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# Random walks in a quarter plane with zero drifts. I : Ergodicity and null recurrence

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**Abstract**

In this paper, we solve the problem of non ergodicity and null recurrence for random walks in the quarter plane with zero drifts in the interior of the domain. A general criterion for null recurrence is given and then used to construct sub and supermartingales by means of Lyapounov functions, which are here functionals of quadratic forms.

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# Marches aléatoires dans un quart de plan avec dérives nulles I : Ergodicité et recurrence nulle

Guy Fayolle[*], Vadim A. Malyshev[†], Michael V. Menshikov[†]

**Résumé**

Dans cet article nous résolvons le problème de non ergodicité et de recurrence nulle pour des marches aléatoires dans le quart de plan, lorsque les dérives moyennes sont nulles à l'intérieur de cette région. Un critère général de récurrence nulle est donné et utilisé pour construire des (sous) surmartingales par l'intermédiaire de fonctions de Lyapounov, qui sont ici en général des fonctionnelles de formes quadratiques.

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# 1 Introduction and preliminaries

We consider a discrete time homogeneous irreducible and aperiodic Markov chain $\mathcal{L} = \{\xi_\eta, n \geq 0\}$, with state space the lattice in the positive quarter plane $\mathbf{Z}_+^2 = \{(i,j) : i,j \geq 0 \ are\ integers\}$, and satisfying the recursive equation

$$\xi_{n+1} = [\xi_n + \theta_{n+1}]^+,$$

where the distribution of $\theta_{n+1}$ depends only on the position of $\xi_n$ in the following way ("maximal " space homogeneity)

$$p\{\theta_{n+1} = (i,j)/\xi_\eta = (k,l)\} = \begin{cases} p_{ij}, & for \quad k,l \geq 1, \\ p'_{ij}, & for \quad k \geq 1, l = 0, \\ p''_{ij}, & for \quad k = 0, l \geq 1, \\ p^0_{ij}, & for \quad k = l = 0. \end{cases}$$

Moreover we assume, for the one step transition probabilities, the following conditions :

A (*Lower boundedness*)

$$\begin{array}{llll} p_{ij} = 0, & if \quad i < -1 & or & j < -1 \ ; \\ p'_{ij} = 0, & if \quad i < -1 & or & j < 0 \ ; \\ p''_{ij} = 0, & if \quad i < 0 & or & j < -1 \ . \end{array}$$

B (*Moment condition*)

$$E[\| \theta_{n+1} \|^3 / \xi_n = (k,l)] \leq B < \infty, \ \ \forall (k,l) \in \mathbf{Z}_+^2,$$

where $\| z \|$, $z \in \mathbf{Z}_+^2$, denotes the euclidian norm and $\epsilon$ is an arbitrary but strictly positive number. In fact, as remarked later in section 3, only the existence of second moments is necessary. But the technical derivations are then more involved and will not be given here for the sake of clarity.

**Notation** : We shall use lower case greek letters $\alpha, \beta, ...$ to denote arbitrary, points of $\mathbf{Z}_+^2$, and then $p_{\alpha\beta}$ will mean the one step transition probabilities of the Markov $\mathcal{L}$. Also $\alpha > 0$ is equivalent to

$$\alpha_x > 0, \alpha_y > 0 \ for \ \alpha = (\alpha_x, \alpha_y).$$

Define the vector

$$M(\alpha) = (M_x(\alpha), M_y(\alpha))$$

of the one step mean jumps (drifts) from the point $\alpha$.

Setting

$$\alpha = (\alpha_x, \alpha_y), \quad \beta = (\beta_x, \beta_y),$$

we have

$$M_x(\alpha) = \sum_\beta p_{\alpha\beta}(\beta_x - \alpha_x),$$

$$M_y(\alpha) = \sum_\beta p_{\alpha\beta}(\beta_y - \alpha_y).$$

Condition $B$ ensures the existence of $M(\alpha)$, for all $\alpha \in \mathbf{Z}_+^2$. By the homogeneity condition $A$, only 4 different drift vectors take place :

$$M(\alpha) = \begin{cases} M, & for \quad \alpha = (\alpha_x, \alpha_y) > 0 \ , \\ M', & for \quad \alpha = (\alpha_x, 0), \ \ \alpha_x > 0 \ ; \\ M'', & for \quad \alpha = (0, \alpha_y), \ \ \alpha_y > 0 \ ; \\ M_0, & for \quad \alpha = (0,0) \ . \end{cases}$$

**Remark** :
i) all our results will be valid if one changes arbitrarily a finite number of transition probabilities.

ii) given $\xi_n = \alpha$, the components of $\theta_{n+1}$ might be taken bounded from below not by $-1$, but by some arbitrary number $-K > -\infty$ provided

- first, that we keep the maximal homogeneity for the drift vectors $M(\alpha)$ introduced above (i.e. 4 of them only are different) ;

- secondly, that the second moments and the covariance of the one step jumps inside $\mathbf{Z}_+^2$ (i.e. from any point $\alpha > 0$) are maintened constant. These last facts will emerge more clearly in the course of the study.

The classification problem for these random walks was studies first in [1] and, in the case of bounded jumps, completely solved except for the case $M = 0$. This was generalized later, in many papers, for unbounded jumps [3, 4, 5, 11].

Nothing was precisely known for the case $M = 0$, up to now, and, in fact, this problem in many aspects is of a very different nature. In particular, intuition does not give us the evidence that the random walk can exhibit an ergodic behaviour when $M = 0$.

To the author's knowledge there exist at least 3 methods which would allow to solve some particular cases of the problem :

1. Analytic approach [8]. Now there are only some preliminary results into this direction.

2. Semi-analytic approach, using well-known explicit results about one-dimensional random walks. E.g. when the random walks inside the quarter-plane is the "composition" of independent random walks along both axis, then it is easy to prove that the mean time to reach the boundary is infinite and thus, for any parameters $p'_{ij}, p''_{ij}$, the chain is not ergodic when $M = 0$.

3. Method of Lyapounov functions. It seems to be the most general approach and we use it here.

One of the main differences between the cases $M \neq 0$ and $M = 0$ is the following : the case $M \neq 0$ is in a sense locally linear and $M = 0$ is locally quadratic. The local second order effects are well caught by quadratic Lyapounov functions first introduced in [5]. But the global Lyapounov function is not quadratic and this causes additional difficulties. There are some general questions about constructive criteria of ergodicity, nonergodicity etc. We tried unsuccessfully to use the existing criteria for non ergodicity. They all seem to work only either for l-dimensional problems or for linear functions. One of our difficulties was to find a completely new one. We give it in section 2. We do not known whether it could be essentially generalized, but it allows to get the complete solution in our case.

For $M = 0$, we get the ergodicity conditions in terms of the second moments and the covariance of the one-step jumps inside $\mathbf{Z}_+^2$,

$$\lambda_x = \sum_{i,j} i^2 p_{ij} \ , \quad \lambda_y = \sum_{i,j} j^2 p_{ij} \ ,$$

$$R = \sum_{i,j} ij \ p_{ij} \ ,$$

and of the angles (counter clockwise oriented) $\phi_x, \phi_y$ (See Figure 1).

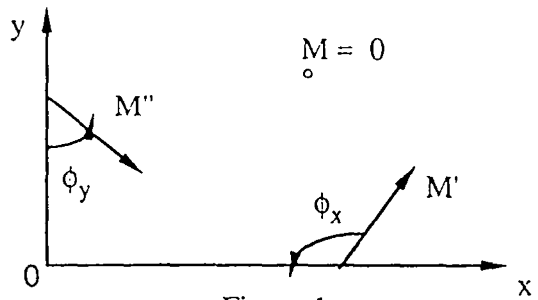


Figure 1

Here $\phi_x$ is the angle between $M'$ and the negative x-axis, $\phi_y$ is the angle between $M''$ and negative y-axis. So, if $\phi_x \neq \frac{\pi}{2}$ and $\phi_y \neq \frac{\pi}{2}$, then

$$tg \ \phi_x = -\frac{M'_y}{M'_x} \ , \quad tg \ \phi_y = -\frac{M''_x}{M''_y} \ .$$

**Theorem 1.1** *If at least one of the following 4 conditions holds*

$$(i) \quad \phi_x > \frac{\pi}{2},$$

$$(ii) \quad \phi_y > \frac{\pi}{2},$$

$$(iii) \quad \phi_x = \frac{\pi}{2}, \quad \phi_y \neq 0,$$

$$(iv) \quad \phi_y = \frac{\pi}{2}, \quad \phi_x \neq 0,$$

*then the random walk $\mathcal{L}$ is not ergodic.*

*If $\phi_x, \phi_y < \frac{\pi}{2}$, then the random walk $\mathcal{L}$ is ergodic if*

$$\lambda_x \ tg \ \phi_x + \lambda_y \ tg \ \phi_y + 2R \equiv -\lambda_x \frac{M'_y}{M'_x} - \lambda_y \frac{M''_x}{M''_y} + 2R < 0 \ . \qquad (1.1)$$

*and non ergodic if*

$$\lambda_x \, tg \, \phi_x + \lambda_y \, tg \, \phi_y + 2R > 0 \, . \tag{1.2}$$

*Moreover if (1.2) holds together with $\phi_x + \phi_y < \frac{\pi}{2}$, then the random walk is null recurrent.*

**Remark 1** : It follows easily from the statement of the theorem that the mean first entrance time of $\mathcal{L}$ into the boundary, when starting from some arbitrary point $\alpha > 0$ at finite distance, is finite [resp. infinite] if $R < 0$ [resp. $R > 0$], since in this case the vectors $M'$ and $M''$ can be properly chosen to satisfy (1.1) [resp. (1.2)].

**Remark 2** : It is clear from the formulation of the theorem that we do not consider the cases when either $\phi_x = \frac{\pi}{2}, \phi_y = 0$ or $\phi_y = \frac{\pi}{2}, \phi_x = 0$. Also, we miss the case

$$\lambda_x \, tg \, \phi_x + \lambda_y \, tg \, \phi_y + 2R = 0 \, .$$

The proof will be given in section 3.

# 2 General criteria for the null recurrence of countable Markov chains

Let us consider a discrete time homogeneous Markov chain $\mathcal{L}$ with a countable state space $X = \{x_n\}, n = 1, 2, \ldots$ The one step transition probabilites are denoted by $q_{x_i x_j}$. $\mathcal{L}$ is assumed to be irreducible and aperiodic. Let $\xi_n$ be the state of $\mathcal{L}$ at time $n$.
In this section, we recall some known criteria for recurrence, ergodicity and non ergodicity which will be used in the sequel. Meanwhile, theorem 2.5 is completely new and enables us to get explicit results in section 2.

**Theorem 2.1** *For the irreducible Markov chain $\mathcal{L}$ to be recurrent, it is sufficient there exists a positive function $f(x), x \in X$, such that*
*$E[f(\xi_{m+1}) - f(\xi_m)/\xi_m = x_i] \leq 0$, $i \notin A$, $f(x_i) \to \infty$ when $i \to \infty$, and $A$ is a finite set.*

**Proof** : see [7] ■

**Theorem 2.2** *(Foster). An irreducible aperiodic Markov chain $\mathcal{L}$ is ergodic if and only if there exists a positive function $f(x), x \in X, \epsilon > 0$ and a finite set $A$ such that*

$$E(f(\xi_{m+1}) - f(\xi_m)/\xi_m = x_i) \leq -\epsilon \ , \ i \notin A \ ,$$

$$E(f(\xi_{m+1})/\xi_m = x_i) < \infty \ , \ i \in A \ .$$

**Proof** : See [10]. For non trivial generalisations of this criterion, see [2].

**Theorem 2.3** *For an irreducible Markov chain $\mathcal{L}$ to be non ergodic, it is sufficient that there exist a function $f(x)$, $x \in X$, and constants $c, m$ such that :*

1. *$E(f(\xi_{n+1}) - f(\xi_n)/\xi_n = x) \geq 0$ , for every $n$, all $x \in \{x : f(x) > c\}$, where the sets $\{x : f(x) > c\}$ and $\{x : f(x) \leq c\}$ are non empty.*
2. *$E(\mid f(\xi_{n+1}) - f(\xi_n) \mid /\xi_n = x) \leq m$ for every $n$, $x \in X$.*

Under the stronger condition

$$\mid f(\xi_{n+1}) - f(\xi_n) \mid \leq \mathcal{D} \ ,$$

with probability 1, for some fixed constant $\mathcal{D}, 0 < \mathcal{D} < \infty$, this theorem was first proved in [9].

**Proof**: In its present form, it was apparently given in [12], by using explicitly the Markov context. It seems interesting here to point out that it is in fact a direct consequence of the following general

**Theorem 2.4** *Let us consider an arbitrary sequence of random variables $(S_n), n \geq 0$. Let $\mathcal{F}_n$ the $\sigma$-algebra generated by $S_0, S_1, \ldots S_n$, where $S_0$ will be taken constant with probability one (this does not restrict the generality). Let $\tau$ be the following $\mathcal{F}_n$-stopping time, $C$ being a real constant,*

$$\tau = inf \ \{n > 0, \ S_n < C/S_0 \geq C\} \ .$$

*Introduce the stopped sequence*

$$\tilde{S}_n = S_{n\wedge\tau} ,$$

*where*

$$n_\wedge\tau = \begin{cases} n, & if \quad n \leq \tau \\ \tau, & if \quad n > \tau, \end{cases}$$

*Suppose that, for* $n \geq 1$*,*

$$E(\tilde{S}_n/\mathcal{F}_{n-1}) \geq \tilde{S}_{n-1} \quad , \quad a.s \quad , \tag{2.1}$$

$$E(\mid \tilde{S}_n - \tilde{S}_{n-1} \mid /\mathcal{F}_{n-1}) \leq M \quad , \quad a.s \quad , \tag{2.2}$$

*Then* $E(\tau) = \infty$.

**Proof** : For all $k \geq 1$, we get from (2.1)

$$E(\mid \tilde{S}_k - \tilde{S}_{k-1} \mid) = E(E(\mid \tilde{S}_k - \tilde{S}_{k-1} \mid /\mathcal{F}_{k-1})) \leq MP(\tau > k-1) .$$

Thus, for any $n$, $l, 1 \leq l \leq n$,

$$E(\mid \tilde{S}_n - \tilde{S}_l \mid) = E(\mid \sum_{k=l+1}^{n} \mid \tilde{S}_k - \tilde{S}_{k-1}) \mid) \leq \sum_{k=l+1}^{n} E(\mid \tilde{S}_k - \tilde{S}_{k-1} \mid) .$$

$$\leq M \sum_{k=l+1}^{n} P(\tau \geq k) \quad , \tag{2.3}$$

whence, immediately,

$$E(\mid \tilde{S}_n \mid) \leq M \sum_{k=1}^{n} P(\tau > k) + S_0 . \tag{2.4}$$

Assume $E(\tau) < \infty$. Then, from (2.3), (2.4) and Cauchy criterion, it follows that $\tilde{S}_n$ is a submartingale converging almost surely (a.s) and in $L_1$. [The convergence a.s. is here obvious since, by the hypothesis, $P(\tau < \infty) = 1$ and thus $\tilde{S}_n = S_{n\wedge\tau} \xrightarrow{a.s.} S_\tau$]. Thus we have

$$E(S_\tau) = \lim_{n\to\infty} E(\tilde{S}_n) \geq E(S_0) \geq C .$$

But, by the definition of $\tau$, $E(S_\tau) < C$, which yields a contradiction. Hence $E(\tau) = \infty$ and the proof of theorem 2.4 is concluded ■

The proof of theorem 2.3 becomes now straightforward, by choosing

$$\tau = inf\ \{n > 0, f(\xi_n) \le c / f(\xi_0) > c\}\ ,$$

$\xi_0$ being taken constant, and

$$S_n = f(\xi_{n\wedge\tau}),\ S_0 = f(\xi_0)\ .$$

The proof of theorem 2.3 is concluded. ■

One of our main results is the following

**Theorem 2.5** *For an irreducible Markov chain $\mathcal{L}$ to be null recurrent, it is sufficient that there exist two functions $f(x)$ and $\varphi(x), x \in X$, and a finite subset $A \in X$, such that the following conditions hold :*

1. $f(x) \ge 0,\ \varphi(x) \ge 0,\ \forall x \in X$ ;
2. *For some positive* $\alpha, \gamma, \epsilon$, *with* $0 < \epsilon \le \alpha$ *and* $1 \le \alpha \le 2$,
   $f(x) \le \gamma(\varphi(x))^{\alpha-\epsilon},\ \forall x \in X$ ;
3. $\varphi(x_i) \to \infty$, *for* $i \to \infty$,
   $\sup_{x \notin A} f(x) > \sup_{x \in A} f(x)$ ;
4. (a) $E[f(\xi_{n+1}) - f(\xi_n)/\xi_n = x] \ge 0,\ \ \forall x \notin A$ ;
   (b) $E[\varphi(\xi_{n+1}) - \varphi(\xi_n)/\xi_n = x] \le 0,\ \ \forall x \notin A$ ;
   (c) $\sup_{x \in X} E[\mid \varphi(\xi_{n+1}) - \varphi(\xi_n) \mid^\alpha /\xi_n = x] = C < \infty$

**Proof** : Let us suppose that such functions exist. Conditions 1), 3) and 4b) on $\varphi(x)$ show immediately, by using theorem 4.1, that $\mathcal{L}$ is recurrent. We shall now assume that $\mathcal{L}$ is ergodic and then come to a contradiction, thus proving the null recurrence.

Let us denote

$$a_n = \varphi(\xi_n),\ \ b_n = f(\xi_n)\ ,$$

$$\tau = inf\ \{n > 0,\ \ \xi_n \in A / \xi_0 \notin A\}\ ,$$

$$\tilde{a}_n = a_{n \wedge \tau}\ ,\ \ \tilde{b}_n = b_{n \wedge \tau}\ .$$

Since $\mathcal{L}$ is assumed to be ergodic, $E(\tau) < \infty$.

It will be inconvenient throughout this study to choose $\xi_0$ to be a constant such that $\xi_0 \notin A$. The two following auxiliary lemmas will be useful for us.

**Lemma 2.6** *Let $\beta_n$, $n \geq 1$, be a sequence of random variables such that $\beta_n \to \beta$ a.s. and*

$$E \mid \beta_n \mid^r \leq c,\ \ \forall n \geq 0,\ \ for\ some\ c, r > 0\ .$$

*Then, for any $s$, $0 \leq s < r$,*

$$\lim_{n \to \infty} E \mid \beta_n - \beta \mid^s \to 0\ ,$$

*and, in particular,*

$$\lim_{n \to \infty} E \mid \beta_n \mid^s = E \mid \beta \mid^s$$

**Proof :** This is a classical result. See for instance [13]. ■

**Lemma 2.7** *Let the $S_n'$s of theorem 4 be now positive random variables and $\tau$ an arbitrary $\mathcal{F}_n$-stopping time. If, for all $n \geq 1$ and $\alpha, \epsilon$ real numbers such that $1 \leq \alpha \leq 2$, $0 < \epsilon \leq \alpha$,*

$$E(\tilde{S}_{n+1} - \tilde{S}_n / \mathcal{F}_n) \leq 0\ \ ,\ \ a.s. \tag{2.5}$$

$$E(\mid \tilde{S}_{n+1} - \tilde{S}_n \mid^\alpha / \mathcal{F}_n) \leq M\ \ ,\ \ a.s \tag{2.6}$$

$$E(\tau) < \infty\ ,$$

*then*

$$\sup_n E(\tilde{S}_n^\alpha) < \infty\ \ , \tag{2.7}$$

$$\tilde{S}_n^{\alpha-\epsilon} \xrightarrow{L_1} S_\tau^{\alpha-\epsilon}\ . \tag{2.8}$$

**Proof** : Define

$$\Delta\tilde{S}_n = \tilde{S}_{n+1} - \tilde{S}_n \ .$$

The following estimate takes place, from Taylor's formula,

$$\tilde{S}^{\alpha}_{n+1} - \tilde{S}^{\alpha}_n = \alpha\Delta\tilde{S}_n(\tilde{S}_n + \theta_n\Delta\tilde{S}_n)^{\alpha-1} \ , \tag{2.9}$$

where $0 < \theta_n < 1$ , $\forall n \geq 0$.

The right member of (2.9) can be rewritten as

$$\alpha\ \tilde{S}^{\alpha-1}_n\Delta\tilde{S}_n + \alpha\ \tilde{S}^{\alpha-1}_n\Delta\tilde{S}_n\ [(1 + \frac{\theta_n\Delta\tilde{S}_n}{\tilde{S}_n})^{\alpha-1} - 1] \leq$$

$$\alpha\ \tilde{S}^{\alpha-1}_n\Delta\tilde{S}_n + \alpha \mid \Delta\tilde{S}_n \mid^{\alpha} ,$$

where we have used the elementary inequalities

$$\mid 1 + v \mid^q \leq 1 + v^q \ , \ \mid 1 - v \mid^q \geq 1 - v^q \ , \ \forall q, \ 0 \leq q \leq 1 \ , \ \forall v \geq 0 \ .$$

Thus taking conditional expectation in (2.9) and using (2.5) and (2.6). we get

$$E[\tilde{S}^{\alpha}_{n+1} - \tilde{S}^{\alpha}_n / \mathcal{F}_n] \leq \alpha M\ \mathbf{1}_{\{\tau > n\}}\ , a.s. \tag{2.10}$$

$$where\ \mathbf{1}_A = \begin{cases} 1 & \text{if A is true,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (2.10) that

$$E(\tilde{S}^{\alpha}_{n+1}) \leq \alpha M \sum_{k=0}^{n} P(\tau > k) + S^{\alpha}_0 = \alpha M E(\tau) + S^{\alpha}_0 \ ,$$

The finiteness of $E(\tau)$ yields (2.7). The convergence in $L_1$ of $\tilde{S}^{\alpha-\epsilon}_n$ to $S^{\alpha-\epsilon}_\tau$ is now a direct consequence of lemma 2.6, since $\tilde{S}_n$ is a positive supermartingale and $\tilde{S}_n \overset{a.s.}{\to} S_\tau$.

Lemma 2.7 is proved ■

Let us return, to the proof of the theorem. Since $A$ is a finite set,

$$\sup_{x\in A}\varphi(x) < \infty \ , \ \sup_{x\in A} f(x) < \infty \ .$$

Since $\mathcal{L}$ is assumed to be ergodic, $P(\tau < \infty) = 1$ and there exist two random variables $\tilde{a}$ and $\tilde{b}$ such that

$$\tilde{a}_n^{\alpha-\epsilon} = \varphi^{\alpha-\epsilon}(\xi_{n\wedge\tau}) \xrightarrow{a.s} \tilde{a} \ , \ 0 \leq \tilde{a} \leq \sup_{x\in A}\varphi_x^{\alpha-\epsilon} \ ,$$

$$\tilde{b}_n = f(\xi_{n\wedge\tau}) \xrightarrow{a.s} \tilde{b} \ , \ 0 \leq \tilde{b} \leq \sup_{x\in A} f(x) \ .$$

Moreover, lemmas 2.6 and 2.7 entail that the r.v.'s $\tilde{a}_n^{\alpha-\epsilon}$ are uniformly integrable and converge to $\tilde{a}$ in the $L_1$-sense. Using condition 2) of theorem 2.5, we have

$$\tilde{b}_n = f(\xi_{n\wedge\tau}) \leq \gamma\, \tilde{a}_n^{\alpha-\epsilon} \ .$$

Thus the family $(\tilde{b}_n), n \geq 0$, dominated by a uniformly integrable family, is also uniformly integrable. This shows that $\tilde{b}$ is the $L_1$-limit of $\tilde{b}_n$ and

$$\lim_{n\to\infty} E\tilde{b}_n = E\tilde{b} \leq \sup_{x\in A} f(x) \tag{2.11}$$

On the other hand, condition 4a) shows that $\tilde{b}_n$ is a submartingale and

$$E[\tilde{b}_n/\xi_0 = i] \geq f(\xi_0) = f(i) \ , \ \forall i \notin A, \ \forall n \geq 0 \ . \tag{2.12}$$

From condition 3), we can choose $i$ so that

$$f(i) > \sup_{x\in A} f(x) \ .$$

Doing so, we get from the estimate (2.11), which does not depend on the initial position $\xi_0$,

$$\lim_{n\to\infty} E(\tilde{b}_n/\xi_0 = i) \leq \sup_{x\in A} f(x) \ ,$$

and this last inequality contradicts (2.12). Thus necessarily $E(\tau) = \infty$ and the proof of theorem 2.5 is completed. ■

# 3 Proof of the main result (theorem 1.1)

Let us introduce the linear function $\varphi : R_+^2 \to R_+$,

$$\varphi(x,y) = px + qy \quad , \quad p > 0 \, , q > 0 \, .$$

We also shall write, for any vector $\gamma = (x,y)$

$$\varphi(\gamma) \equiv \varphi(x,y) \, .$$

**Lemma 3.1** *Let $p, q > 0$, be such that the vectors $M'$ and $M''$ have the following properties (see fig. 2)*

$$\begin{cases} \varphi(M') = pM'_x + qM'_y & \geq 0 \, , \\ \varphi(M'') = pM''_x + qM''_y & \geq 0 \, . \end{cases} \tag{3.1}$$

*Then, for all $(x,y) \in \mathbf{Z}_+^2, (x,y) \neq (0,0)$,*

$$E[\varphi(\xi_{n+1})/\xi_n = (x,y)] \geq \varphi(x,y) \, , \tag{3.2}$$

*i.e. $\varphi(\xi_n)$ is a positive submartingale.*

**Proof** : Immediate by using the linearity of $\varphi$ and the fact that $M = (M_x, M_y) = (0,0)$ for $x, y > 0$. ■

We claim that the conditions of lemma 3.1 hold in each of the cases (i)-(iv) of theorem 1.1. They are also valid in the case

**a)** $\quad \phi_{\mathbf{x}} < \frac{\pi}{2} \, , \, \phi_{\mathbf{y}} < \frac{\pi}{2} \, , \, \phi_{\mathbf{x}} + \phi_{\mathbf{y}} \geq \frac{\pi}{2} \, ,$

which yields

$$M'_x \, M''_y - M'_y \, M''_x \leq 0 \, . \tag{3.3}$$

**Lemma 3.2** *If, for the random walk $\mathcal{L}$, the conditions of lemma 3.1 are fulfilled, then $\mathcal{L}$ is not ergodic.*

**Proof :** It is a direct consequence of theorem 2.3. It is worth mentioning that this result holds under the mere assumption.

$$E[\| \theta_{n+1} \| / \xi_n = (x,y)] \leq K < \infty ,$$

which is weaker than the condition B stated in section 1. ∎

Let us consider now the case

**b)** $\phi_{\mathbf{x}} + \phi_{\mathbf{y}} < \frac{\pi}{2}$ .

Then we have the property opposite to that of lemma 3.1, since now the vectors $M'$ and $M''$ look inside the simplex bounded by the two positive axis and the line $px + qy = D$. (see fig 3).

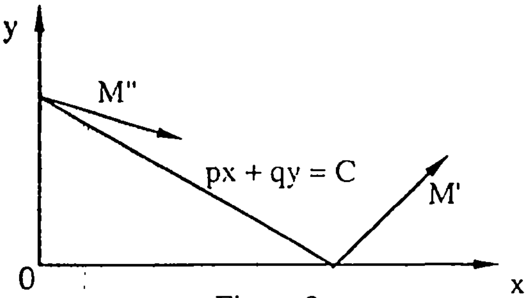


Figure 2

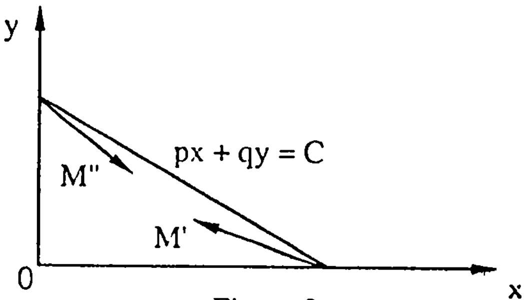


Figure 3

It means that there exist $p > 0, q > 0$, such that the linear function $\phi(x_t, y_t)$ be a positive supermartingale. Thus the random walk $\mathcal{L}$ is recurrent. Our goal is to distinguish between positive and null recurrence. To that end, we introduce the following functional of quadratic form

$$f(x,y) = (ux^2 + vy^2 + xy)^\delta \quad , \quad x,y \in \mathbf{Z}_+^2 ,$$

where $u \geq 0,\ v \geq 0,\ 0 < \delta < 1$.

We want to adjust $u, v$ and $\delta$ to satisfy the conditions of theorem 2.5. Define, for the sake of brevity,

$$\begin{aligned} Q(x,y) &= ux^2 + vy^2 + xy \ , \\ \Delta Q(x,y) &= Q(x + \theta_x, y + \theta_y) - Q(x,y) \\ &= u\theta_x^2 + v\theta_y^2 + \theta_x\theta_y + (y + 2ux)\theta_x + (x + 2vy)\theta_y \ , \end{aligned} \tag{3.4}$$

where (see section 1),

$$\theta_{n+1} = (\theta_x, \theta_y), \ \ given \ \ \xi_n = (x,y) \geq 0 \ ,$$

in which case, *ad libitum*, $f(x,y)$ will be rewritten as $f(\xi_n)$.
We shall estimate the quantity

$$\begin{aligned} H(x,y) &\stackrel{\text{def}}{\equiv} E[f(\xi_{n+1}) - f(\xi_n)/\xi_n = (x,y)] \\ &= E[(Q(x,y) + \Delta Q(x,y))^\delta - Q^\delta(x,y)] \end{aligned} \tag{3.5}$$

**Lemma 3.3** *There exists $\delta > 0$ and a constant $D$ such that*

$$H(x,y) = \delta Q^{\delta-1}(x,y)[E(\Delta Q(x,y)) + (\delta - 1)\mathcal{O}(1) + \mathfrak{o}(1)] \tag{3.6}$$

*for all $(x,y)$, such that $(x^2+y^2) > D^2$,*

*where, as usual, $\mid \mathcal{O}(z) \mid < K \mid z \mid$ and $\frac{\mathfrak{o}(z)}{z} \to 0$ as $z$ tends to a given limit. In particular, $\mathfrak{o}(1)$ means a function which tends to zero when $D \to \infty$.*

**Proof** : From Taylor's formula we have

$$\begin{aligned} H(x,y) &= E[\Delta Q(x,y)(Q(x,y) + \gamma(x,y)\Delta Q(x,y))^{\delta-1}] \\ &= \delta Q^{\delta-1}(x,y)[E[\Delta Q(x,y)] + \psi(x,y)] \ , \end{aligned} \tag{3.7}$$

where $\gamma(x,y)$ is a random variable such that $0 < \gamma(x,y) < 1$ (note that $Q + \gamma\Delta Q \geq 0$) and

$$\psi(x,y) = E[\Delta Q(x,y)[(1 + \frac{\gamma(x,y)\Delta Q(x,y)}{Q(x,y)})^{\delta-1} - 1]] \ .$$

It suffices to prove $\psi(x,y) = (\delta - 1)\mathcal{O}(1) + \mathfrak{o}(1)$.

i) **Case of bounded jumps.** The result is immediate from the definition (3.4), after using $\mid (1+z)^{\delta-1} - 1 \mid = (\delta - 1)\mathcal{O}(z)$, for $z$ sufficiently small, since, from the boundedness of the jumps

$$\frac{\Delta^2 Q(x,y)}{Q(x,y)} < A < \infty \ \ , \ \ \forall (x,y) \neq (0,0) \ .$$

ii) **Case of unbounded jumps.** Let us write $\psi(x,y)$ under the form

$$\psi(x,y) = \psi_1(x,y) + \psi_2(x,y) \ ,$$

with

$$
\begin{aligned}
\psi_1(x,y) &= E[\Delta Q(x,y)T(x,y)\mathbf{1}_{\{|\theta_x+\theta_y|\le z\}}] \ , \\
\psi_2(x,y) &= E[\Delta Q(x,y)T(x,y)\mathbf{1}_{\{|\theta_x+\theta_y|> z\}}] \ ,
\end{aligned}
\tag{3.8}
$$

where $z$ is some positive real number and

$$
T(x,y) \stackrel{def}{\equiv} [1 + \frac{\gamma(x,y)\Delta Q(x,y)}{Q(x,y)}]^{\delta-1} - 1 \ .
$$

We have, according to (3.4),

$$
\Delta Q(x,y) = x(2u\ \theta_x + \theta_y) + y(2v\ \theta_y + \theta_x) + u\theta_x^2 + v\theta_y^2 + \theta_x\theta_y \tag{3.9}
$$

Hence, using condition $A$ of section 1 on the lower boundedness of jumps, we have for fixed $z > 0$

$$
|\ \Delta Q(x,y)\ | \le (ax+by)z + c^2z^2 \ , \ for \ \ |\ \theta_x + \theta_y\ | < z
$$

where $a$, $b$, $c$ are positive constants depending only $u$, $v$ and it follows that. for $(x^2+y^2) > D^2$, $z < \sqrt{D}$,

$$
\begin{aligned}
|\ \psi_1(x,y)\ | \ &\le E[|\ \Delta Q(x,y)T(x,y)\ |\ \mathbf{1}_{\{|\theta_x+\theta_y|\le z\}}] \\
&= (1-\delta)\mathcal{O}[\frac{((ax+by)z+c^2z^2)^2}{Q(x,y)}] = (1-\delta)\mathcal{O}(1)
\end{aligned}
\tag{3.10}
$$

which gives an estimate for $\psi_1(x,y)$.
On the other hand, on $\{|\ \theta_x + \theta_y > z\}$, $\Delta Q(x,y)$ is positive, as emerges from (3.9), whence

$$
-1 \le T(x,y) \le 0 \ \ and \ \ |\ T(x,y)\ | \le 1 \ .
$$

It follows that

$$
|\ \psi_2(x,y)\ | \le E[\Delta Q(x,y)\mathbf{1}_{\{|\theta_x+\theta_y|>z\}}] \ .
$$

Hence

$$
\psi_2(x,y) \le E[Q(\theta_x,\theta_y)\mathbf{1}_{\{|\theta_x+\theta_y>z\}}] + (ax+by)E[|\ \theta_x+\theta_y\ |\ \mathbf{1}_{\{|\theta_x+\theta_y|>z\}}] \ . \tag{3.11}
$$

To estimate the right member of (3.11), we use the following simple result, valid for any random variable $X$, such that $E[\mid X \mid^r] < \infty$,

$$E[\mid X \mid^r \mathbf{1}_{\{X > z\}}] = o \mid z^{s-r} \mid \ , \ \ \forall \ 0 \leq s \leq r \ .$$

Therefore, taking into account the moment condition $B$ ensuring $E[\mid \theta_x + \theta_y \mid^3] < \infty$, we get

$$\mid \psi_2(x, y) \mid \leq o(\frac{1}{z}) + (ax + by)\ o(\frac{1}{z^2}) \ . \tag{3.12}$$

Choosing again $z < \sqrt{D}$ in (3.12) yields

$$\psi_2(x, y) = o(1) \ . \tag{3.13}$$

Finally, (3.10) and (3.13) together imply $\mid \psi(x, y) \mid = (1 - \delta)\mathcal{O}(1) + o(1)$ and the proof of lemma 3 is concluded ■

**Remark** : It is possible to refine this proof, assuming only the existence of moments of order two. We omit it to avoid unnecessary excursions, which would obscure the readability and the general ideas.

We continue with the proof of theorem 1.1.

It follows from (3.6), that one can find $D$ and $\delta, 0 < \delta < 1$, such that, if $E[\Delta Q(x, y)] > 0$, then $H(x, y) \geq 0$, for any $x, y$ satisfying $x^2 + y^2 > D$, or equivalently, from (3.5),

$$E[f(\xi_{n+1}) - f(\xi_n) \mid \xi_n = (x, y)] \geq 0 \ .$$

But $E[\Delta Q(x, y)] > 0, \ x^2 + y^2 > D^2$, is equivalent, by using (3.9), to the following system of inequalities

$$\begin{cases} u_x + v\lambda_y + R & > 0 \ , \\ 2vM_y'' + M_x'' & > 0 \ , \\ 2uM_x' + M_y' & > 0 \ , \end{cases} \tag{3.14}$$

for some $u, v > 0$.

Thus if (3.14) is satisfied, there exist two functions $f(x, y) = (ux^2 + vy^2 + xy)^\delta$ and $\varphi(x, y) = px + qy$, such that, when $\phi_x + \phi_y < \frac{\pi}{2}$, the conditions 1, 3, and 4 of theorem 2.5 hold, simply taking $\alpha = 2$ in the statement of the theorem. Moreover in this case, condition 2 of theorem 2.5 becomes immediately fulfilled, since for $x$ and $y$ sufficiently large, $0 < \delta < 1$,

$$0 < (ux^2 + vy^2 + xy)^\delta < K(x^2 + y^2) .$$

But, since from the assumptions $M'_y \geq 0$, $M'_x < 0$, $M''_x \geq 0$, $M''_y < 0$, we conclude that (??) holds for some $u, v > 0$, if

$$-\lambda_x \frac{M'_y}{M'_x} - \lambda_y \frac{M''_x}{M''_y} + 2R > 0 ,$$

which is simply on other way of rewriting (1.2).
We have proved the "null recurrence" part of theorem 1.1.

To prove the ergodicity part of the theorem, we proceed as in [ ], by introducing again the quadratic form

$$Q(x, y) = ux^2 + vy^2 + xy$$

and showing that Foster's criterion (2.2) can be satisfied, for some $u, v, \epsilon > 0$. With the notation above, we have, from (3.9),

$$\begin{aligned} K(x,y) &\stackrel{\text{def}}{=} E[Q(\xi_{n+1}) - Q(\xi_n)/\xi_n = (x,y)] \\ &= xE[2u\theta_x + \theta_y] + yE[2v\theta_y + \theta_x] + E[Q(\theta_x\theta_y)] . \end{aligned} \tag{3.15}$$

Since $E[Q(\theta_x, \theta_y)] = \mathcal{O}(1)$, $\forall (x, y) \in \mathbf{Z}_+^2$, we get from (??), after taking into account the boundary conditions on the axes,

$$K(x,y) = \begin{cases} \lambda_x u + \lambda_y v + R, & (x,y) > 0 , \\ x(2uM'_x + M'_y) + \mathcal{O}(1), & x > 0,\ y = 0 , \\ y(2vM''_y + M''_x) + \mathcal{O}(1), & x = 0,\ y > 0 . \end{cases}$$

Thus, for some $\epsilon > 0$ and some finite subset $\mathbf{E} \in \mathbf{Z}_+^2$, we have $K(x, y) < -\epsilon$ , $\forall x \notin \mathbf{E}$, provided that the following system can be satisfied, for some $u, v > 0$,

$$\begin{aligned} &\lambda_x u + \lambda_y v + R && < 0 , \\ &2uM'_x + M'_y && < 0 , \\ &2vM''_y + M''_x && < 0 . \end{aligned} \tag{3.16}$$

The inequalities $M_x'' \geq 0$, $M_y'' < 0$, $M_y' \geq 0$, $M_x' < 0$, show at once that (3.16) can be satisfied for $u > 0$, $v > 0$, if (1.1) holds, that is

$$-\lambda_x \frac{M_y'}{M_x'} - \lambda_y \frac{M_x''}{M_y''} + 2R < 0 \, .$$

In this case the remaining conditions of Foster's criterion are clearly fulfilled and the random walk is ergodic.

The proof of theorem (1.1) is terminated. ■

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