Stochastic ordering of random processes with an imbedded point process

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ORDONNANCEMENT STOCHASTIQUE DE PROCESSUS ALEATOIRES AVEC UN PROCESSUS PONCTUEL INCLUS

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RESUME

On introduit plusieurs ordres stochastiques, liés à la probabilité de Palm et à la probabilité stationnaire d'un processus ponctuel. On montre comment ces ordres permettent d'étudier les propriétés d'ordonnancement de processus stationnaires avec des suites incluses. On étudie notamment les conséquences pour le processus stationnaire d'une propriété d'ordonnancement vérifiée par la suite incluse. Les cas des propriétés d'ordonnancement intégral et d'association sont successivement examinés.

Mots-Clés: Ordres stochastiques intégraux, probabilités de Palm, processus stochastiques avec un processus ponctuel inclus, association de variables aléatoires.

STOCHASTIC ORDERING OF RANDOM PROCESSES WITH AN IMBEDDED POINT PROCESS

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ABSTRACT

We introduce multivariate partial orderings related with the Palm and time-stationary probabilities of a point process. Using these orderings, we give conditions for the monotonicity of a random sequence, with respect to some integral stochastic ordering, to be inherited by a continuous time process in which this sequence is imbedded. This type of inheritance is also discussed for the property of association.

Keywords: Integral Stochastic Ordering, Palm Probabilities, Stochastic Processes with an Imbedded Point Process, Association of Random variables.

1. Introduction

There exist several methods for deriving stochastic comparison results on random sequences defined by recursions, like for instance the actual waiting times in a $G/G/1$ queue (see [3]). Within the framework of queueing theory, such sequences are most often imbedded in a continuous time stochastic process (e.g. the virtual waiting time process for $G/G/1$ queues), for which these comparison methods, which rely in a crucial way on the recursive nature of the sequences, do not apply directly. The aim of the present paper is to investigate the conditions under which an ordering property initially established for such an imbedded sequence might extend to the continuous time process as well. The framework that is used to address these questions is that of a stochastic process with an imbedded point process, as introduced by Franken, König, Arndt and Schmidt in their book. The basic tools that are used to answer this question are multivariate extensions of stochastic orderings related with the stationary excess operator of renewal theory, that were initially defined and analyzed in [9] and [4].

Consider a probability space (Ω, F, P, θ_t), where P is θ_t- invariant, and a stationary point process

$$N = \sum_{i \in \mathbb{Z}} \delta_{T_n}$$

on this space, where $T_1 = T$ is the first non-negative point of N. It is assumed that the intensity λ of N is finite and that there are no double points. The Palm space of N will be denoted by (Ω, F, P^0, θ), where $\theta = \theta_T$.

Let y be a $I\!R^K$-valued random variable on (Ω, F, P^0, θ) satisfying the relation

$$y \circ \theta = h(y, \eta) \tag{1.1}$$

where η is some integrable $I\!R^J$-valued random variable and where h is some Borel mapping $h : I\!R^K \times I\!R^J \to I\!R^K$.

Let $y(t)$, $t \in I\!R$, be the $I\!R^K$-valued stochastic process on (Ω, F, P^0) defined by

$$y(t) = g(t - T_n, y \circ \theta^n, \xi \circ \theta^n), \ T_n \le t < T_{n+1}, \ P^0 - a.s., \tag{1.2}$$

where ξ is some integrable $I\!R^M$-valued random variable, and where g is some measurable mapping $g : I\!R^{K+M+1} \to I\!R^K$, which is assumed to admit left-hand limits with respect to the variable t. The random variable ξ and the function g are assumed to satisfy the consistency constraint

$$g(T^-, y, \xi) = h(y, \eta),$$

or equivalently

$$y(T_n -) = y \circ \theta^n, \quad P - a.s.$$

In queueing theory, this function typically describes the way the workload decreases with time between arrival epochs. A typical example is provided by the sequence of waiting times in the $G/G/1$ queue, which is obtained when taking $\eta = (\sigma, \tau) \in IR^2$, and $h(y, \eta) = (y + \sigma - \tau)^+$, where σ_n is the n-th service time and τ_n the n-th inter-arrival time. This sequence is imbedded in the continuous time workload process defined by $\xi = \sigma \in IR$ and $g(t, y, \xi) = (y + \sigma - t)^+,\ 0 \leq t < T$.

The stochastic process $y(t)$, initially defined on the Palm space of N, also admits a time stationary version that satisfies the relation

$$y(\omega, t) = y(\theta_t(\omega), 0), \quad P - a.s.$$

(see [1]).

Consider a second point process $\tilde{N}$ on $(\tilde{\Omega}, \tilde{F}, \tilde{P}, \tilde{\theta}_t)$, on which are also defined a stationary sequence $\tilde{y}_n$ satisfying the relation

$$\tilde{y} \circ \tilde{\theta} = h(\tilde{y}, \tilde{\xi})),$$

(with the same function h as in (1.1)), and a stochastic process $\tilde{y}(t)$ such that

$$\tilde{y}(t) = g(t - \tilde{T}_n, y\tilde{\circ}\tilde{\theta}^n, \tilde{\xi} \circ \tilde{\theta}^n),\ \tilde{T}_n \leq t < \tilde{T}_{n+1},\ \tilde{P}^0\ a.s.,$$

(with the sane function g as in (1.2)). All the objects initially defined for the first processes are also supposed to exist for the second ones, and will be denoted by the same symbol with a tilde.

Let $x\ [P]$ denote the distribution function of the random variable x with respect to the probability P.

The general question under investigation can now be formulated as follows: what ordering relations should hold on the distribution functions $(\xi_n, y_n)\ [P^0]$ and $(\tilde{\xi}_n, \tilde{y}_n)\ [\tilde{P}^0]$ and on the function g for ensuring the relation

$$y(0)\ [P] \leq_{\mathcal{L}} \tilde{y}(0)\ [\tilde{P}], \tag{1.3}$$

where $\leq_{\mathcal{L}}$ denotes some integral ordering associated with the family of Borel mappings $\mathcal{L}$ (eg. *st*, *cx*, *icx*)?

The paper is structured as follows. In §2.. we review basic definitions on integral ordering, and we give some examples of the techniques that are used in order to establish comparison results on stationary sequences of random variables, when these sequences are defined through recursions. The Palm and Stationary stochastic orderings are introduced in §3., and their relationship with classical integral orderings is investigated. In §4., these stochastic orderings are then used to answer the question mentioned above. In §5. we see how this type of technique can also be applied to check whether the property of *association* can be inherited from imbedded sequences to continuous time processes.

2. Integral Ordering and Comparison of Sequences

2.1. Integral Stochastic Orderings

Let $\mathcal{D}(I\!R^n)$ denote the space of distribution functions on $I\!R^n$ and $\mathcal{L}$ be a set of borel mappings from $I\!R^n$ onto $I\!R$. Consider the binary relation $\leq_{\mathcal{L}}$ on $\mathcal{D}(I\!R^n)$ defined by

$$F \leq_{\mathcal{L}} G \quad \text{iff} \quad \int_{I\!R^n} f(x)F(dx) \leq \int_{I\!R^n} f(x)G(dx),$$

for all $f \in \mathcal{L}$ such that the integrals are well defined.

The binary relation $\leq_{\mathcal{L}}$ is clearly reflexive and transitive, so that it always defines a partial semiordering on $\mathcal{D}(I\!R^n)$. If the set $\mathcal{L}$ is rich enough to imply the anti-symmetry property then $\leq_{\mathcal{L}}$ defines a partial ordering on this space.

Let $X_0,\ X_1, \cdots$ and $Y_0,\ Y_1, \cdots$ be $I\!R$-valued stochastic sequences on the probability space $(\Omega, \mathcal{F}, P)$. The sequence Y will be said to dominate X for the $\leq_{\mathcal{L}}$ ordering if for all $n \geq 1$, the distribution functions F and G on $I\!R^n$ defined by $F(x) = P[X_0 \leq x_0, \cdots, X_n \leq x_n]$ and $G(x) = P[Y_0 \leq x_0, \cdots, Y_n \leq x_n]$ satisfy the ordering relation $F \leq_{\mathcal{L}} G$, or equivalently if

$$(2.1.1), \qquad E[f(X_1, \cdots, X_n)] \leq E[f(Y_1, \cdots, Y_n)]$$

for all $f : I\!R^n \to I\!R$ in $\mathcal{L}$.

Two basic instances of sets $\mathcal{L}$ are first considered. The set

$$(2.1.2) \qquad \{st\} = \{f : I\!R^n \to I\!R \mid f \text{ non} - \text{decreasing } \}$$

will be said to generate the *strong ordering* on $\mathcal{D}(I\!R^n)$, while the set

$$(2.1.3) \qquad \{cx\} = \{f : I\!R^n \to I\!R \mid f \text{ convex } \},$$

generates the so called *convex ordering* on $\mathcal{D}(I\!R^n)$. The set

$$(2.1.4) \qquad \{icx\} = \{st\} \bigcap \{cx\}$$

generates the *increasing convex* ordering.

If one replaces convex functions by concave functions in the above definitions, one gets the *concave* and *increasing concave* orderings, which are respectively denoted by $\leq_{cv}$ and $\leq_{icv}$.

The fact that $f(x)$ is convex iff $-f$ is concave (resp. $f(x)$ is icx iff $-f(-x)$ is icv) implies the relations

$$(2.1.7) \qquad X \leq_{cx} Y \Leftrightarrow Y \leq_{cv} X$$

and

$$X \leq_{icx} Y \Leftrightarrow -Y \leq_{icv} -X. \tag{2.1.8}$$

2.2. Comparison of Stochastic Recursive Sequences

Although the techniques to prove that stationary sequences satisfy some ordering property are not our main focus, it seems useful to review some basic results on the matter.

Consider a $I\!R^K$-valued sequence $y_0,\ y_1,\ \cdots$ generated by the recursion

$$y_{n+1} = h(y_n, \eta_n), \quad n = 0, 1, \ldots \tag{2.2.1}$$

for some Borel mapping $h : I\!R^K \times I\!R^J \to I\!R^K$, some exogeneous sequence of $I\!R^M$-valued random variables $\eta_0,\ \eta_1,\ \cdots$ and initial condition $y_0 = \beta$ all defined on the probability space $(\Omega, \mathcal{F}, P^0, \theta)$.

Within the general framework mentioned above, the questions pertaining to the stochastic comparison of two sequences with respect to the integral orderings of §1.1: can be formulated as follows consider another recursion, say

$$\tilde{y}_{n+1} = h(\tilde{y}_n, \tilde{\eta}_n), \quad n = 0, 1, \ldots$$

with the same dynamics h, and which only differs from the first one by its initial condition $\tilde{\beta}$ and its exogeneous sequence $\tilde{\eta}_0,\ \tilde{\eta}_1,\ \cdots$. Under what conditions on h does the ordering relation

$$(\beta, \eta_0, \eta_1, \cdots) \leq_{\mathcal{C}} (\tilde{\beta}, \tilde{\eta}_0,\ \tilde{\eta}_1, \cdots) \tag{2.2.3}$$

(we mean here that all finite dimensional distributions satisfy the ordering relation) imply that

$$(y_0,\ y_1, \cdots) \leq_{\mathcal{C}} (\tilde{y}_0, \tilde{y}_1, \cdots). \tag{2.2.4}$$

Assuming in addition that the sequences y_n and $\tilde{y}_n$ converge weakly to the stationary limits y_∞ and $\tilde{y}_\infty$ respectively, when does the relation

$$y_\infty \leq_{\mathcal{C}} \tilde{y}_\infty \tag{2.2.5}$$

hold? Assume now that the sequence y_n can be made stationary for the shift θ for an appropriate choice of the initial condition, namely there exists a $I\!R^K$-valued random variable y on $(\Omega, \mathcal{F}, P^0)$, distributed like y_∞ and satisfying the relation

$$y \circ \theta = h(y, \eta). \tag{2.2.6}$$

The following and more general question can also be considered. When does the relation

$$(y, y \circ \theta^1, \cdots, y \circ \theta^n) \leq_{\mathcal{L}} (\tilde{y}, y\tilde{\circ}\tilde{\theta}^1, \cdots, \tilde{y} \circ \tilde{\theta}^n) \tag{2.2.7}$$

also hold for all $n \geq 1$?

Concerning the first question, the following results, which are mainly based on Strassen's theorems ([8]), can be found partly in [3] and with more detail in [2],

Theorem (2.2.8) *Assume that $(y, \eta) \to h(y, \eta)$ is non decreasing. If*

$$(\beta, \eta_0, \ \eta_1, \cdots) \leq_{st} (\tilde{\beta}, \tilde{\eta}_0, \ \tilde{\eta}_1, \cdots)$$

then

$$(y_0, y_1, \cdots) \leq_{st} (\tilde{y}_0, \ \tilde{y}_1, \cdots).$$

Theorem (2.2.9)*Assume that the random variables $(\beta, \eta_0, \ \eta_1, \cdots)$ are integrable, and that*

(i) The function $y \to h(y, \eta)$ is non-decreasing;

(ii) The function $(y, \eta) \to h(y, \eta)$ is convex (resp. convex and non decreasing); Then,

$$(\beta, \eta_0, \ \eta_1, \cdots) \leq_{cx} \ (\text{resp. } \leq_{icx})(\tilde{\beta}, \tilde{\eta}_0, \ \tilde{\eta}_1, \cdots)$$

implies

$$(y_0, \ y_1, \cdots) \leq_{icx} (\tilde{y}_0, \tilde{y}_1, \cdots).$$

These results can be seen as extensions to the non i.i.d case of Stoyan's mapping method (see p. 47 of his book). The transient bounds of (2.2.8) and (2.2.9) can then be extended to steady state in several different ways: let $\leq_{\mathcal{L}}$ be some integral stochastic ordering on $\mathcal{D}(I\!R^K)$, and let y_n and $\tilde{y}_n$, $n \geq 0$, be two $I\!R^K$ valued sequences of random variables that converge weakly to the random variables y_∞ and $\tilde{y}_\infty$ respectively.

The conditions under which the property

$$y_n \leq_{\mathcal{L}} \tilde{y}_n \tag{2.2.10}$$

for all $n \geq 0$ implies

$$y_\infty \leq_{\mathcal{L}} \tilde{y}_\infty \tag{2.2.11}$$

may be obtained analytically (see the book of D. Stoyan for instance).

This type of approach is not always needed. One can often associate with the initial sequence a so called Loynes' schema that allows one to solve this problem directly. Roughly speaking, within the setting of §2.1, such a schema exists whenever the mapping $y \to h(y, \eta)$ is non-decreasing (see [1]).

In this case, it is possible to associate with the transient sequence y_n, $n \geq 0$ with initial condition 0, a Loynes' sequence, say z_n, $n \geq, 0$ such that for all $n \geq 0$, the distribution functions of z_n and y_n coincide, and where z_n is non-decreasing in n a.s.. Therefore, the weak convergence of y_n is equivalent to the a.s. convergence of the increasing sequence z_n to a finite limit z_∞ (see [1] for more details on this technique).

When such a schema exists, (2.2.10) reads hence

$$E[f(z_n)] \leq E[f(\tilde{z}_n)] \quad \forall f \in \mathcal{L}, \tag{2.2.12}$$

for all $n \geq 0$, and when letting n go to ∞ in (2.2.12), a direct application of the monotone convergence theorem yields

$$E[f(z_\infty)] \leq E[f(\tilde{z}_\infty)],$$

for all $f \in \mathcal{L}$ such that the expectations exist, which is equivalent to (2.2.11).

The more general question (2.2.7) can also be answered whenever the equation of interest admits a Loynes' schema. In that case, $y = z_\infty$ is a solution to (2.2.6). Using the technique proposed in [3], it is easy to see that under the assumption (2.2.3) the property

$$(z_\infty, \eta_0,\ \eta_1, \cdots) \leq_{st} (\tilde{z}_\infty, \tilde{\eta}_0,\ \tilde{\eta}_1, \cdots)$$

will hold under the assumptions of (2.2.8), and similarly,

$$(z_\infty, \eta_0,\ \eta_1, \cdots) \leq_{icx} (\tilde{z}_\infty, \tilde{\eta}_0,\ \tilde{\eta}_1, \cdots)$$

will hold under the assumptions of (2.2.9), provided z_∞ is integrable. The proof of (2.2.7) is then concluded when using (2.2.8) and (2.2.9) for the appropriate initial condition.

3. Stochastic Orderings Related with Point Processes

3.1. The Palm and Stationary Stochastic Orders

Let N be a stationary point processes defined on the probability space $(\Omega, \mathcal{F}, P, \theta_t)$. Assume that N has no double points and a finite intensity, λ. Let $(\Omega, \mathcal{F}, P^0, \theta)$ denote the Palm probability space of N. We denote by T the first positive point of N and by X some IR^n-valued mark of N associated with the first positive point. Denote by $F^0(t, x)$ the function of $\mathcal{D}(IR^{n+1})$ defined by

$$F^0(t, x) = P^0[T \leq t, X \leq x], \quad t \geq 0,\ x \in IR^n. \tag{3.1.1}$$

To this distribution function, one can associate the stationary law of the couple (T, X). Its distribution function $F(t,x) \in \mathcal{D}(I\!R^{n+1})$ is defined by

$$F(t,x) = P[T \leq t, X \leq x], \quad t \geq 0, \; x \in I\!R^n. \tag{3.1.2}$$

It is obtained from the Palm inversion formula (see [1])

$$\begin{aligned} F(t,x) &= \lambda E^0[\int_0^T 1_{T \leq t+u, X \leq x} du] \\ &= \lambda E^0[T \wedge t \; 1_{X \leq x}]. \end{aligned}$$

An integration by parts yields then the relation

$$F(t,x) = \lambda \int_0^t (F^0(\infty, x) - F^0(u,x))du. \tag{3.1.3}$$

The distribution function $F \in \mathcal{D}(I\!R^{n+1})$ will be denoted $\mathcal{S}F^0$. Clearly,
the mapping $t \to F(t,x)$ is differentiable on $(0,\infty)$, with derivative $\lambda(F^0(\infty,x) - F^0(t,x)$;
the derivative at $t = 0$ is equal to $\lambda F^0(\infty, x)$ (use the property that $F^0(0,x) = 0$);
the mapping $(t,x) \to F_t'(0,x) - F_t'(t,x)$ is non-decreasing (use the fact that $F_t'(0,x) - F_t'(t,x) = \lambda F^0(t,x)$).

Conversely, if F is the stationary distribution function of (T, X) under P, then the Palm distribution of (T, X) is given by the formula $F^0 = \mathcal{P}F \in \mathcal{D}(I\!R^{n+1})$, where

$$F^0(t,x) = \frac{F_t'(0,x) - F_t'(t,x)}{F_t'(0,\infty)}. \tag{3.1.4}$$

It is interesting to look at the consequences of these definitions on the marginal distribution functions

$$F_T^0(t) = F^0(t,\infty), \quad F_T(t) = F(t,\infty), \quad t \in I\!R^+$$

and

$$F_X^0(x) = F^0(\infty, x), \quad F_X(x) = F(\infty, x), \quad x \in I\!R^n.$$

In the case where there are no marks, (3.1.2) reads

$$F_T(t) = (\mathcal{S}F^0)_T(t) = \frac{\int_0^t (1 - F_T^0(u))du}{\int_0^\infty (1 - F_T^0(u))du}, \quad t \geq 0. \tag{3.1.5}$$

while (3.1.4) boils down to

$$F_T^0(t) = (\mathcal{P}F)_T(t) = 1 - \frac{F_t'(t)}{F_t'(0)}, \quad t \geq 0. \tag{3.1.6}$$

Similarly,

$$(3.1.7) \qquad F_X(x) = (\mathcal{S}F^0)_X(x) = \frac{\int_0^\infty (F_X^0(x) - F^0(u,x))du}{\int_0^\infty (1 - F_T^0(u))du}, \quad x \in I\!R^n,$$

and

$$(3.1.8) \qquad F_X^0(x) = (\mathcal{S}F)_X(x) = \frac{F_t'(0,x)}{F_t'(0,\infty)} \quad x \in I\!R^n,$$

We are now in a position to give the definition of the Palm and stationary orderings. Let $\tilde{N}$ be a second stationary process as defined above. Any object that was defined for N is supposed to exist for $\tilde{N}$, and will be denoted by the same symbol with a tilde.

Within this setting, the $S - \mathcal{L}$ and $P - \mathcal{L}$ orderings are defined on $\mathcal{D}(I\!R^{n+1})$ as follows.

Definition (3.1.8) $F^0 \leq_{S-\mathcal{L}} \tilde{F}^0$ *if* $\mathcal{S}F^0 \leq_{\mathcal{L}} \mathcal{S}\tilde{F}^0$,

and

Definition (3.1.9) $F \leq_{P-\mathcal{L}} \tilde{F}$ *if* $\mathcal{P}F \leq_{\mathcal{L}} \mathcal{P}\tilde{F}$.

In words, the Palm distribution functions of T, X and $\tilde{T}, \tilde{X}$ compare for $\leq_{S-\mathcal{L}}$ iff their stationary distribution compare for $\leq_{\mathcal{L}}$, with a similar definition for $P - \mathcal{L}$. Observe that the antisymmetry of the $\leq_{\mathcal{L}}$ ordering is is inherited by the $\leq_{S-\mathcal{L}}$ and the $\leq_{P-\mathcal{L}}$ orderings. This follows immediately from the bijective nature of the mappings $\mathcal{S}$ and $\mathcal{P}$.

3.2. Analytical Properties

S-L.

For all set $\mathcal{L}$ of Borel mappings from $I\!R^+ \times I\!R^n$ onto $I\!R$, denote by $I - \mathcal{L}$ the set of mappings $\phi : I\!R^{n+1} \to I\!R$ that admit an integral representation of the form

$$\phi(t, x_1, \cdots, x_n) = \int_0^t f(u, x_1, \cdots, x_n)du,$$

for some mapping f in $\mathcal{L}$. For instance, on $\mathcal{D}(I\!R^+)$, the set $I - st$ coincides with the set of cx functions on $I\!R^+$ that vanish at the origin. More generally, $\phi(t,x)$: $I\!R^+ \times I\!R^n \to I\!R$ is in $I - st$ iff $(t,x) \to \phi_t'(t,x)$ exists, and is in $\{st\}$.

Similarly define $I^+ - \mathcal{L}$ to be the set of Borel mappings $\phi(t,x) : I\!R^+ \times I\!R^n \to I\!R$ that vanish at the origin and such that $\phi_t'(t,x)$ exists, is *non-negative*, and is in st. On $\mathcal{D}(I\!R^+)$, $I^+ - st$ boils down to the set of icx functions that vanish at the origin.

Theorem (3.2.1) $F^0 \leq_{S-\mathcal{L}} \tilde{F}^0$ *iff the relation*

$$(3.2.2) \qquad \frac{E^0[\phi(T,X)]}{E^0[T]} \leq \frac{\tilde{E}^0[\phi(\tilde{T},\tilde{X})]}{\tilde{E}^0[\tilde{T}]}$$

holds for all ϕ in $I - \mathcal{L}$.

Proof

From the definition (3.1.7), $F^0 \leq_{S-\mathcal{L}} \tilde{F}^0$ iff the relation

$$E[f(T, X)] \leq \tilde{E}[f(\tilde{T}, \tilde{X})]$$

holds for all f in $\mathcal{L}$. Owing to the Palm inversion formula, this in turn reads

$$\lambda E^0[\int_0^T f(T-u, X)du] \leq \tilde{\lambda}\tilde{E}^0[\int_0^{\tilde{T}} f(\tilde{T}-u, \tilde{X})du],$$

where we used the assumption that X (resp. $\tilde{X}$) was a mark of N (resp. $\tilde{N}$). Equivalently, $F^0 \leq_{S-\mathcal{L}} \tilde{F}^0$ iff the relation

$$\lambda E^0[\int_0^T f(u, X)du] \leq \tilde{\lambda}\tilde{E}^0[\int_0^{\tilde{T}} f(u, \tilde{X})du],$$

holds for all f in $\mathcal{L}$, which concludes the proof.

In dimension one, namely when there are no marks, we have the following simpler characterizations,

Corollary (3.2.3) *For F^0_T and $\tilde{F}^0_T$ in $\mathcal{D}$, $F^0_T \leq_{S-st} \tilde{F}^0_T$ iff*

(3.2.4) $$\frac{1}{E^0[T]}\int_x^\infty (1 - F^0_T(u))du \leq \frac{1}{E^0[\tilde{T}]}\int_x^\infty (1 - \tilde{F}^0_T(u))du,$$

which is equivalent to

(3.2.5) $$\frac{E^0[(T-x)^+]}{E^0[T]} \leq \frac{E^0[(\tilde{T}-x)^+]}{E^0[\tilde{T}]},$$

for all $x \geq 0$.

Corollary (3.2.6) *For F^0_T and $\tilde{F}^0_T$ in $\mathcal{D}$, $F^0_T \leq_{S-icx} \tilde{F}^0_T$ if and only if*

(3.2.7) $$\frac{1}{E^0[T]}\int_x^\infty \int_u^\infty (1 - F^0_T(v))dvdu \leq \frac{1}{E^0[\tilde{T}]}\int_x^\infty \int_u^\infty (1 - \tilde{F}^0_T(v))dvdu,$$

which is equivalent to

(3.2.8) $$\frac{1}{E^0[T]}E^0[((T-u)^+)^2] \leq \frac{1}{E^0[\tilde{T}]}E^0[((\tilde{T}-u)^+)^2].$$

The property that two distribution functions F^0 and $\tilde{F}_0$ compare with respect to $\leq_{S-st}$ implies the ordering of the first moments of T:

Lemma (3.2.9) *$F^0 \leq_{S-st} \tilde{F}^0$ implies $E^0[T] \leq \tilde{E}^0[\tilde{T}]$.*

Proof. $F^0 \leq_{S-st} \tilde{F}^0$ implies $F^0_T \leq_{S-st} \tilde{F}^0_T$. From (3.2.4), this in turn implies

$$\frac{1}{E[T]}\frac{1}{x}\int_0^x (1 - F_t^0(u))du \geq \frac{1}{E[\tilde{T}]}\frac{1}{x}\int_0^x (1 - \tilde{F}_t^0(u))du.$$

Letting x tend to zero and using the assumption that $F_t^0(0) = \tilde{F}_t^0(0) = 0$ yields (3.2.9).

As for the marginal distribution of the marks $F^0_X \in \mathcal{D}(IR^n)$, observe that the relation $F^0_X \leq_{S-\mathcal{L}} \tilde{F}^0_X$ reads

$$\frac{E^0[Tf(X)]}{E^0[T]} \leq \frac{\tilde{E}^0[\tilde{T}f(\tilde{X})]}{\tilde{E}^0[\tilde{T}]}, \tag{3.2.10}$$

for all $f : IR^n \to IR$ in $\mathcal{L}$.

Using (3.2.9), one gets the following additional property on $\mathcal{D}(IR^{n+1})$:

Theorem (3.2.11) $F^0 \leq_{I-st} \tilde{F}^0 \;\Rightarrow\; F^0 \leq_{S-st} \tilde{F}^0 \;\Rightarrow\; F^0 \leq_{I^+-st} \tilde{F}^0$.

Proof. $F^0 \leq_{I-st} \tilde{F}^0$ implies $F^0_T \leq_{cx} \tilde{F}^0_T$, which in turn implies $E^0[T] = \tilde{E}^0[\tilde{T}]$. From the very definition, $F^0 \leq_{I-st} \tilde{F}^0$ reads

$$\int_{IR^{n+1}} f(t,x)F^0[dt,dx] \leq \int_{IR^{n+1}} f(t,x)\tilde{F}^0[dt,dx],$$

for all $f \in I - st$. Dividing this by $E^0[T] = \tilde{E}^0[\tilde{T}]$ yields then that $F^0 \leq_{S-st} \tilde{F}^0$ in view of (3.2.1).

As for the second implication, $F^0 \leq_{S-st} \tilde{F}^0$ implies $E^0[T] \leq \tilde{E}^0[\tilde{T}]$. If $f \in I^+ - st$, we have then

$$E^0[f(T,X)] \leq \frac{E^0[T]}{\tilde{E}^0[\tilde{T}]}\tilde{E}^0[f(\tilde{T},\tilde{X})] \leq \tilde{E}^0[f(\tilde{T},\tilde{X})],$$

which concludes the proof.

In dimension one, we see that the stochastic ordering $\leq_{S-st}$ is located right inbetween $\leq_{cx}$ and $\leq_{icx}$.

Corollary (3.2.12) *In $\mathcal{D}$, $\leq_{cx}\Rightarrow\leq_{S-st}\Rightarrow\leq_{icx}$.*

Both implications in (3.2.12) are only oneway (see [4]).

Similarly, using the fact that $I^+ - st \subset st$, one gets the following property

Lemma (3.2.13) $F^0 \leq_{st} \tilde{F}^0 \Rightarrow F^0 \leq_{I^+-st} \tilde{F}^0$.

In dimension one, this property boils down to the well known relation $\leq_{st} \Rightarrow \leq_{icx}$.

P-L.

Similarly, we get the following analytical characterizations from (3.1.4):

Lemma (3.2.14) *$F_X \leq_{P-\mathcal{L}} \tilde{F}_X$ iff the relation*

$$F'_T(0,\infty)^{-1} \int_{I\!R^n} f(x) F'_T(0,dx) \leq \tilde{F}'_T(0,\infty)^{-1} \int_{I\!R^n} f(x) \tilde{F}'_T(0,dx), \tag{3.2.15}$$

holds for all $f \in \mathcal{L}$.

For F_T, the following results follow immediately from (3.1.6)

Lemma (3.2.16) *$F_T \leq_{P-st} \tilde{F}_T$ in C iff for all $t > 0$*

$$\frac{F'_T(t)}{F'_T(0)} \leq \frac{\tilde{F}'_T(t)}{\tilde{F}'_T(0)},$$

Lemma (3.2.17) *$F_T \leq_{P-icx} \tilde{F}_T$ in C iff for all $t > 0$*

$$\frac{\bar{F}_T(t)}{F'_T(0)} \leq \frac{\bar{\tilde{F}}_T(t)}{\tilde{F}'_T(0)},$$

and

Lemma (3.2.18) *$F_T \leq_{P-cx} \tilde{F}_T$ in C iff $F_T \leq_{st} \tilde{F}_T$ and $F'_T(0) = \tilde{F}'_T(0)$.*

Let $\mathcal{L}$ be a class of Borel mappings from $I\!R^{n+1}$ onto $I\!R$ being differentiable with respect to the first variable (e.g. cx or icx). For such a class, define the set $D-\mathcal{L}$ as the set of functions $f : I\!R^+ \times I\!R^n \to I\!R$ that admit a representation of the form $f(t,x) = \phi'_t(t,x)$, where ϕ is in $\mathcal{L}$.

Concerning joint distributions, and for $\mathcal{L}$ as above, we have the following sufficient condition

Theorem (3.2.19) *For the relation $F \leq_{P-\mathcal{L}} \tilde{F}$ to hold, it is enough to have $F_x \leq_{P-\mathcal{L}} \tilde{F}_x$ and*

$$\frac{E[f(T,X)]}{\lambda} \leq \frac{\tilde{E}[f(\tilde{T},\tilde{X})]}{\tilde{\lambda}}, \tag{3.2.20}$$

for all f in $D-\mathcal{L}$.

Proof Using (3.1.4), we get that that for all ϕ in $\mathcal{L}$,

$$E^0[\phi(T,X)] = E^0[\phi(0,X)] + \lambda^{-1} E[\phi'_t(T,X)]. \tag{3.2.21}$$

Under our two assumptions

$$E^0[\phi(0,X)] \leq \tilde{E}^0[\phi(0,\tilde{X})],$$

and

$$\lambda^{-1}E[\phi_t'(T,X)] \leq \tilde{\lambda}^{-1}E[\phi_t'(\tilde{T},\tilde{X})],$$

which concludes the proof.

As a direct consequence of (3.2.11), we also have

Corollary (3.2.22) *In $\mathcal{C}$, $\leq_{P-I-st} \Rightarrow \leq_{st} \Rightarrow \leq_{P-I^+st}$.*

The relationships between all these orderings are summarized in the following chart, which can be viewed as a continuation of (2.2.8)-(2.2.9):

$$\begin{tikzcd}
 & & S-I-st \arrow[r, Rightarrow] & S-S-st \arrow[d, Rightarrow] \\
 & I-st \arrow[r, Rightarrow] & S-st \arrow[r, Rightarrow] \arrow[d, Rightarrow] & S-I^+-st \\
P-I-st \arrow[r, Rightarrow] & st \arrow[r, Rightarrow] \arrow[d, Rightarrow] & I^+-st & \\
P-st \arrow[r, Rightarrow] \arrow[d, Rightarrow] & P-I^+-st & & \\
P-P-I^+-st & & &
\end{tikzcd}$$

It should be kept in mind that the distribution functions under comparison should be in some appropriate sets for these relations to hold.

4. From Palm to Stationary Measures

The discussion will be conducted under the assumption that the mapping $t \rightarrow g(t,y,\xi)$ is non-increasing (e.g the workload process, or the number of customers).

4.1. Strong Bounds

Theorem (4.1.1) *Assume $(t,-y,-\xi) \rightarrow g(t,y,\xi)$ is non-increasing, and*

$$(T,-y,-\xi)[P^0] \geq_{S-st} (\tilde{T},-\tilde{y},-\tilde{\xi})\ [\tilde{P}^0].$$

Then

$$y(0)\ [P] \leq_{st} \tilde{y}(0)\ [\tilde{P}].$$

Proof. Let $f : I\!R^{+K} \to I\!R^+$ be a non-decreasing function such that the random variable $f(y(0))$ is P-integrable (resp. $f(\tilde{y}(0))$ is $\tilde{P}$-integrable). The Palm inversion formula reads

$$E_P[f(y(0))] = \lambda E^0[\int_0^T f \circ g(t, y, \xi) dt]. \tag{4.1.2}$$

The function

$$(T, -y, -\xi) \to \phi(T, -y, -\xi) = \int_0^T (-f \circ g(t, y, \xi)) dt$$

is in $I - st$. Using now the characterization (3.2.2) of $S - st$, together with the ordering assumption, we get

$$\lambda E^0[\phi(T, -y, -\xi)] \geq \tilde{\lambda} \tilde{E}^0[\phi(\tilde{T}, -\tilde{y}, -\tilde{\xi})],$$

which in turn implies

$$E[f(y(0))] \leq \tilde{E}[f(\tilde{y}(0))].$$

In the renewal case, this condition factorizes as follows.

Corollary (4.1.3) *Assume that $(t, -y, -\xi) \to g(t, y, \xi)$ is non-increasing, and that, T and (y, ξ) (resp. $\tilde{T}$ and $(\tilde{y}, \tilde{\xi})$) are P^0 (resp. $\tilde{P}^0$)-independent. If, for all t,*

$$(y, \xi)\ [P^0] \leq_{st} (\tilde{y}, \tilde{\xi})\ [\tilde{P}^0]$$

and

$$T\ [P^0] \geq_{S-st} \tilde{T}\ [\tilde{P}^0],$$

then

$$y(0)\ [P] \leq_{st} \tilde{y}(0)\ [\tilde{P}].$$

Proof. An immediate proof is obtained when observing that $X \leq_{st} \tilde{X}$ and $T \leq_{S-st} \tilde{T}$ imply $(T, X) \leq_{S-st} (\tilde{T}, \tilde{X})$ whenever X and T (resp. $\tilde{T}$ and $\tilde{X}$) are independent. We give another proof based on purely analytical arguments. Let f be as in the preceding proof. Owing to the independence assumption, (4.1.2) rewrites

$$E_P[f(y(0))] = \lambda \int_0^\infty P^0[T > t] E^0[f \circ g(t, y, \xi)] dt,$$

which in turn implies that $E^0[f \circ g(t, y, \xi)] < \infty$ for all t, owing to the decreasingness of $t \to g(t, y, \xi)$. Similarly, the assumption that $\tilde{E}[f(\tilde{y}(0))] < \infty$, implies $\tilde{E}^0[f \circ g(t, \tilde{y}, \tilde{\xi})] < \infty$ for all t.

The assumption $(y,\xi)\ [P^0] \leq_{st} (\tilde{y},\tilde{\xi})\ [\tilde{P}^0]$ implies $g(t,y,\xi)\ [P^0] \leq_{st} g(t,\tilde{y},\tilde{\xi})\ [\tilde{P}^0]$, which in turn implies

$$E_P[f(y(0))] \leq \int_0^\infty \lambda P^0[T > t]\tilde{E}^0[f \circ g(t,\tilde{y},\tilde{\xi})]dt.$$

Integrating by parts the RHS of the above relation yields

$$\begin{aligned} E_P[f(y(0))] &\leq \lambda E^0[T]\tilde{E}^0[f \circ g(\infty,\tilde{y},\tilde{\xi})] \\ &- \int_0^\infty \lambda(\int_0^t P^0[T > u]du)\tilde{E}^0[f \circ g(dt,\tilde{y},\tilde{\xi})] \\ &= \tilde{E}^0[f \circ g(\infty,\tilde{y},\tilde{\xi})] \\ &- \int_0^\infty \lambda(\int_0^t P^0[T > u]du)\tilde{E}^0[f \circ g(dt,\tilde{y},\tilde{\xi})], \end{aligned}$$

where we used the fact that $t \to f \circ g(t,y,\xi)$ is monotone in order to define the Stieltjes integral with respect to $\tilde{E}^0[f \circ g(dt,y,\xi)]$. Observe that the assumptions that T is P^0-integrable and $f(y(0))$ is P-integrable imply that the integral $\int_0^\infty \int_0^t P^0[T > t]E^0[f \circ g(dt,y,\xi)]$ converges.
The assumption that $\tilde{T} \leq_{S-st} T$ and (3.2.4) imply in turn that

$$\lambda \int_0^t P^0[T > u]du \leq \tilde{\lambda} \int_0^t \tilde{P}^0[\tilde{T} > u]du.$$

This together with the assumption that $t \to g(t,y,\xi)$ is non-increasing imply

$$\begin{aligned} E_P[f(y(0))] &\leq \tilde{E}^0[f \circ g(\infty,\tilde{y},\tilde{\xi})] \\ &- \tilde{\lambda} \int_0^\infty (\int_0^t \tilde{P}^0[\tilde{T} > u]du)\tilde{E}^0[f \circ g(dt,\tilde{y},\tilde{\xi})] \\ &= \tilde{\lambda}\tilde{E}^0[\tilde{T}]\tilde{E}^0[f \circ g(\infty,\tilde{y},\tilde{\xi})] \\ &+ \tilde{\lambda} \int_0^\infty (\int_0^t \tilde{P}^0[\tilde{T} > u]du)\tilde{E}^0[f \circ g(dt,\tilde{y},\tilde{\xi})] \\ &= \tilde{\lambda} \int_0^\infty \tilde{E}^0[1(t \leq \tilde{T})f \circ g(t,\tilde{y},\tilde{\xi})dt] \\ &= E_{\tilde{P}}[f(\tilde{y}(0))], \end{aligned}$$

where we used the independence assumption between $\tilde{T}$ and $g(t,\tilde{y},\tilde{\xi})$.

4.2. Bounds by Projection

Based on the same type of proofs, we get the following theorems:

Theorem (4.2.1) *Assume $(t, -y, -\xi) \to g(t, y, \xi)$ is non-increasing and convex, and*

$$(T, -y, -\xi)[P^0] \geq_{S-icv} (\tilde{T}, -\tilde{y}, -\tilde{\xi})\ [\tilde{P}^0].$$

Then

$$y(0)\ [P] \leq_{icx} \tilde{y}(0)\ [\tilde{P}].$$

Proof

The proof is based on the fact that if f is icx, the function

$$(T, -y, -\xi) \to \phi(T, -y, -\xi) = \int_0^T (-f \circ g(t, y, \xi))dt$$

is in $I - icv$.

Corollary (4.2.2) *Assume that for all t, the mapping $(y, \xi) \to g(t, y, \xi)$ is non-negative, non decreasing and convex, and that, for all (y, ξ), $t \to g(t, y, \xi)$ is non-increasing. Assume in addition that, for all t, the random variables T and (y, ξ) (resp. $\tilde{T}$ and $(\tilde{y}, \tilde{\xi})$) are P^0 (resp. $\tilde{P}^0$)-independent. If for all t, the relations*

$$g(t, y, \xi)\ [P^0] \leq_{icx} g(t, \tilde{y}, \tilde{\xi})\ [\tilde{P}^0]$$

and

$$T\ [P^0] \geq_{S-icx} \tilde{T}\ [\tilde{P}^0]$$

hold, then

$$y(0)\ [P] \leq_{icx} \tilde{y}(0)\ [\tilde{P}].$$

5. Association of Random Variables

5.1. Definition and Basic Properties of Association ([5])

The $I\!R$-valued random variables $\{X_1, ..., X_n\}$ are said to be *associated* if the inequality

$$E[f(X)g(X)] \geq E[f(X)]E[g(X)] \tag{3.1.1}$$

holds for all *non-decreasing* mappings $f, g : I\!R^n \to I\!R$ for which the expectations $E[f(X)]$, $E[g(X)]$ and $E[f(X)g(X)]$ exist.

The sequence of $I\!R$-valued random random variables $\{X_1, X_2, \cdots\}$ is said to be associated if the random variables $\{X_1, ..., X_n\}$ are associated for all $n \geq 1$.

The association property can often be established without computing explicitly the joint distribution of the variables. This can be done for instance by using the following "calculus"

(i) *The set consisting of a single random variable is associated;*

(ii) *The union of independent sets of associated random variables forms a set of associated random variables;*

(iii) *Independent random variables are always associated;*

(iv) *Any subset of a set of associated random variables forms a set of associated random variables;*

(v) *For any non-decreasing function* $f : \mathbb{R}^n \to \mathbb{R}$, *and any set of associated random variables* $\{X_1, \cdots, X_n\}$, *the variables* $\{f(X), X_1, \cdots, X_n\}$ *are associated, where* $f(X) \equiv f(X_1, \ldots, X_n)$.

The property of association has several interesting consequences. For instance, if the random variables $\{X_1, ..., X_n\}$ are associated, then

$$\max_{\{i=1,\cdots,n\}} X_i \leq_{st} \max_{\{i=1,\cdots,n\}} \overline{X}_i, \tag{5.1.3}$$

where $\overline{X}$ is the 'product form vector associated with X, namely the vector with mutually independent components and such that $\overline{X}_i$ equals X_i in distribution for all $i = 1, \ldots, n$.

5.2. Association of Stationary Sequences

We now outline how the association properties can be established for the solutions of (2.2.1) (see [3] for the proof).

Theorem (5.2.1) *Assume that the function* $(y, \xi) \to h(y, \eta)$ *is non-decreasing. If the exogeneous sequence* $\{\eta_0, \eta_1, \cdots\}$ *and the initial condition* β *form a set of associated random variables , then the random variables* $\{y_0, y_1, \cdots\}$ *are also associated.*

Again, the existence of a Loynes' schema implies that the transient comparison result of (5.2.1) extends to steady state as well.

5.3. From Palm to Stationary

The main result of this section is the following theorem:

Theorem (5.3.1) *Assume that* $(y, \xi) \to g(t, y, \xi)$ *is non-decreasing and that* $t \to g(t, y, \xi)$ *is monotone. Assume in addition that* T *and* (y, ξ) *are mutually independent with respect to* P^0. *If the components of the vectors* (y, ξ) *form a set of associated random variables with respect to* P^0, *then the components of the vector* $y(0)$ *form a set of associated random variables with respect to* P.

Proof Let f_1 and f_2 be non-decreasing functions $IR^{+K} \to IR$ such that $f_1(y(0))$, $f_2(y(0))$ and $f_1(y(0))f_2(y(0))$, are P-integrable. The Palm inversion formula reads

$$E_P[f_1(y(0))f_2(y(0))] = \lambda E_{P^0}[\int_0^\infty 1(t \leq T) f_1 \circ g(t,y,\xi) f_2 \circ g(t,y,\xi)dt]$$

Owing to the assumption that $(y,\xi) \to g(t,y,\xi)$ is non-decreasing and to properties (iv) and (v) of association, it follows that the random variables $f_1 \circ g(t,y,\xi)$ and $f_2 \circ g(t,y,\xi)$ are associated. This together with the independence assumption immediately implies

$$E_P[f_1(y(0))f_2(y(0))] \geq \lambda \int_0^\infty P^0[t \leq T]E^0[f_1 \circ g(t,y,\xi)]E^0[f_2 \circ g(t,y,\xi)]dt$$

Consider now the function $t \to h(t) = \lambda P^0[T \geq t]$, as the density of the non-negative random variable T, on the time-stationary probability space (Ω, F, P). Denote by $t \to \phi_1(t)$ and $t \to \phi_2(t)$ the real valued functions

$$\phi_1(t) = E^0[f_1 \circ g(t,y,\xi)]$$

and

$$\phi_2(t) = E^0[f_2 \circ g(t,y,\xi)]$$

respectively. We have

$$\lambda \int_0^\infty P^0[t \leq T]E^0[f_1 \circ g(t,y,\xi)]E^0[f_2 \circ g(t,y,\xi)]dt = \int_0^\infty \phi_1(t)\phi_2(t)h(t)dt$$
$$= E_P[\phi_1(T)\phi_2(T)].$$

Observe that the functions $\phi_1(t)$ and $\phi_2(t)$ are either both non-decreasing or non-increasing. This together with the property that the set consisting of the single variable T is associated imply

$$E_P[\phi_1(T)\phi_2(T)] \geq E_P[\phi_1(T)]E_P[\phi_2(T)],$$

namely

$$\lambda \int_0^\infty P^0[t \leq T]E^0[f_1 \circ g(t,y,\xi)]E^0[f_2 \circ g(t,y,\xi)]dt$$
$$\geq \lambda \int_0^\infty P^0[t \leq T]E^0[f_1 \circ g(t,y,\xi)]dt \ \lambda \int_0^\infty P^0[t \leq T]E^0[f_2 \circ g(t,y,\xi)]dt.$$

Using once more the independence assumption between T and $g(t,y,\xi)$ allows one to rewrite this inequality as

$$E_P[f_1(y(0))f_2(y(0))] \geq \lambda E^0[\int_0^T f_1 \circ g(t,y,\xi)dt] \ \lambda E^0[\int_0^T f_2 \circ g(t,y,\xi)dt].$$
$$= E_P[f_1(y(0))]E_P[f_2(y(0))]$$

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