψ, ϕ. These equations are described in more details in Appendix 2. From these two equations we may get an equation in ψ, ϕ by writing the constraint:

$$u_1 v_2 - v_2 u_1 = 0 \qquad (9)$$

which yield to:

$$\begin{aligned} &p_1 \sin^3\psi + p_2 \cos^3\psi + (p_{31}\cos\psi + p_{32})\sin^2\psi + \\ &(p_{41}\sin\psi + p_{42})\cos^2\psi + \sin\psi(p_{51}\cos\psi + p_{52}) + p_6\cos\psi = 0 \end{aligned} \qquad (10)$$

where the coefficients p_i are independent from ψ. These coefficients are described in Appendix 3. If we define $x = \tan\frac{\psi}{2}$ equation (10) is then a sixth order polynomial in x. Let us consider this equation for a given $\phi = \phi_s$. We get then at most 6 solutions in ψ, ψ_{s_i}, $1 \leq i \leq 6$. For each pair $\phi_s, \psi s_i$ equation (8) yields two solutions in θ, $(\theta_s, -\theta_s)$. Thus for a given ϕ_s we get at most 6 pairs of possible solutions $(\phi_s, \psi s_i, \theta_s)$, $(\phi_s, \psi s_i, -\theta_s)$ which in turn yields to six pairs of solution for C, $\underline{OC_{1_i}}$,$\underline{OC_{2_i}}$ with:

$$\underline{OC_{1_i}} = \begin{pmatrix} x_{0_1} \\ y_{0_1} \\ z_{0_1} \end{pmatrix}_i \qquad \underline{OC_{2_i}} = \begin{pmatrix} x_{0_2} \\ y_{0_2} \\ z_{0_2} \end{pmatrix}_i \qquad (11)$$

But by using the remark done during the resolution of the system $\mathcal{S}$ we know that $x_{0_2} = x_{0_1}, y_{0_2} = y_{0_1}, z_{0_2} = -z_{0_1}$. Thus the second set of solution $\underline{OC_{2_i}}$ represents simply the symmetrical configuration with respect to the fixed base of the first one. Thus for every ϕ we get at most a set of 6 possible solutions

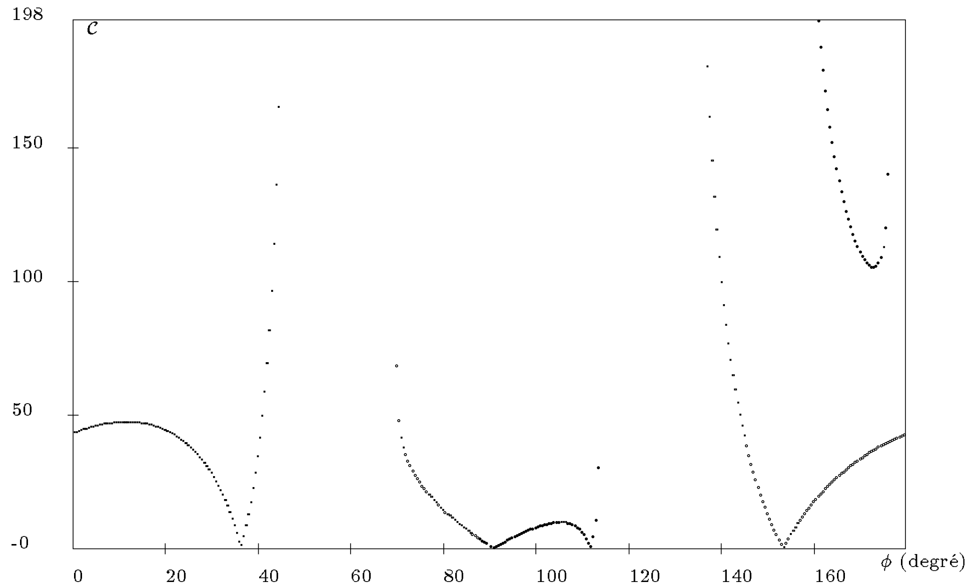

Figure 2: A typical ϕ-curve (ϕ in degree)

of the forward kinematics. They are only possible solutions because during the resolution we have used only five equations (i.e. $\rho_{12}, \rho_{45}, \rho_{65}, \rho_{24}, \rho_{36}$) among the set of 6 independent equations defined by (2) and therefore we have to verify if the links lengths associated to these solutions are identical to the initial set. In order to verify the validity of a solution we define a performance index $\mathcal{C}$:

$$\mathcal{C} = \sum_{1}^{6} ||\rho_{s_i} - \rho_i|| \tag{12}$$

where ρ_{s_i} denotes the length of link i for a possible solution. This index will vanished for each solution of the forward kinematics. By the use of a discretization of ϕ we are able to draw a plotting of $\mathcal{C}$ as a function of ϕ which is called a *ϕ-curve.* It is shown in Appendix 4 that the discretization of ϕ has not to be done between $[0, 2\pi]$ but only between $[0, \pi]$ because any solution in the range $[\pi, 2\pi]$ will give identical solutions to those find in the interval $[0, \pi]$. By looking at the ϕ-curve we can determine among the possible solutions those which have a performance index close to zero. These solutions are then fed to a least square algorithm which enables to get the exact solutions. For example if we consider the ϕ-curve described in figure 2 it appears that 4 solutions have an index close to zero. Taking into account the symmetrical solutions we will thus have 8 solutions for this particular case to which we have to add the eventual solutions corresponding to the particular cases of the resolution of the system $\mathcal{S}$.

4 Particular cases

The previous section does not deal with particular cases for which the determinant of the system $\mathcal{S}$ vanishes.

First we consider the case where $\sin\theta = 0$. In this case ρ_{12} is linear in term of x_0. Then ρ_{34} becomes linear in term of y_0. By using this results ρ_6^2 can be written as $z_0^2 + c = 0$. Thus we get two possible solutions and have only to verify if the corresponding links lengths are the same as the original set.

If $\sin\psi(yb_3 - yb_2) = -\sin\phi(ya_3 - ya_2)$ then equation ρ_{65} does not contain any term in z_0. It is possible to show that the ϕ-curve will be identical if we choose any other equation instead of ρ_{65}. Thus the only change compared with the general case is that we use another equation to compute the value of z_0.

If $\sin\psi = \sin\phi = 0$ then ρ_{12} is linear in term of x_0. Then ρ_{24}, ρ_{65} are linear in term of y_0, z_0. We expand ρ_1^2 and if we define $x = \tan\frac{\theta}{2}$ we get at most fourth order polynomial in x. Thus we get a four possible solutions and have only to verify if the corresponding links lengths are the same as the original set.

5 Maximum number of solutions

The solution of the direct kinematic problem is the solution of the set of equations:

$$u_1 \cos\theta + u_2 = 0 \tag{13}$$

$$v_1 \cos\theta + v_2 = 0 \tag{14}$$

$$\rho_1^2(\psi, \theta, \phi) = \rho_1^2 \tag{15}$$

If we define $x = \tan\frac{\psi}{2}$,$y = \tan\frac{\phi}{2}$, $z = \tan\frac{\theta}{2}$ this set is a set of multi-variable polynomials. The higher order term of the first equation is $x^2y^2z^2$ (term u_{23} in u_2) and therefore the order of this equation is 6. The higher order term of the second equation is $x^4y^4z^2$ and therefore its order is 10. The order of the last equation can be tediously found as 22 (see Appendix 5). Using Bezout's theorem we can thus state that there will be at most 6x10x22=1320 solutions.

6 Algorithm and Numerical example

The following algorithm has been implemented:

- verify if the initial set of links lengths can satisfied the particular cases and find the corresponding configurations.
- compute the performance index $\mathcal{C}$ for a discretization of ϕ in the range $[0, \pi]$ (a step of one degree seems to be sufficient).

- if $\mathcal{C}$ is sufficiently low use the possible solution as an estimate for a least-square method.

This algorithm has been used for a manipulator with the following characteristics:

number	x_a	y_a	x_b	y_b
0	-9.7	9.1	-3	7.3
2	12.76	3.9	7.822	-1.052
3	3	-13	4.822	-6.248

The initial set of links lengths is determined for the configuration $x_0 = -5$, $y_0 = 5$, $z_0 = 17$, $\psi = 0$, $\theta = 30$, $\phi = 0$. The six over-the-base solutions are given in table 1 and the corresponding configurations are presented in figure 3.

7 Conclusion

The proposed algorithm enables to find all the solutions of the forward kinematics problem for a general 6 d.o.f. parallel manipulator. Although the computation time is rather important (about thirty seconds on a SUN 3-60 workstation) it is a first step toward a general resolution of the difficult forward kinematics problem. The upper bound of the number of solutions is probably overestimated: we plan to continue to work on the equations to decrease it.

x_0	y_0	z_0	ψ	θ	ϕ
-5.0	5.0	17.0	0.0	30.0	0.0
4.8641	3.2024	14.6063	323.627375	95.320208	36.371860
-10.993397	1.780824	12.329258	206.593364	-77.993466	153.406322
-5.0	-7.648977	11.288760	0.0	-118.179036	0.0
5.502910	-4.708340	8.390066	68.130481	127.378302	111.871026
-4.693844	-2.020516	5.186273	88.941651	-82.951268	91.057316

Table 1: Solution of the forward kinematics problem (all the angles are in degree)

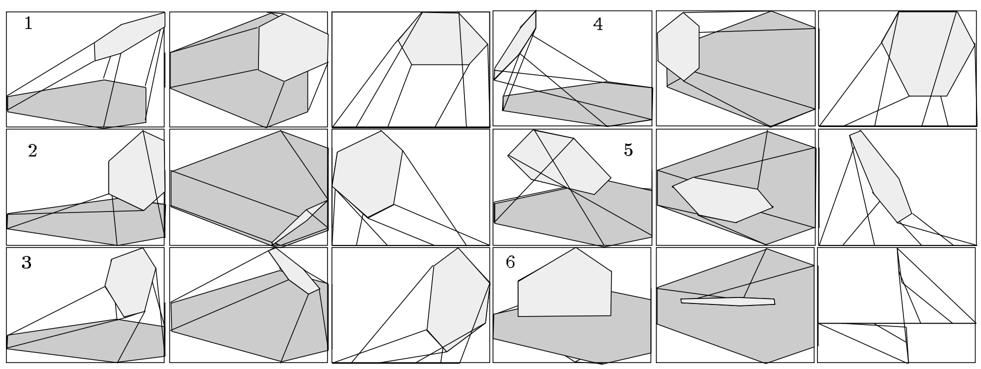

Figure 3: The 6 configurations for which the links lengths are identical (perspective, top and side view). We would get the 6 others solutions by putting the mobile plate in a symmetrical position with respect to the base.

8 Appendix 1: Resolution of the system $\mathcal{S}$

We suppose that the rotation matrix is defined by:

$$R = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix} \tag{16}$$

From the three equations $\rho_{12}, \rho_{45}, \rho_{65}$ we are able to compute x_0, y_0, z_0. We get:

$$x_0 = -\frac{x_{b_0}\rho_5^2 - x_{b_0}\rho_4^2 + 2x_{b_0}S_{45} - 2x_{b_3}S_{12} + x_{b_3}\rho_1^2 - x_{b_3}\rho_2^2}{4x_{b_3}x_{a_0} - 4x_{b_0}x_{a_3}}$$

$$\begin{aligned}
y_0 \quad = \quad & -x_{b_0}x_{a_2}\rho_5^2 v_3 - y_{b_1}x_{b_3}\rho_2^2 v_4 v_3 \\
& +2\rho_6^2 x_{b_3}x_{a_0}v_3 - 4S_{65}x_{b_3}x_{a_0}v_3 - 2U_{65}x_{b_3}x_{a_0}v_3 + 4y_{b_2}x_{b_3}S_{12}v_6 v_1 \\
& +2y_{b_1}x_{b_3}S_{12}v_6 v_1 - 2S_{45}x_{b_2}x_{a_0}v_3 - 2y_{b_2}x_{b_3}\rho_2^2 v_4 v_3 + 2y_{b_2}x_{b_3}\rho_1^2 v_4 v_3 - 4y_{b_2}x_{b_3}S_{12}v_4 v_3 \\
& -2y_{b_1}x_{b_3}S_{12}v_4 v_3 + 2x_{b_0}y_{b_2}\rho_5^2 v_4 v_3 + 2x_{b_0}y_{b_1}S_{45}v_4 v_3 - 2x_{b_0}\rho_6^2 x_{a_3}v_3 - y_{b_1}x_{a_3}\rho_2^2 v_6 \\
& +y_{b_1}x_{b_3}\rho_2^2 v_6 v_1 - y_{b_1}x_{b_3}\rho_1^2 v_6 v_1 + y_{b_1}x_{a_3}\rho_1^2 v_6 + 4y_{b_2}S_{45}x_{a_0}v_6 - 2x_{b_0}y_{b_2}\rho_4^2 v_4 v_3 \\
& +4x_{b_0}y_{b_2}S_{45}v_4 v_3 + 2x_{b_0}x_{a_2}S_{45}v_3 + 2x_{b_2}x_{a_3}S_{12}v_3 - 2x_{b_0}S_{45}x_{a_3}v_3 - x_{b_0}y_{b_1}\rho_4^2 v_4 v_3 \\
& +4x_{b_0}S_{65}x_{a_3}v_3 + 2x_{b_0}U_{65}x_{a_3}v_3 + 2S_{45}x_{b_3}x_{a_0}v_3 - 2y_{b_2}x_{b_3}\rho_1^2 v_6 v_1 + 2x_{b_0}y_{b_2}\rho_4^2 v_6 v_1 \\
& -2x_{b_0}y_{b_1}S_{45}v_6 v_1 - 4x_{b_0}y_{b_2}S_{45}v_6 v_1 - 2x_{a_2}x_{b_3}S_{12}v_3 + y_{b_1}\rho_5^2 x_{a_0}v_6 - y_{b_1}\rho_4^2 x_{a_0}v_6 \\
& -2x_{b_0}y_{b_2}\rho_5^2 v_6 v_1 + 2y_{b_2}\rho_5^2 x_{a_0}v_6 - 2y_{b_2}\rho_4^2 x_{a_0}v_6 + 2y_{b_1}S_{45}x_{a_0}v_6 \\
& -2y_{b_2}x_{a_3}\rho_2^2 v_6 + x_{b_0}y_{b_1}\rho_4^2 v_6 v_1 - x_{b_0}y_{b_1}\rho_5^2 v_6 v_1 + x_{a_2}x_{b_3}\rho_1^2 v_3 \\
& +2y_{b_2}x_{a_3}\rho_1^2 v_6 - 4y_{b_2}x_{a_3}S_{12}v_6 - 2y_{b_1}x_{a_3}S_{12}v_6 + 2y_{b_2}x_{b_3}\rho_2^2 v_6 v_1 \\
& -\rho_5^2 x_{b_2}x_{a_0}v_3 + x_{b_0}\rho_4^2 x_{a_3}v_3 + y_{b_1}x_{b_3}\rho_1^2 v_4 v_3 + x_{b_0}y_{b_1}\rho_5^2 v_4 v_3 \\
& -x_{a_2}x_{b_3}\rho_2^2 v_3 + x_{b_2}x_{a_3}\rho_2^2 v_3 - \rho_4^2 x_{b_3}x_{a_0}v_3 - x_{b_2}x_{a_3}\rho_1^2 v_3 \\
& -x_{b_0}x_{a_2}\rho_4^2 v_3 - \rho_5^2 x_{b_3}x_{a_0}v_3 + \rho_4^2 x_{b_2}x_{a_0}v_3 + x_{b_0}\rho_5^2 x_{a_3}v_3 \\
& /8y_{b_2}x_{b_3}x_{a_0}v_6 v_2 + 4y_{b_1}x_{b_3}x_{a_0}v_6 v_2 - 4x_{b_0}x_{a_3}v_6 y_{b_1}v_2 - 8x_{b_0}x_{a_3}v_6 y_{b_2}v_2 \\
& -8y_{b_2}x_{b_3}x_{a_0}v_5 v_3 - 4y_{b_1}x_{b_3}x_{a_0}v_5 v_3 + 8x_{b_0}x_{a_3}y_{b_2}v_5 v_3 + 4x_{b_0}x_{a_3}y_{b_1}v_5 v_3 \\
& +8y_{a_2}x_{b_3}x_{a_0}v_3 + 4y_{a_1}x_{b_3}x_{a_0}v_3 - 8x_{b_0}y_{a_2}x_{a_3}v_3 - 4x_{b_0}y_{a_1}x_{a_3}v_3
\end{aligned}$$

$$\begin{aligned}
z_0 \quad = \quad & -2x_{a_0}y_{a_2}\rho_5^2 - 2x_{a_0}y_{a_1}S_{45} - 2x_{a_3}y_{a_2}\rho_1^2 \\
& +2x_{a_3}y_{a_1}S_{12} - 4x_{a_0}y_{a_2}S_{45} + 2x_{a_0}y_{a_2}\rho_4^2 + 4x_{a_3}y_{a_2}S_{12} \\
& +4x_{a_0}y_{b_2}S_{45}v_5 + 2x_{a_0}y_{b_1}S_{45}v_5 - 4x_{a_3}y_{b_2}S_{12}v_5 - 2x_{a_3}y_{b_1}S_{12}v_5 \\
& -2x_{a_0}y_{b_2}\rho_4^2 v_5 - 2x_{a_3}y_{b_2}\rho_2^2 v_5 - x_{b_3}y_{a_1}\rho_2^2 v_1 + x_{b_3}y_{a_1}\rho_1^2 v_1 \\
& +x_{b_0}y_{b_1}\rho_4^2 v_5 v_1 - x_{a_2}x_{b_3}\rho_2^2 v_2 + x_{b_2}x_{a_3}\rho_2^2 v_2 - \rho_5^2 x_{b_2}x_{a_0}v_2 \\
& +x_{b_0}\rho_4^2 x_{a_3}v_2 + y_{b_1}x_{b_3}\rho_1^2 v_4 v_2 - 4S_{65}x_{b_3}x_{a_0}v_2 - 2U_{65}x_{b_3}x_{a_0}v_2 \\
& -2S_{45}x_{b_2}x_{a_0}v_2 - 2y_{b_2}x_{b_3}\rho_2^2 v_4 v_2 + 2y_{b_2}x_{b_3}\rho_1^2 v_4 v_2 + 2x_{b_0}y_{b_2}\rho_5^2 v_4 v_2 \\
& -2x_{b_0}y_{b_2}\rho_4^2 v_4 v_2 + 2x_{b_0}x_{a_2}S_{45}v_2 + 4y_{b_2}x_{b_3}S_{12}v_5 v_1 + 2y_{b_1}x_{b_3}S_{12}v_5 v_1 \\
& -2x_{b_0}y_{b_2}\rho_5^2 v_5 v_1 + 2x_{b_0}y_{b_2}\rho_4^2 v_5 v_1 + 2x_{b_0}U_{65}x_{a_3}v_2 + 2S_{45}x_{b_3}x_{a_0}v_2 \\
& +2\rho_6^2 x_{b_3}x_{a_0}v_2 - 2x_{b_3}y_{a_1}S_{12}v_1 + 2x_{b_0}y_{a_2}\rho_5^2 v_1 - 2x_{b_0}y_{a_2}\rho_4^2 v_1 \\
& +4x_{b_0}y_{a_2}S_{45}v_1 + 2x_{b_0}y_{a_1}S_{45}v_1 + x_{a_0}y_{a_1}\rho_4^2 - x_{a_0}y_{a_1}\rho_5^2
\end{aligned}$$

$$
\begin{aligned}
&-x_{a_3}y_{a_1}\rho_1^2 + x_{a_3}y_{a_1}\rho_2^2 - 4x_{b_0}y_{b_2}S_{45}v_5v_1 - 2x_{b_0}y_{b_1}S_{45}v_5v_1\\
&-2x_{b_3}y_{a_2}\rho_2^2v_1 + 2x_{b_3}y_{a_2}\rho_1^2v_1 - 4x_{b_3}y_{a_2}S_{12}v_1 - x_{b_0}y_{b_1}\rho_4^2v_4v_2\\
&+x_{b_0}y_{a_1}\rho_5^2v_1 - x_{b_0}y_{a_1}\rho_4^2v_1 + \rho_4^2x_{b_2}x_{a_0}v_2 + 2x_{a_3}y_{b_2}\rho_1^2v_5\\
&+2x_{b_2}x_{a_3}S_{12}v_2 - 2x_{b_0}S_{45}x_{a_3}v_2 - 2x_{b_0}\rho_6^2x_{a_3}v_2 + 4x_{b_0}S_{65}x_{a_3}v_2\\
&+x_{b_0}\rho_5^2x_{a_3}v_2 - x_{b_2}x_{a_3}\rho_1^2v_2 + x_{a_2}x_{b_3}\rho_1^2v_2 - x_{b_0}x_{a_2}\rho_4^2v_2\\
&-\rho_5^2x_{b_3}x_{a_0}v_2 + x_{a_0}y_{b_1}\rho_5^2v_5 - x_{a_0}y_{b_1}\rho_4^2v_5 - 4y_{b_2}x_{b_3}S_{12}v_4v_2\\
&-2y_{b_1}x_{b_3}S_{12}v_4v_2 + 4x_{b_0}y_{b_2}S_{45}v_4v_2 + 2x_{b_0}y_{b_1}S_{45}v_4v_2 + 2y_{b_2}x_{b_3}\rho_2^2v_5v_1\\
&-2y_{b_2}x_{b_3}\rho_1^2v_5v_1 + x_{b_0}y_{b_1}\rho_5^2v_4v_2 - \rho_4^2x_{b_3}x_{a_0}v_2 + y_{b_1}x_{b_3}\rho_2^2v_5v_1\\
&-y_{b_1}x_{b_3}\rho_1^2v_5v_1 + 2x_{a_3}y_{a_2}\rho_2^2 - x_{a_3}y_{b_1}\rho_2^2v_5 - y_{b_1}x_{b_3}\rho_2^2v_4v_2\\
&+x_{b_0}x_{a_2}\rho_5^2v_2 - 2x_{a_2}x_{b_3}S_{12}v_2 - x_{b_0}y_{b_1}\rho_5^2v_5v_1 + 2x_{a_0}y_{b_2}\rho_5^2v_5 + x_{a_3}y_{b_1}\rho_1^2v_5\\
&/8y_{b_2}x_{b_3}x_{a_0}v_6v_2 + 4y_{b_1}x_{b_3}x_{a_0}v_6v_2 - 4x_{b_0}x_{a_3}v_6y_{b_1}v_2 - 8x_{b_0}x_{a_3}v_6y_{b_2}v_2\\
&-8y_{b_2}x_{b_3}x_{a_0}v_5v_3 - 4y_{b_1}x_{b_3}x_{a_0}v_5v_3 + 8x_{b_0}x_{a_3}y_{b_2}v_5v_3 + 4x_{b_0}x_{a_3}y_{b_1}v_5v_3\\
&+8y_{a_2}x_{b_3}x_{a_0}v_3 + 4y_{a_1}x_{b_3}x_{a_0}v_3 - 8x_{b_0}y_{a_2}x_{a_3}v_3 - 4x_{b_0}y_{a_1}x_{a_3}v_3
\end{aligned}
$$

9 Appendix 2: Factorization of ρ_{24}, ρ_{36}

The equations (8) are of the form:

$$u_1 \cos\theta + u_2 = 0 \tag{17}$$

$$v_1 \cos\theta + v_2 = 0 \tag{18}$$

We have:

$$u_1 = u_{11}\cos\phi\sin\psi + u_{12}\sin\phi\cos\psi \tag{19}$$

with u_{11}, u_{12} constants defined by:

$$
\begin{aligned}
u_{11} \;=\; &-4x_{a_2}x_{a_3}x_{b_0}y_{b_2} - 4x_{a_0}x_{a_2}x_{b_3}y_{b_0} + 4x_{a_0}x_{a_3}x_{b_2}y_{b_0} - 4x_{a_0}x_{a_3}x_{b_2}y_{b_3}\\
&+4x_{a_2}x_{a_3}x_{b_0}y_{b_3} + 4x_{a_0}x_{a_2}x_{b_3}y_{b_2}
\end{aligned}
$$

$$
\begin{aligned}
u_{12} \;=\; &4x_{a_2}x_{b_0}x_{b_3}y_{a_0} + 4x_{a_0}x_{b_2}x_{b_3}y_{a_3} + 4x_{a_3}x_{b_0}x_{b_2}y_{a_2} - 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}\\
&-4x_{a_2}x_{b_0}x_{b_3}y_{a_3} - 4x_{a_0}x_{b_2}x_{b_3}y_{a_2}
\end{aligned}
$$

Then

$$u_2 = u_{21}\cos\phi\sin\psi + u_{22}\sin\phi\cos\psi + u_{23} \tag{20}$$

with u_{21}, u_{22}, u_{23} constants defined by:

$$
\begin{aligned}
u_{21} \;=\; &4x_{a_2}x_{b_0}x_{b_3}y_{a_0} + 4x_{a_0}x_{b_2}x_{b_3}y_{a_3} + 4x_{a_3}x_{b_0}x_{b_2}y_{a_2} - 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}\\
&-4x_{a_2}x_{b_0}x_{b_3}y_{a_3} - 4x_{a_0}x_{b_2}x_{b_3}y_{a_2}
\end{aligned}
$$

$$
\begin{aligned}
u_{22} \;=\; &-4x_{a_2}x_{a_3}x_{b_0}y_{b_2} - 4x_{a_0}x_{a_2}x_{b_3}y_{b_0} + 4x_{a_0}x_{a_3}x_{b_2}y_{b_0} - 4x_{a_0}x_{a_3}x_{b_2}y_{b_3}\\
&+4x_{a_2}x_{a_3}x_{b_0}y_{b_3} + 4x_{a_0}x_{a_2}x_{b_3}y_{b_2}
\end{aligned}
$$

$$
\begin{aligned}
u_{23} \;=\; &\rho_1^2x_{a_2}x_{b_3} - \rho_2^2x_{a_2}x_{b_3} + \rho_6^2x_{a_0}x_{b_3} - \rho_3^2x_{a_0}x_{b_3}\\
&+\rho_2^2x_{a_3}x_{b_2} - \rho_1^2x_{a_3}x_{b_2} - \rho_5^2x_{a_0}x_{b_2} + \rho_4^2x_{a_0}x_{b_2}\\
&-\rho_6^2x_{a_3}x_{b_0} + \rho_3^2x_{a_3}x_{b_0} + \rho_5^2x_{a_2}x_{b_0} - \rho_4^2x_{a_2}x_{b_0}
\end{aligned}
$$

Then

$$v_1 = v_{11} \sin^2 \psi + v_{12} \cos^2 \psi + v_{13} \tag{21}$$

with:

$$v_{11} = v_{111} \cos \phi + v_{112} \sin \phi \tag{22}$$

where v_{111}, v_{112} are constants defined by:

$$\begin{aligned}
v_{111} \;=\;& 4x_{a_2}x_{a_3}x_{b_0}y_{b_3}^2 - 4x_{a_0}x_{a_3}x_{b_2}y_{b_3}^2 + 4x_{a_0}x_{a_2}x_{b_3}y_{b_2}y_{b_3} - 4x_{a_2}x_{a_3}x_{b_0}y_{b_2}y_{b_3} \\
& -4x_{a_0}x_{a_2}x_{b_3}y_{b_0}y_{b_3} + 8x_{a_0}x_{a_3}x_{b_2}y_{b_0}y_{b_3} - 4x_{a_2}x_{a_3}x_{b_0}y_{b_0}y_{b_3} - 4x_{a_0}x_{a_2}x_{b_3}y_{b_0}y_{b_2} \\
& +4x_{a_2}x_{a_3}x_{b_0}y_{b_0}y_{b_2} + 4x_{a_0}x_{a_2}x_{b_3}y_{b_0}^2 - 4x_{a_0}x_{a_3}x_{b_2}y_{b_0}^2
\end{aligned}$$

$$\begin{aligned}
v_{112} \;=\;& 4x_{a_0}^2x_{b_0}x_{b_3}y_{b_3} - 4x_{a_0}x_{a_2}x_{b_2}x_{b_3}y_{b_3} + 4x_{a_2}x_{a_3}x_{b_0}x_{b_2}y_{b_3} - 4x_{a_0}x_{a_3}x_{b_0}^2y_{b_3} \\
& +4x_{a_0}x_{a_3}x_{b_3}^2y_{b_2} - 4x_{a_3}^2x_{b_0}x_{b_3}y_{b_2} - 4x_{a_0}^2x_{b_0}x_{b_3}y_{b_2} + 4x_{a_0}x_{a_3}x_{b_0}^2y_{b_2} - 4x_{a_0}x_{a_3}x_{b_3}^2y_{b_0} \\
& +4x_{a_0}x_{a_2}x_{b_2}x_{b_3}y_{b_0} + 4x_{a_3}^2x_{b_0}x_{b_3}y_{b_0} - 4x_{a_2}x_{a_3}x_{b_0}x_{b_2}y_{b_0}
\end{aligned}$$

Then

$$v_{12} = \sin \phi v_{121} \tag{23}$$

with v_{121} a constant:

$$\begin{aligned}
v_{121} \;=\;& 4x_{b_0}x_{b_3}y_{a_0}y_{a_3}y_{b_3} - 4x_{b_0}x_{b_3}y_{a_2}y_{a_3}y_{b_3} + 4x_{b_0}x_{b_3}y_{a_0}y_{a_2}y_{b_3} - 4x_{b_0}x_{b_3}y_{a_0}^2y_{b_3} \\
& +4x_{b_0}x_{b_3}y_{a_3}^2y_{b_2} - 8x_{b_0}x_{b_3}y_{a_0}y_{a_3}y_{b_2} + 4x_{b_0}x_{b_3}y_{a_0}^2y_{b_2} - 4x_{b_0}x_{b_3}y_{a_3}^2y_{b_0} \\
& +4x_{b_0}x_{b_3}y_{a_2}y_{a_3}y_{b_0} + 4x_{b_0}x_{b_3}y_{a_0}y_{a_3}y_{b_0} - 4x_{b_0}x_{b_3}y_{a_0}y_{a_2}y_{b_0}
\end{aligned}$$

and:

$$v_{13} = v_{131} \sin^2 \phi + v_{132} \tag{24}$$

and

$$v_{131} = v_{1311} \sin \psi + v_{1312} \tag{25}$$

where v_{1311}, v_{1312} are constants defined by:

$$\begin{aligned}
v_{1311} \;=\;& 4x_{a_0}x_{a_3}y_{a_2}y_{b_3}^2 - 4x_{a_0}x_{a_3}y_{a_0}y_{b_3}^2 - 4x_{a_0}x_{a_3}y_{a_3}y_{b_2}y_{b_3} + 4x_{a_0}x_{a_3}y_{a_0}y_{b_2}y_{b_3} \\
& +4x_{a_0}x_{a_3}y_{a_3}y_{b_0}y_{b_3} - 8x_{a_0}x_{a_3}y_{a_2}y_{b_0}y_{b_3} + 4x_{a_0}x_{a_3}y_{a_0}y_{b_0}y_{b_3} + 4x_{a_0}x_{a_3}y_{a_3}y_{b_0}y_{b_2} \\
& -4x_{a_0}x_{a_3}y_{a_0}y_{b_0}y_{b_2} - 4x_{a_0}x_{a_3}y_{a_3}y_{b_0}^2 + 4x_{a_0}x_{a_3}y_{a_2}y_{b_0}^2 - 4x_{a_0}x_{a_2}x_{b_2}x_{b_3}y_{a_3} \\
& +4x_{a_0}^2x_{b_0}x_{b_3}y_{a_3} + 4x_{a_2}x_{a_3}x_{b_0}x_{b_2}y_{a_3} - 4x_{a_0}x_{a_3}x_{b_0}^2y_{a_3} + 4x_{a_0}x_{a_3}x_{b_3}^2y_{a_2} \\
& -4x_{a_3}^2x_{b_0}x_{b_3}y_{a_2} - 4x_{a_0}^2x_{b_0}x_{b_3}y_{a_2} + 4x_{a_0}x_{a_3}x_{b_0}^2y_{a_2} - 4x_{a_0}x_{a_3}x_{b_3}^2y_{a_0} \\
& +4x_{a_0}x_{a_2}x_{b_2}x_{b_3}y_{a_0} + 4x_{a_3}^2x_{b_0}x_{b_3}y_{a_0} - 4x_{a_2}x_{a_3}x_{b_0}x_{b_2}y_{a_0}
\end{aligned}$$

$$\begin{aligned}
v_{1312} \;=\;& 4x_{a_0}x_{b_2}x_{b_3}y_{a_3}^2 - 4x_{a_2}x_{b_0}x_{b_3}y_{a_3}^2 - 4x_{a_0}x_{b_2}x_{b_3}y_{a_2}y_{a_3} + 4x_{a_3}x_{b_0}x_{b_2}y_{a_2}y_{a_3} \\
& -4x_{a_0}x_{b_2}x_{b_3}y_{a_0}y_{a_3} + 8x_{a_2}x_{b_0}x_{b_3}y_{a_0}y_{a_3} - 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}y_{a_3} + 4x_{a_0}x_{b_2}x_{b_3}y_{a_0}y_{a_2} \\
& -4x_{a_3}x_{b_0}x_{b_2}y_{a_0}y_{a_2} - 4x_{a_2}x_{b_0}x_{b_3}y_{a_0}^2 + 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}^2
\end{aligned}$$

Then

$$v_{132} = \cos \phi(q_1 \sin \phi + q_2) + q_3 \sin \phi + q_4 \tag{26}$$

with

$$
\begin{aligned}
q_1 &= q_{11}\cos\psi + q_{12}\sin\psi \\
q_2 &= q_{21}\cos\psi\sin\psi \\
q_3 &= q_{31}\cos\psi\sin\psi \\
q_4 &= q_{41}\sin\psi
\end{aligned}
$$

The q_{ij} are constants defined by:

$$
\begin{aligned}
q_{11} = {} & 4(y_{a_3} - y_{a_0})(x_{a_3}x_{b_0}y_{a_2}y_{b_3} - x_{a_3}x_{b_0}y_{a_0}y_{b_3} - x_{a_0}x_{b_3}y_{a_3}y_{b_2} \\
& + x_{a_0}x_{b_3}y_{a_2}y_{b_2} - x_{a_3}x_{b_0}y_{a_2}y_{b_2} \\
& + x_{a_3}x_{b_0}y_{a_0}y_{b_2} + x_{a_0}x_{b_3}y_{a_3}y_{b_0} - x_{a_0}x_{b_3}y_{a_2}y_{b_0})
\end{aligned}
$$

$$
\begin{aligned}
q_{12} = {} & -4(y_{a_3} - y_{a_0})(x_{a_0}x_{a_3}x_{b_2}y_{b_3} - x_{a_2}x_{a_3}x_{b_0}y_{b_3} - x_{a_0}x_{a_2}x_{b_3}y_{b_2} \\
& + x_{a_2}x_{a_3}x_{b_0}y_{b_2} + x_{a_0}x_{a_2}x_{b_3}y_{b_0} - x_{a_0}x_{a_3}x_{b_2}y_{b_0})
\end{aligned}
$$

$$
\begin{aligned}
q_{21} = {} & 4(y_{b_3} - y_{b_0})(x_{a_3}x_{b_0}y_{a_2}y_{b_3} - x_{a_3}x_{b_0}y_{a_0}y_{b_3} - x_{a_0}x_{b_3}y_{a_3}y_{b_2} + x_{a_0}x_{b_3}y_{a_2}y_{b_2} \\
& - x_{a_3}x_{b_0}y_{a_2}y_{b_2} + x_{a_3}x_{b_0}y_{a_0}y_{b_2} + x_{a_0}x_{b_3}y_{a_3}y_{b_0} - x_{a_0}x_{b_3}y_{a_2}y_{b_0})
\end{aligned}
$$

$$
\begin{aligned}
q_{31} = {} & 4(x_{a_0}x_{b_2}x_{b_3}y_{a_3} - x_{a_2}x_{b_0}x_{b_3}y_{a_3} - x_{a_0}x_{b_2}x_{b_3}y_{a_2} + x_{a_3}x_{b_0}x_{b_2}y_{a_2} \\
& + x_{a_2}x_{b_0}x_{b_3}y_{a_0} - x_{a_3}x_{b_0}x_{b_2}y_{a_0})(y_{b_3} - y_{b_0})
\end{aligned}
$$

$$
q_{41} = -4x_{a_0}x_{a_3}(y_{b_3} - y_{b_0})(y_{a_2}y_{b_3} - y_{a_0}y_{b_3} - y_{a_3}y_{b_2} + y_{a_0}y_{b_2} + y_{a_3}y_{b_0} - y_{a_2}y_{b_0})
$$

Finally v_2 is defined by:

$$
v_2 = v_{21}\sin^2\psi + v_{22}\cos^2\psi + v_{23} \tag{27}
$$

with:

$$
\begin{aligned}
v_{21} &= v_{211}\sin\phi + v_{212}\cos\phi && (28) \\
v_{22} &= v_{221}\sin\phi && (29) \\
v_{23} &= g_1\sin\psi + g_2\cos\psi + g_3\sin\psi\cos\psi + g_4\cos\phi + g_5\sin\phi && (30)
\end{aligned}
$$

with

$$
\begin{aligned}
v_{211} = {} & 4x_{a_0}x_{b_3}y_{a_3}y_{b_2}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_3}y_{b_2}y_{b_3} - 4x_{a_0}x_{b_3}y_{a_2}y_{b_2}y_{b_3} + 4x_{a_3}x_{b_0}y_{a_2}y_{b_2}y_{b_3} \\
& - 4x_{a_0}x_{b_3}y_{a_3}y_{b_0}y_{b_3} + 4x_{a_3}x_{b_0}y_{a_3}y_{b_0}y_{b_3} + 4x_{a_0}x_{b_3}y_{a_0}y_{b_0}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_0}y_{b_0}y_{b_3} \\
& + 4x_{a_0}x_{b_3}y_{a_2}y_{b_0}y_{b_2} - 4x_{a_3}x_{b_0}y_{a_2}y_{b_0}y_{b_2} - 4x_{a_0}x_{b_3}y_{a_0}y_{b_0}y_{b_2} + 4x_{a_3}x_{b_0}y_{a_0}y_{b_0}y_{b_2}
\end{aligned}
$$

$$
\begin{aligned}
v_{212} = {} & 4x_{a_0}x_{b_2}x_{b_3}y_{a_3}y_{b_3} - 4x_{a_2}x_{b_0}x_{b_3}y_{a_3}y_{b_3} - 4x_{a_0}x_{b_2}x_{b_3}y_{a_2}y_{b_3} + 4x_{a_3}x_{b_0}x_{b_2}y_{a_2}y_{b_3} \\
& + 4x_{a_2}x_{b_0}x_{b_3}y_{a_0}y_{b_3} - 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}y_{b_3} - 4x_{a_0}x_{b_2}x_{b_3}y_{a_3}y_{b_0} + 4x_{a_2}x_{b_0}x_{b_3}y_{a_3}y_{b_0} \\
& + 4x_{a_0}x_{b_2}x_{b_3}y_{a_2}y_{b_0} - 4x_{a_3}x_{b_0}x_{b_2}y_{a_2}y_{b_0} - 4x_{a_2}x_{b_0}x_{b_3}y_{a_0}y_{b_0} + 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}y_{b_0}
\end{aligned}
$$

$$
\begin{aligned}
v_{221} = {} & 4x_{a_3}x_{b_0}y_{a_2}y_{b_3}^2 - 4x_{a_3}x_{b_0}y_{a_0}y_{b_3}^2 - 4x_{a_3}x_{b_0}y_{a_3}y_{b_2}y_{b_3} + 4x_{a_3}x_{b_0}y_{a_0}y_{b_2}y_{b_3} \\
& + 4x_{a_3}x_{b_0}y_{a_3}y_{b_0}y_{b_3} - 4x_{a_0}x_{b_3}y_{a_2}y_{b_0}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_2}y_{b_0}y_{b_3} + 4x_{a_0}x_{b_3}y_{a_0}y_{b_0}y_{b_3} \\
& + 4x_{a_0}x_{b_3}y_{a_3}y_{b_0}y_{b_2} - 4x_{a_0}x_{b_3}y_{a_0}y_{b_0}y_{b_2} - 4x_{a_0}x_{b_3}y_{a_3}y_{b_0}^2 + 4x_{a_0}x_{b_3}y_{a_2}y_{b_0}^2
\end{aligned}
$$

and

$$
\begin{aligned}
g_1 &= g_{11}\cos^2\phi + g_{12}\cos\phi\sin\phi + g_{13} && (31)\\
g_2 &= g_{21}\sin^2\phi + g_{23}\cos\phi\sin\phi + g_{24} && (32)\\
g_3 &= g_{31}\sin\phi + g_{32}\cos\phi && (33)
\end{aligned}
$$

where g_{ij} are constants defined by:

$$
\begin{aligned}
g_{11} = {} & 4x_{a_3}x_{b_0}y_{a_2}y_{a_3}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_0}y_{a_3}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_0}y_{a_2}y_{b_3} + 4x_{a_3}x_{b_0}y_{a_0}^2y_{b_3}\\
& -4x_{a_0}x_{b_3}y_{a_3}^2y_{b_2} + 4x_{a_0}x_{b_3}y_{a_2}y_{a_3}y_{b_2} - 4x_{a_3}x_{b_0}y_{a_2}y_{a_3}y_{b_2} + 4x_{a_0}x_{b_3}y_{a_0}y_{a_3}y_{b_2}\\
& +4x_{a_3}x_{b_0}y_{a_0}y_{a_3}y_{b_2} - 4x_{a_0}x_{b_3}y_{a_0}y_{a_2}y_{b_2} + 4x_{a_3}x_{b_0}y_{a_0}y_{a_2}y_{b_2} - 4x_{a_3}x_{b_0}y_{a_0}^2y_{b_2}\\
& +4x_{a_0}x_{b_3}y_{a_3}^2y_{b_0} - 4x_{a_0}x_{b_3}y_{a_2}y_{a_3}y_{b_0} - 4x_{a_0}x_{b_3}y_{a_0}y_{a_3}y_{b_0} + 4x_{a_0}x_{b_3}y_{a_0}y_{a_2}y_{b_0}
\end{aligned}
$$

$$
\begin{aligned}
g_{12} = {} & 4x_{a_0}x_{b_2}x_{b_3}y_{a_3}^2 - 4x_{a_2}x_{b_0}x_{b_3}y_{a_3}^2 - 4x_{a_0}x_{b_2}x_{b_3}y_{a_2}y_{a_3} + 4x_{a_3}x_{b_0}x_{b_2}y_{a_2}y_{a_3}\\
& -4x_{a_0}x_{b_2}x_{b_3}y_{a_0}y_{a_3} + 8x_{a_2}x_{b_0}x_{b_3}y_{a_0}y_{a_3} - 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}y_{a_3} + 4x_{a_0}x_{b_2}x_{b_3}y_{a_0}y_{a_2}\\
& -4x_{a_3}x_{b_0}x_{b_2}y_{a_0}y_{a_2} - 4x_{a_2}x_{b_0}x_{b_3}y_{a_0}^2 + 4x_{a_3}x_{b_0}x_{b_2}y_{a_0}^2
\end{aligned}
$$

$$
\begin{aligned}
g_{13} = {} & \rho_4^2x_{a_0}x_{b_2}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_2}y_{a_3}y_{b_3} - \rho_1^2x_{a_0}x_{b_3}y_{b_3} + \rho_2^2x_{a_3}x_{b_2}y_{b_3}\\
& -\rho_1^2x_{a_3}x_{b_2}y_{b_3} - \rho_5^2x_{a_0}x_{b_2}y_{b_3} + 2U_{65}x_{a_0}x_{b_3}y_{b_0} - 2\rho_6^2x_{a_0}x_{b_3}y_{b_0}\\
& -2U_{65}x_{a_3}x_{b_0}y_{b_0} + 2\rho_6^2x_{a_3}x_{b_0}y_{b_0} - \rho_5^2x_{a_0}x_{b_3}y_{b_2} - \rho_4^2x_{a_0}x_{b_3}y_{b_2}\\
& -\rho_1^2x_{a_2}x_{b_3}y_{b_0} + \rho_2^2x_{a_0}x_{b_3}y_{b_2} + \rho_1^2x_{a_0}x_{b_3}y_{b_2} + \rho_5^2x_{a_3}x_{b_0}y_{b_2}\\
& -2U_{65}x_{a_0}x_{b_3}y_{b_3} + \rho_5^2x_{a_2}x_{b_0}y_{b_3} - \rho_4^2x_{a_2}x_{b_0}y_{b_3} - 4x_{a_0}x_{b_3}y_{a_0}y_{a_3}y_{b_3}\\
& +2\rho_6^2x_{a_0}x_{b_3}y_{b_3} + 2U_{65}x_{a_3}x_{b_0}y_{b_3} - 2U_{24}x_{a_3}x_{b_0}y_{b_3} + \rho_4^2x_{a_3}x_{b_0}y_{b_2}\\
& -\rho_2^2x_{a_3}x_{b_0}y_{b_2} - \rho_1^2x_{a_3}x_{b_0}y_{b_2} + \rho_2^2x_{a_2}x_{b_3}y_{b_0} - \rho_5^2x_{a_3}x_{b_0}y_{b_0} - \rho_4^2x_{a_3}x_{b_0}y_{b_0}\\
& -\rho_5^2x_{a_2}x_{b_0}y_{b_0} + \rho_4^2x_{a_2}x_{b_0}y_{b_0} + 4x_{a_0}x_{b_3}y_{a_2}y_{a_3}y_{b_3} - 2U_{24}x_{a_0}x_{b_3}y_{b_2}\\
& +2U_{24}x_{a_3}x_{b_0}y_{b_2} - 4x_{a_0}x_{b_3}y_{a_2}y_{a_3}y_{b_2} + 4x_{a_3}x_{b_0}y_{a_2}y_{a_3}y_{b_2} + \rho_5^2x_{a_0}x_{b_3}y_{b_0}\\
& +\rho_4^2x_{a_0}x_{b_3}y_{b_0} - \rho_2^2x_{a_3}x_{b_2}y_{b_0} + \rho_1^2x_{a_3}x_{b_2}y_{b_0} + \rho_5^2x_{a_0}x_{b_2}y_{b_0}\\
& -\rho_4^2x_{a_0}x_{b_2}y_{b_0} - 2\rho_6^2x_{a_3}x_{b_0}y_{b_3} - 4x_{a_3}x_{b_0}y_{a_0}y_{a_3}y_{b_0} + 4x_{a_3}x_{b_0}y_{a_0}y_{a_2}y_{b_0}\\
& +\rho_1^2x_{a_2}x_{b_3}y_{b_3} - \rho_2^2x_{a_0}x_{b_3}y_{b_3} - \rho_2^2x_{a_2}x_{b_3}y_{b_3} + 2U_{24}x_{a_0}x_{b_3}y_{b_3}\\
& +\rho_2^2x_{a_3}x_{b_0}y_{b_3} + \rho_1^2x_{a_3}x_{b_0}y_{b_3} + 4x_{a_3}x_{b_0}y_{a_0}y_{a_3}y_{b_3} + 4x_{a_0}x_{b_3}y_{a_0}y_{a_3}y_{b_0}\\
& +4x_{a_0}x_{b_3}y_{a_0}y_{a_2}y_{b_2} - 4x_{a_3}x_{b_0}y_{a_0}y_{a_2}y_{b_2} - 4x_{a_0}x_{b_3}y_{a_0}y_{a_2}y_{b_0}
\end{aligned}
$$

$$
\begin{aligned}
g_{21} = {} & 4x_{a_2}x_{a_3}x_{b_0}y_{a_3}y_{b_3} - 4x_{a_0}x_{a_3}x_{b_2}y_{a_3}y_{b_3} + 4x_{a_0}x_{a_3}x_{b_2}y_{a_0}y_{b_3} - 4x_{a_2}x_{a_3}x_{b_0}y_{a_0}y_{b_3}\\
& +4x_{a_0}x_{a_2}x_{b_3}y_{a_3}y_{b_2} - 4x_{a_2}x_{a_3}x_{b_0}y_{a_3}y_{b_2} - 4x_{a_0}x_{a_2}x_{b_3}y_{a_0}y_{b_2} + 4x_{a_2}x_{a_3}x_{b_0}y_{a_0}y_{b_2}\\
& -4x_{a_0}x_{a_2}x_{b_3}y_{a_3}y_{b_0} + 4x_{a_0}x_{a_3}x_{b_2}y_{a_3}y_{b_0} + 4x_{a_0}x_{a_2}x_{b_3}y_{a_0}y_{b_0} - 4x_{a_0}x_{a_3}x_{b_2}y_{a_0}y_{b_0}
\end{aligned}
$$

$$
\begin{aligned}
g_{23} = {} & 4x_{a_0}x_{a_3}y_{a_0}y_{b_3}^2 - 4x_{a_0}x_{a_3}y_{a_2}y_{b_3}^2 + 4x_{a_0}x_{a_3}y_{a_3}y_{b_2}y_{b_3} - 4x_{a_0}x_{a_3}y_{a_0}y_{b_2}y_{b_3}\\
& -4x_{a_0}x_{a_3}y_{a_3}y_{b_0}y_{b_3} + 8x_{a_0}x_{a_3}y_{a_2}y_{b_0}y_{b_3} - 4x_{a_0}x_{a_3}y_{a_0}y_{b_0}y_{b_3} - 4x_{a_0}x_{a_3}y_{a_3}y_{b_0}y_{b_2}\\
& +4x_{a_0}x_{a_3}y_{a_0}y_{b_0}y_{b_2} + 4x_{a_0}x_{a_3}y_{a_3}y_{b_0}^2 - 4x_{a_0}x_{a_3}y_{a_2}y_{b_0}^2 + 4x_{a_0}x_{a_2}x_{b_2}x_{b_3}y_{a_3}\\
& -4x_{a_0}^2x_{b_0}x_{b_3}y_{a_3} - 4x_{a_2}x_{a_3}x_{b_0}x_{b_2}y_{a_3} + 4x_{a_0}x_{a_3}x_{b_0}^2y_{a_3} - 4x_{a_0}x_{a_3}x_{b_3}^2y_{a_2}\\
& +4x_{a_3}^2x_{b_0}x_{b_3}y_{a_2} + 4x_{a_0}^2x_{b_0}x_{b_3}y_{a_2} - 4x_{a_0}x_{a_3}x_{b_0}^2y_{a_2} + 4x_{a_0}x_{a_3}x_{b_3}^2y_{a_0}\\
& -4x_{a_0}x_{a_2}x_{b_2}x_{b_3}y_{a_0} - 4x_{a_3}^2x_{b_0}x_{b_3}y_{a_0} + 4x_{a_2}x_{a_3}x_{b_0}x_{b_2}y_{a_0}
\end{aligned}
$$

$$
\begin{aligned}
g_{24} \quad = \quad & \rho_1^2 x_{b_3} y_{a_2} y_{b_3} - \rho_2^2 x_{b_3} y_{a_2} y_{b_3} + \rho_5^2 x_{b_0} y_{a_2} y_{b_3} - \rho_4^2 x_{b_0} y_{a_2} y_{b_3} \\
& + \rho_2^2 x_{b_3} y_{a_0} y_{b_3} - \rho_1^2 x_{b_3} y_{a_0} y_{b_3} - \rho_5^2 x_{b_0} y_{a_0} y_{b_3} + \rho_4^2 x_{b_0} y_{a_0} y_{b_3} \\
& + \rho_2^2 x_{b_3} y_{a_3} y_{b_2} - \rho_1^2 x_{b_3} y_{a_3} y_{b_2} - \rho_5^2 x_{b_0} y_{a_3} y_{b_2} + \rho_4^2 x_{b_0} y_{a_3} y_{b_2} \\
& - \rho_2^2 x_{b_3} y_{a_0} y_{b_2} + \rho_1^2 x_{b_3} y_{a_0} y_{b_2} + \rho_5^2 x_{b_0} y_{a_0} y_{b_2} - \rho_4^2 x_{b_0} y_{a_0} y_{b_2} \\
& - \rho_2^2 x_{b_3} y_{a_3} y_{b_0} + \rho_1^2 x_{b_3} y_{a_3} y_{b_0} + \rho_5^2 x_{b_0} y_{a_3} y_{b_0} - \rho_4^2 x_{b_0} y_{a_3} y_{b_0} \\
& + \rho_2^2 x_{b_3} y_{a_2} y_{b_0} - \rho_1^2 x_{b_3} y_{a_2} y_{b_0} - \rho_5^2 x_{b_0} y_{a_2} y_{b_0} + \rho_4^2 x_{b_0} y_{a_2} y_{b_0}
\end{aligned}
$$

$$
\begin{aligned}
g_{31} \quad = \quad & 4x_{a_2} x_{a_3} x_{b_0} y_{b_3}^2 - 4x_{a_0} x_{a_3} x_{b_2} y_{b_3}^2 + 4x_{a_0} x_{a_2} x_{b_3} y_{b_2} y_{b_3} - 4x_{a_2} x_{a_3} x_{b_0} y_{b_2} y_{b_3} \\
& - 4x_{a_0} x_{a_2} x_{b_3} y_{b_0} y_{b_3} + 8x_{a_0} x_{a_3} x_{b_2} y_{b_0} y_{b_3} - 4x_{a_2} x_{a_3} x_{b_0} y_{b_0} y_{b_3} - 4x_{a_0} x_{a_2} x_{b_3} y_{b_0} y_{b_2} \\
& + 4x_{a_2} x_{a_3} x_{b_0} y_{b_0} y_{b_2} + 4x_{a_0} x_{a_2} x_{b_3} y_{b_0}^2 - 4x_{a_0} x_{a_3} x_{b_2} y_{b_0}^2
\end{aligned}
$$

$$
\begin{aligned}
g_{32} \quad = \quad & 4x_{b_0} x_{b_3} y_{a_0} y_{a_3} y_{b_3} - 4x_{b_0} x_{b_3} y_{a_2} y_{a_3} y_{b_3} + 4x_{b_0} x_{b_3} y_{a_0} y_{a_2} y_{b_3} - 4x_{b_0} x_{b_3} y_{a_0}^2 y_{b_3} \\
& + 4x_{a_0} x_{a_2} x_{b_2} x_{b_3} y_{b_3} - 4x_{a_0}^2 x_{b_0} x_{b_3} y_{b_3} - 4x_{a_2} x_{a_3} x_{b_0} x_{b_2} y_{b_3} + 4x_{a_0} x_{a_3} x_{b_0}^2 y_{b_3} \\
& + 4x_{b_0} x_{b_3} y_{a_3}^2 y_{b_2} - 8x_{b_0} x_{b_3} y_{a_0} y_{a_3} y_{b_2} + 4x_{b_0} x_{b_3} y_{a_0}^2 y_{b_2} - 4x_{a_0} x_{a_3} x_{b_3}^2 y_{b_2} \\
& + 4x_{a_3}^2 x_{b_0} x_{b_3} y_{b_2} + 4x_{a_0}^2 x_{b_0} x_{b_3} y_{b_2} - 4x_{a_0} x_{a_3} x_{b_0}^2 y_{b_2} - 4x_{b_0} x_{b_3} y_{a_3}^2 y_{b_0} \\
& + 4x_{b_0} x_{b_3} y_{a_2} y_{a_3} y_{b_0} + 4x_{b_0} x_{b_3} y_{a_0} y_{a_3} y_{b_0} - 4x_{b_0} x_{b_3} y_{a_0} y_{a_2} y_{b_0} + 4x_{a_0} x_{a_3} x_{b_3}^2 y_{b_0} \\
& - 4x_{a_0} x_{a_2} x_{b_2} x_{b_3} y_{b_0} - 4x_{a_3}^2 x_{b_0} x_{b_3} y_{b_0} + 4x_{a_2} x_{a_3} x_{b_0} x_{b_2} y_{b_0}
\end{aligned}
$$

$$
\begin{aligned}
g_4 \quad = \quad & \rho_2^2 x_{a_3} y_{a_2} y_{b_3} - \rho_1^2 x_{a_3} y_{a_2} y_{b_3} - \rho_5^2 x_{a_0} y_{a_2} y_{b_3} + \rho_4^2 x_{a_0} y_{a_2} y_{b_3} \\
& - \rho_2^2 x_{a_3} y_{a_0} y_{b_3} + \rho_1^2 x_{a_3} y_{a_0} y_{b_3} + \rho_5^2 x_{a_0} y_{a_0} y_{b_3} \\
& - \rho_4^2 x_{a_0} y_{a_0} y_{b_3} - \rho_2^2 x_{a_3} y_{a_3} y_{b_2} + \rho_1^2 x_{a_3} y_{a_3} y_{b_2} \\
& + \rho_5^2 x_{a_0} y_{a_3} y_{b_2} - \rho_4^2 x_{a_0} y_{a_3} y_{b_2} + \rho_2^2 x_{a_3} y_{a_0} y_{b_2} \\
& - \rho_1^2 x_{a_3} y_{a_0} y_{b_2} - \rho_5^2 x_{a_0} y_{a_0} y_{b_2} + \rho_4^2 x_{a_0} y_{a_0} y_{b_2} \\
& + \rho_2^2 x_{a_3} y_{a_3} y_{b_0} - \rho_1^2 x_{a_3} y_{a_3} y_{b_0} - \rho_5^2 x_{a_0} y_{a_3} y_{b_0} \\
& + \rho_4^2 x_{a_0} y_{a_3} y_{b_0} - \rho_2^2 x_{a_3} y_{a_2} y_{b_0} + \rho_1^2 x_{a_3} y_{a_2} y_{b_0} + \rho_5^2 x_{a_0} y_{a_2} y_{b_0} - \rho_4^2 x_{a_0} y_{a_2} y_{b_0} \\
g_5 \quad = \quad & 2U_{24} x_{a_3} x_{b_0} y_{a_2} - \rho_4^2 x_{a_0} x_{b_2} y_{a_0} - \rho_5^2 x_{a_3} x_{b_0} y_{a_0} \\
& - \rho_4^2 x_{a_3} x_{b_0} y_{a_0} - \rho_5^2 x_{a_2} x_{b_0} y_{a_0} + \rho_4^2 x_{a_2} x_{b_0} y_{a_0} \\
& - \rho_1^2 x_{a_3} x_{b_2} y_{a_3} + \rho_2^2 x_{a_3} x_{b_0} y_{a_3} + \rho_4^2 x_{a_0} x_{b_2} y_{a_3} \\
& - \rho_5^2 x_{a_0} x_{b_3} y_{a_2} - \rho_4^2 x_{a_0} x_{b_3} y_{a_2} + \rho_2^2 x_{a_0} x_{b_3} y_{a_2} \\
& - \rho_1^2 x_{a_3} x_{b_0} y_{a_2} + \rho_2^2 x_{a_2} x_{b_3} y_{a_0} + 2U_{24} x_{a_0} x_{b_3} y_{a_3} \\
& + 2U_{65} x_{a_0} x_{b_3} y_{a_0} - 2\rho_6^2 x_{a_0} x_{b_3} y_{a_0} - 2U_{65} x_{a_3} x_{b_0} y_{a_0} \\
& + 2\rho_6^2 x_{a_3} x_{b_0} y_{a_0} + \rho_1^2 x_{a_3} x_{b_0} y_{a_3} + \rho_5^2 x_{a_2} x_{b_0} y_{a_3} \\
& - \rho_4^2 x_{a_2} x_{b_0} y_{a_3} - \rho_1^2 x_{a_2} x_{b_3} y_{a_0} + \rho_5^2 x_{a_0} x_{b_3} y_{a_0} \\
& - 2U_{65} x_{a_0} x_{b_3} y_{a_3} + 2\rho_6^2 x_{a_0} x_{b_3} y_{a_3} + 2U_{65} x_{a_3} x_{b_0} y_{a_3} \\
& - 2U_{24} x_{a_3} x_{b_0} y_{a_3} - 2\rho_6^2 x_{a_3} x_{b_0} y_{a_3} - 2U_{24} x_{a_0} x_{b_3} y_{a_2} \\
& + \rho_1^2 x_{a_0} x_{b_3} y_{a_2} + \rho_5^2 x_{a_3} x_{b_0} y_{a_2} + \rho_4^2 x_{a_3} x_{b_0} y_{a_2} \\
& - \rho_2^2 x_{a_3} x_{b_0} y_{a_2} + \rho_4^2 x_{a_0} x_{b_3} y_{a_0} - \rho_2^2 x_{a_3} x_{b_2} y_{a_0} \\
& + \rho_1^2 x_{a_3} x_{b_2} y_{a_0} + \rho_5^2 x_{a_0} x_{b_2} y_{a_0} - \rho_2^2 x_{a_2} x_{b_3} y_{a_3} \\
& + \rho_1^2 x_{a_2} x_{b_3} y_{a_3} - \rho_1^2 x_{a_0} x_{b_3} y_{a_3} - \rho_2^2 x_{a_0} x_{b_3} y_{a_3} - \rho_5^2 x_{a_0} x_{b_2} y_{a_3} + \rho_2^2 x_{a_3} x_{b_2} y_{a_3}
\end{aligned}
$$

We notice that:

$$
\begin{aligned}
u_{11} &= u_{22} && (34)\\
u_{12} &= u_{21} && (35)\\
q_{11} &= g_{11} && (36)\\
q_{12} &= g_{21} && (37)\\
v_{111} &= g_{31} && (38)
\end{aligned}
$$

10 Appendix 3: Equation 10

From the system:

$$u_1 \cos\theta + u_2 = 0 \tag{39}$$

$$v_1 \cos\theta + v_2 = 0 \tag{40}$$

we get the equation in the variables ψ, ϕ:

$$u_1 v_2 - u_2 v_1 = 0 \tag{41}$$

which can be expressed in ψ by:

$$
\begin{aligned}
& p_1 \sin^3\psi + p_2 \cos^3\psi + (p_{31}\cos\psi + p_{32})\sin^2\psi \\
& +(p_{41}\sin\psi + p_{42})\cos^2\psi + \sin\psi(p_{51}\cos\psi + p_{52}) + p_6\cos\psi = 0 && (42)\\
& && (43)
\end{aligned}
$$

The coefficients p_i can be written as:

$$\Longrightarrow \quad p_1 = x_1 \cos^2\phi + x_2 \cos\phi \sin\phi \tag{44}$$

with:

$$
\begin{aligned}
x_1 &= u_{11}v_{212} - u_{12}v_{111}\\
x_2 &= u_{11}v_{211} - u_{12}v_{112}
\end{aligned}
$$

$$\Longrightarrow \quad p_2 = x_3 \sin^2\phi \tag{45}$$

where:

$$x_3 = u_{12}v_{221} - u_{11}v_{121}$$

$$\Longrightarrow \quad p_{31} = x_4 \cos^2\phi + x_5 \sin^2\phi + x_6 \cos\phi \sin\phi \tag{46}$$

with:

$$
\begin{aligned}
x_4 &= g_{32}u_{11} - q_{21}u_{12}\\
x_5 &= u_{12}v_{211} - u_{11}v_{112}\\
x_6 &= u_{12}v_{212} - q_{31}u_{12}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad p_{32} &= x_7 \cos^3\phi + x_8 \sin\phi \cos^2\phi + x_9 \cos\phi \sin^2\phi + \\
& \quad x_{10}\cos\phi + x_{11}\sin\phi && (47)
\end{aligned}
$$

with

$$
\begin{aligned}
x_7 &= q_{11}u_{11} \\
x_8 &= g_{12}u_{11} - q_{12}u_{12} \\
x_9 &= -u_{12}v_{1311} \\
x_{10} &= -u_{23}v_{111} - q_{41}u_{12} + g_{13}u_{11} \\
x_{11} &= -u_{23}v_{112}
\end{aligned}
$$

and

$$
\Longrightarrow \quad p_{41} = x_{12}\sin^2\phi + x_{13}\cos\phi\sin\phi \tag{48}
$$

where

$$
\begin{aligned}
x_{12} &= u_{12}v_{111} - q_{31}u_{11} \\
x_{13} &= u_{11}v_{221} - u_{12}v_{121} + g_{32}u_{12} - q_{21}u_{11}
\end{aligned}
$$

$$
\Longrightarrow \quad p_{42} = x_{14}\sin^3\phi + x_{15}\cos\phi\sin^2\phi + x_{16}\sin\phi \tag{49}
$$

with:

$$
\begin{aligned}
x_{14} &= q_{12}u_{12} - u_{11}v_{1312} \\
x_{15} &= g_{23}u_{12} - q_{11}u_{11} \\
x_{16} &= g_{24}u_{12} - u_{23}v_{121}
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \quad p_{51} &= x_{17}\sin^3\phi + x_{18}\cos\phi\sin^2\phi + x_{19}\sin\phi\cos^2\phi \\
&\quad + x_{20}\sin\phi + x_{21}\cos\phi
\end{aligned} \tag{50}
$$

where:

$$
\begin{aligned}
x_{17} &= -u_{11}v_{1311} \\
x_{18} &= g_{12}u_{12} - u_{12}v_{1312} \\
x_{19} &= g_{23}u_{11} \\
x_{20} &= g_{13}u_{12} - q_{31}u_{23} - q_{41}u_{11} \\
x_{21} &= g_{24}u_{11} - q_{21}u_{23}
\end{aligned}
$$

$$
\Longrightarrow \quad p_{52} = x_{22}\sin^2\phi + x_{23}\cos^2\phi + x_{24}\cos\phi\sin\phi + x_{25} \tag{51}
$$

with:

$$
\begin{aligned}
x_{22} &= -u_{23}v_{1311} \\
x_{23} &= g_4u_{11} \\
x_{24} &= g_5u_{11} - q_{12}u_{23} \\
x_{25} &= -q_{41}u_{23}
\end{aligned}
$$

$$\Longrightarrow \quad p_6 = x_{26}\sin^2\phi + x_{27}\cos\phi\sin\phi \tag{52}$$

where:

$$\begin{aligned} x_{26} &= g_5 u_{12} - u_{23} v_{1312} \\ x_{27} &= g_4 u_{12} - q_{11} u_{23} \end{aligned}$$

Using the tangent of the the half angle equation (42) yield to a six order polynomial:

$$\sum_{i=0}^{i=6} e_i x^i = 0 \tag{53}$$

with

$$\begin{aligned} e_0 &= p_6 + p_{42} + p_2 \\ e_1 &= 2p_{52} + 2p_{51} + 2p_{41} \\ e_2 &= p_6 - p_{42} + 4p_{32} + 4p_{31} - 3p_2 \\ e_3 &= 4p_{52} - 4p_{41} + 8p_1 \\ e_4 &= -p_6 - p_{42} + 4p_{32} - 4p_{31} + 3p_2 \\ e_5 &= 2p_{52} - 2p_{51} + 2p_{41} \\ e_6 &= p_{42} - p_6 - p_2 \end{aligned}$$

11 Appendix 4: Discretization of the ϕ-curve

We will show that the discretization on ϕ is necessary only in the range $[0,\pi]$. We notice that if $[\phi, \psi]$ is a solution of equation (53) then $[\phi + \pi, \psi + \pi]$ is also a solution of this equation. Then we notice that:

$$\begin{aligned} u_1(\psi,\phi) &= u_1(\psi+\pi, \phi+\pi) \\ u_2(\psi,\phi) &= u_2(\psi+\pi, \phi+\pi) \\ v_1(\psi,\phi) &= -v_1(\psi+\pi, \phi+\pi) \\ v_2(\psi,\phi) &= -v_2(\psi+\pi, \phi+\pi) \end{aligned}$$

and therefore the solution in θ obtained from equation (8) are identical. If $[\psi,\theta,\phi]$, $[\psi,-\theta,\phi]$ are solutions then $[\psi+\pi,\theta,\phi+\pi]$, $[\psi+\pi,-\theta,\phi+\pi]$ are also solutions. The rotation matrix R is such that:

$$R(\psi,\theta,\phi) = R(\psi+\pi,-\theta,\phi+\pi)$$

which yield to:

$$\underline{H} \;\; ((\psi,\theta,\phi) = \underline{H} \;\; (\psi+\pi,-\theta,\phi+\pi)$$

The configurations associated to (ψ,θ,ϕ), $(\psi+\pi,-\theta,\phi+\pi)$ are therefore identical. In the same way:

$$R(\psi,-\theta,\phi) = R(\psi+\pi,\theta,\phi+\pi)$$

and therefore:

$$\underline{H} \;\; ((\psi,-\theta,\phi) = \underline{H} \;\; (\psi+\pi,\theta,\phi+\pi)$$

The configurations associated to $(\psi,-\theta,\phi)$, $(\psi+\pi,\theta,\phi+\pi)$ are therefore identical. This justify that the discretization is necessary only in the range $[0,\pi]$.

12 Appendix 5: Order of equation (15)

We want to determine the order of the equation:

$$\rho_1^2(\psi, \theta, \phi) = \rho_1^2$$

The left part of this equation is defined by the norm of the vector $\underline{A_1 B_1}$. Thus we study each componant of this vector and calculate its higher order term only.

12.1 Study of $A_1B_1[1]$

We have:

$$A_1B_1[1] = \frac{nt1}{dt1}$$

with:

$$dt1 = 4(x_{a_3}x_{b_0} - x_{a_0}x_{b_3})$$

and:

$$nt1 = w_1 \sin\phi + w_2 \cos\phi + w_3$$

with $w_3 = \mathrm{c}^{\mathrm{te}}$.

$$\begin{aligned} w_1 &= \cos\theta(w_{11}\sin\psi + w_{12}\cos\psi) + w_{13}\cos\psi \\ w_2 &= \cos\theta(w_{21}\sin\psi) + w_{22}\cos\psi + w_{23} \end{aligned}$$

with $w_{22} = -w_{11}$, $w_{12} = w_{23}$, $w_{13} = w_{21}$, all these terms being constant.

12.1.1 Degree of $A_1B_1[1]$

$$w_1 = \frac{z^2}{(1+z^2)}\left(\frac{x}{(1+x^2)} + \frac{x^2}{(1+x^2)}\right) + \frac{x^2}{(1+x^2)} = \frac{x^2z^2}{(1+x^2)(1+z^2)}$$

$$w_2 = \frac{z^2}{(1+z^2)}\frac{x}{(1+x^2)} + \frac{x^2}{(1+x^2)} + w_{23} = \frac{x^2z^2}{(1+z^2)(1+x^2)}$$

which yield to:

$$nt1 = \frac{x^2z^2y}{(1+x^2)(1+z^2)(1+y^2)} + \frac{x^2z^2y^2}{(1+x^2)(1+z^2)(1+y^2)} + w_3$$

therefore:

$$\frac{nt1}{dt1} = \frac{x^2y^2z^2}{(1+x^2)(1+y^2)(1+z^2)}$$

12.2 Study of $A_1B_1[2]$

We have:

$$A_1B_1[2] = \frac{nt2}{dt2}$$

with:

$$dt2 = pp_1 \sin\psi + pp_2 \sin\phi$$

where:

$$\begin{aligned}
pp_1 &= 4(x_{a_0}x_{b_3}y_{b_3} - x_{a_0}x_{b_3}y_{b_2} - x_{b_0}x_{a_3}y_{b_3} + x_{b_0}x_{a_3}y_{b_2}) \\
pp_2 &= 4(x_{a_0}x_{b_3}y_{a_3} - x_{a_0}x_{b_3}y_{a_2} - x_{b_0}x_{a_3}y_{a_3} + x_{b_0}x_{a_3}y_{a_2})
\end{aligned}$$

We have:

$$nt2 = pp_3 \sin\phi + pp_4 \cos\phi + pp_5$$

where pp_3, pp_4 are constant.

$$pp_5 = pq_1 \cos\theta + pq_2$$

with:

$$pq_1 = pr_1 \sin^2\phi + \sin\phi(pr_3\cos\phi + pr_4) + pr_6\cos\phi + pr_7$$

where:

$$\begin{aligned}
pr_1 &= ps_1 \sin\psi + ps_2\cos\psi \\
pr_3 &= ps_3 \sin\psi + ps_4\cos\psi \\
pr_4 &= ps_5 \cos^2\psi + ps_6\cos\psi\sin\psi \\
pr_6 &= ps_7 \cos\psi\sin\psi \\
pr_7 &= ps_8 \sin\psi
\end{aligned}$$

in which the terms ps_j are constant. And:

$$pq_2 = qr_1 \sin^2\phi + \sin\phi(qr_3\cos\phi + qr_4) + qr_6\cos\phi + qr_7$$

with:

$$\begin{aligned}
qr_1 &= qs_1 \sin\psi + qs_2\cos\psi \\
qr_3 &= qs_3 \sin\psi + qs_4\cos\psi \\
qr_4 &= qs_5 \sin^2\psi + qs_6 \\
qr_6 &= qs_7 \sin^2\psi + qs_8\sin\psi\cos\psi \\
qr_7 &= qs_9 \sin\psi + qs_{10}\cos\psi
\end{aligned}$$

in which the terms qs_i are constants.

12.2.1 Degree of $A_1B_1[2]$

We have:

$$dt2 = pp1\frac{x}{(1+x^2)} + pp2\frac{y}{(1+y^2)}$$

which yield to:

$$dt2 = \frac{pp1x(1+y^2) + pp2y(1+x^2)}{(1+x^2)(1+y^2)}$$

Then:

$$\begin{aligned}
qr_7 &= \frac{x^2}{(1+x^2)} \\
qr_6 &= \frac{x^3}{(1+x^2)^2}
\end{aligned}$$

$$qr_4 = \frac{x^4}{(1+x^2)^2}$$
$$qr_3 = \frac{x^2}{(1+x^2)}$$
$$qr_1 = \frac{x^2}{(1+x^2)}$$

$$\begin{aligned} pq_2 \quad = \quad & \frac{x^2}{(1+x^2)}\frac{y^2}{(1+y^2)^2} + \frac{y}{(1+y^2)}(\frac{x^2}{(1+x^2)}\frac{y^2}{(1+y^2)} + \\ & \frac{x^4}{(1+x^2)^2}) + \frac{x^3}{(1+x^2)^2}\frac{y^2}{(1+y^2)} + \frac{x^2}{(1+x^2)} \end{aligned}$$

thus:

$$pq_2 = \frac{x^4y^4}{(1+x^2)^2(1+y^2)^2}$$

Then:

$$pr_7 = \frac{x}{(1+x^2)}$$
$$pr_6 = \frac{x^3}{(1+x^2)^2}$$
$$pr_4 = \frac{x^4}{(1+x^2)^2}$$
$$pr_3 = \frac{x^2}{(1+x^2)}$$
$$pr_1 = \frac{x^2}{(1+x^2)}$$

Hence:

$$\begin{aligned} pq_1 \quad = \quad & \frac{x^2}{(1+x^2)}\frac{y^2}{(1+y^2)^2} + \frac{y}{(1+y^2)}(\frac{x^2}{(1+x^2)}\frac{y^2}{(1+y^2)} + \\ & \frac{x^4}{(1+x^2)^2}) + \frac{x^3}{(1+x^2)^2}\frac{y^2}{(1+y^2)} + \frac{x}{(1+x^2)} \end{aligned}$$

and therefore:

$$pq_1 = \frac{x^3y^4 + x^4y^3}{(1+x^2)^2(1+y^2)^2}$$

This yield to:

$$pp_5 = \frac{x^3y^4 + x^4y^3}{(1+x^2)^2(1+y^2)^2}\frac{z^2}{(1+z^2)} + \frac{x^4y^4}{(1+x^2)^2(1+y^2)^2}$$

Hence:

$$pp_5 = \frac{x^4y^4z^2}{(1+x^2)^2(1+y^2)^2(1+z^2)}$$

Therefore:

$$nt2 = \frac{z}{(1+z^2)} + \frac{z^2}{(1+z^2)} + \frac{x^4y^4z^2}{(1+x^2)^2(1+y^2)^2(1+z^2)}$$

which yield to:

$$nt2 = \frac{x^4y^4z^2}{(1+x^2)^2(1+y^2)^2(1+z^2)}$$

The final result is:

$$\frac{nt2}{dt2} = \frac{x^4y^4z^2}{(1+x^2)(1+y^2)(1+z^2)(pp_1xy^2+pp_2x^2y+pp_2y+pp_1x)}$$

12.3 Study of $A_1B_1[3]$

We have:

$$A_1B_1[3] = \frac{nt3}{dt3}$$

where:

$$dt3 = \sin\theta(r_1\sin\phi + r_2\sin\psi)$$

with

$$\begin{aligned}
r_1 &= 4(-x_{a_0}x_{b_3}y_{a_3} + x_{a_0}x_{b_3}y_{a_2} + x_{b_0}x_{a_3}y_{a_3} - x_{b_0}x_{a_3}y_{a_2}) \\
r_2 &= 4(-x_{a_0}x_{b_3}y_{b_3} + x_{a_0}x_{b_3}y_{b_2} + x_{b_0}x_{a_3}y_{b_3} - x_{b_0}x_{a_3}y_{b_2})
\end{aligned}$$

Then:

$$nt3 = qw_1\cos^2\theta + qw_2\cos\theta + qw_3\sin^2\theta + qw_4$$

where:

$$qw_1 = qsw_1\sin^2\phi + qsw_2\cos\phi\sin\phi + qsw_3\cos\phi + qsw_4\sin\phi + qsw_5$$

with:

$$\begin{aligned}
qsw_5 &= psw_1\cos\psi\sin\psi \\
qsw_4 &= psw_2\cos\psi \\
qsw_2 &= psw_4\cos^2\psi + psw_5\cos\psi\sin\psi + psw_6 \\
qsw_3 &= psw_3\sin\psi \\
qsw_1 &= psw_7\cos^2\psi + psw_8\cos\psi\sin\psi
\end{aligned}$$

the terms psw_i being constant. Then we have:

$$qw_2 = rw_1\cos^2\phi + rw_2\cos\phi\sin\phi + rw_3\cos\phi + rw_4\sin\phi + rw_5$$

with:

$$\begin{aligned}
rw_1 &= rsw_1\cos^2\psi + rsw_2\cos\psi\sin\psi + rsw_3 \\
rw_2 &= rsw_4\cos^2\psi + rsw_5\cos\psi\sin\psi + rsw_6 \\
rw_3 &= rsw_7\cos\psi + rsw_8\sin\psi \\
rw_4 &= rsw_9\cos\psi + rsw_{10}\sin\psi \\
rw_5 &= rsw_{11}\cos^2\psi + rsw_{12}\cos\psi\sin\psi + rsw_{13}
\end{aligned}$$

in which the terms rsw_i are constants. Then:

$$qw_3 = tw_1\sin^2\phi + tw_2\cos\phi\sin\phi + tw_3\cos\phi + tw_4\sin\phi$$

where $tw_1 = \mathrm{c}^{\mathrm{te}}$ and $tw_2 = \mathrm{c}^{\mathrm{te}}$

$$
\begin{aligned}
tw_3 &= tw_{31} \sin\psi \\
tw_4 &= tw_{32} \sin\psi
\end{aligned}
$$

with tw_{31}, tw_{32} constants. Finally:

$$qw_4 = yw_1 \sin^2\phi + yw_2 \cos^2\phi + yw_3 \cos\phi \sin\phi + yw_4 \cos\phi + yw_5 \sin\phi + yw_6$$

with

$$
\begin{aligned}
yw_1 &= yw_{11} \cos\psi \sin\psi \\
yw_2 &= yw_{21} \sin^2\psi + yw_{22} \cos\psi \sin\psi \\
yw_3 &= yw_{31} \cos^2\psi + yw_{32} \sin^2\psi + yw_{33} \cos\psi \sin\psi \\
yw_4 &= yw_{41} \cos\psi + yw_{42} \sin\psi \\
yw_5 &= yw_{51} \cos\psi + yw_{52} \sin\psi \\
yw_6 &= \mathrm{c}^{\mathrm{te}}
\end{aligned}
$$

where yw_{ij} are constants.

12.3.1 Degree of $A_1B_1[3]$

$$
\begin{aligned}
yw_5 &= \frac{x^2}{(1+x^2)} \\
yw_4 &= \frac{x^2}{(1+x^2)} \\
yw_3 &= \frac{x^4}{(1+x^2)^2} \\
yw_2 &= \frac{x^3}{(1+x^2)^2} \\
yw_1 &= \frac{x^3}{(1+x^2)^2}
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
qw_4 \quad = \quad & \frac{x^3}{(1+x^2)^2}\frac{y^2}{(1+y^2)^2} + \frac{x^3}{(1+x^2)^2}\frac{y^4}{(1+y^2)^2} + \frac{x^4}{(1+x^2)^2}\frac{y^3}{(1+y^2)^2} \\
& + \frac{x^2}{(1+x^2)}\frac{y^2}{(1+y^2)} + \frac{x^2}{(1+x^2)}\frac{y}{(1+y^2)} + yw_6
\end{aligned}
$$

hence:

$$qw_4 = \frac{x^4y^4}{(1+x^2)^2(1+y^2)^2}$$

Then:

$$
\begin{aligned}
tw_3 &= \frac{x}{(1+x^2)} \\
tw_4 &= \frac{x}{(1+x^2)}
\end{aligned}
$$

which yield to:

$$qw_3 = \frac{y^2}{(1+y^2)^2} + \frac{y^3}{(1+y^2)^2} + \frac{x}{(1+x^2)}\frac{y^2}{(1+y^2)} + \frac{x}{(1+x^2)}\frac{y}{(1+y^2)}$$

therefore:

$$qw_3 = \frac{x^2y^3}{(1+x^2)(1+y^2)^2}$$

Then:

$$rw_5 = \frac{x^4}{(1+x^2)^2}$$
$$rw_4 = \frac{x^2}{(1+x^2)}$$
$$rw_3 = \frac{x^2}{(1+x^2)}$$
$$rw_2 = \frac{x^4}{(1+x^2)^2}$$
$$rw_1 = \frac{x^4}{(1+x^2)^2}$$

$$\begin{aligned} qw_2 &= \frac{x^4}{(1+x^2)^2}\frac{y^4}{(1+y^2)^2} + \frac{x^4}{(1+x^2)^2}\frac{y^3}{(1+y^2)^2} + \frac{x^2}{(1+x^2)}\frac{y^2}{(1+y^2)} \\ &\quad + \frac{x^2}{(1+x^2)}\frac{y}{(1+y^2)} + \frac{x^4}{(1+x^2)^2} \end{aligned}$$

which yield to:

$$qw_2 = \frac{x^4y^4}{(1+x^2)^2(1+y^2)^2}$$

Then:

$$qsw_1 = \frac{x^4}{(1+x^2)^2}$$
$$qsw_3 = \frac{x}{(1+x^2)}$$
$$qsw_2 = \frac{x^4}{(1+x^2)^2}$$
$$qsw_4 = \frac{x^2}{(1+x^2)}$$
$$qsw_5 = \frac{x^3}{(1+x^2)^2}$$

$$\begin{aligned} qw_1 &= \frac{x^4}{(1+x^2)^2}\frac{y^2}{(1+y^2)^2} + \frac{x^4}{(1+x^2)^2}\frac{y^3}{(1+y^2)^2} + \frac{x}{(1+x^2)}\frac{y^2}{(1+y^2)} \\ &\quad + \frac{x^2}{(1+x^2)}\frac{y}{(1+y^2)} + \frac{x^3}{(1+x^2)^2} \end{aligned}$$

Therefore:

$$qw_1 = \frac{x^3y^4 + x^4y^3}{(1+x^2)^2(1+y^2)^2}$$

$$\begin{aligned} nt3 \quad = \quad & \frac{x^3y^4 + x^4y^3}{(1+x^2)^2(1+y^2)^2}\frac{z^4}{(1+z^2)^2} + \frac{x^4y^4}{(1+x^2)^2(1+y^2)^2}\frac{z^2}{(1+z^2)} \\ & + \frac{x^2y^3}{(1+x^2)^2(1+y^2)^2}\frac{z^2}{(1+z^2)^2} + \frac{x^4y^4}{(1+x^2)^2(1+y^2)^2} \end{aligned}$$

which is equivalent to:

$$nt3 = \frac{x^4y^4z^4}{(1+x^2)^2(1+y^2)^2(1+z^2)^2}$$

$$dt3 = \frac{r_1zy(1+x^2) + r_2zx(1+y^2)}{(1+x^2)(1+y^2)(1+z^2)}$$

which yield to:

$$\frac{nt3}{dt3} = \frac{x^4y^4z^4}{z(1+x^2)(1+y^2)(1+z^2)(r_1y(1+x^2) + r_2x(1+y^2))}$$

12.4 Summary

We define:

$$U = (1+x^2)^2(1+y^2)^2(1+z^2)^2$$

We have:

$$A_1B_1[1]^2 = \frac{x^4y^4z^4}{U}$$

$$A_1B_1[2]^2 = \frac{x^8y^8z^4}{U(pp_1x(1+y^2) + pp_2y(1+x^2)y)^2}$$

$$A_1B_1[3]^2 = \frac{x^8y^8z^6}{U(r_2x(1+y^2) + r_1y(1+x^2))^2}$$

But $pp_1 = r_2, pp_2 = r_1$ and therefore the common denominator will be:

$$U_1 = U(r_2x(1+y^2) + r_1y(1+x^2))^2$$

Or, if we define U_{11} as:

$$U_{11} = (r_2x(1+y^2) + r_1y(1+x^2))^2 \quad \Longrightarrow U_1 = UU_{11}$$

Let us calculate:

$$U_2 = ||A_1B_1||^2 - \rho_1^2 = 0$$

We have:

$$U_2 = \frac{x^4y^4z^4U_{11}}{UU_{11}} + \frac{x^8y^8z^4}{UU_{11}} + \frac{x^8y^8z^6}{UU_{11}} - \frac{\rho_1^2UU_{11}}{UU_{11}}$$

We may write:

$$U_{11} = (xy^2 + x^2y)$$

hence we have:

$$x^5y^6z^4 + x^6y^5z^4 + x^8y^8z^4 + x^8y^8z^6 - \rho_1^2x^5y^6z^4 - \rho_1^2x^6y^5z^4 = 0$$

The order of the higher terms are thus: 15, 15, 20, 22, 15, 15. Therefore the order will be 22.

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