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Résolution d'un problème aux limites non linéaire par une méthode spectrale

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Résumé : On étudie un problème aux conditions aux limites non linéaires posé aussi bien dans l'intérieur que dans l'extérieur du disque unité de $\mathbf{R}^2$. Grâce à l'opérateur capacité, nous le transformons en un problème posé sur le cercle unité. En développant la solution dans la base de Fourier, nous construisons par la méthode de Galerkin le problème approché. Nous montrons la convergence du schéma de point-fixe ainsi qu'une estimation en norme L^2 de l'erreur. Finalement, nous présentons des résultats numériques provenant d'un problème de formage électromagnétique.

A Nonlinear Boundary Value Problem Solved by Spectral Methods

Abstract : We study a nonlinear boundary value problem posed in the interior or the exterior of the unit disk in $\mathbf{R}^2$. Using capacity operator, we transform it into a pseudo-differential problem on the unit circle. The Galerkin method together with Fourier expansion, is used to approximate our problem. We show the convergence of the fixed-point scheme and we give an accurate bound of the L^2-norm of the error. Numerical results coming from a problem arising in electromagnetic casting are also presented.

Key-Words : Nonlinear boundary value problem, fixed-point, Galerkin, spectral methods.

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1. Introduction.

The modelization of various problems arising in domains like fluid mechanics, electromagnetism, elasticity, acoustics, etc ... lead to exterior or interior nonlinear elliptic boundary value problem, see [3], [4], [10], [11] and references given there. Usually, these problems are written in integral form and then solved using boundary element methods, see e.g. [1], [2], [6], [12]. Our purpose, here, is to develop a different approach based on the use of *exterior or interior capacity operator combined with spectral methods*.

We begin by transforming the problem, with the help of the capacity operator, into a pseudo-differential problem on the unit circle, and we show, in part 2, the convergence of a fixed point scheme well adapted to this problem.

In part 3, we examine in details the approximate Galerkin scheme defined on a finite dimensional space of trigonometric polynomials. We show, using the same method as in part 2, the convergence of the approximate fixed point scheme and we give an accurate bound for the error in terms of L^2-norm.

We give, in part 4, first a few simple numerical examples to show the efficiency of our method, then we investigate more deeply a physical example arising from a free boundary problem in electromagnetic casting.

2. The continuous problem.

Let Γ be the unit circle of the plane. We are interested, here, in both exterior or interior boundary value problems posed on Γ. In order not to complicate too much the statement, and because it is the case of our concrete application (part 4) we will make explicit all our results, for **exterior** problems only, but they stay, of course, valid for interior problems where the only change lies in the use of interior capacity operator instead of the exterior one.

Let, then, Ω be the exterior of the unit disk :

$$\Omega = \{(x,y) \in \mathbf{R}^2,\ x^2 + y^2 > 1\}$$

and consider the following nonlinear exterior boundary value problem :

$$(2.1) \qquad \begin{cases} \Delta v = 0 & \text{on } \Omega \\ -\dfrac{\partial v}{\partial n} = \beta(x,v) - f & \text{on } \Gamma \\ v \text{ bounded at infinity} \end{cases}$$

with the following assumptions on the nonlinearity β :

(2.2) • $\beta : \Gamma \times \mathbf{R} \to \mathbf{R}$ is continuous and $\frac{\partial \beta}{\partial u}(x,u)$ is continuous on $\Gamma \times \mathbf{R}$.

(2.3) • There exists λ continuous, piecewise C^1 on Γ, $\lambda \geq 0$, $\lambda \not\equiv 0$ such that $\forall x \in \Gamma, \forall u \in \mathbf{R} \quad \frac{\partial \beta}{\partial u}(x,u) \geq \lambda(x) \geq 0$.

Moreover, we suppose f continuous on Γ.

Remarks :

- The fact that the problem is posed on the interior or exterior of the unit disk is not such a restriction, because every nonlinear boundary value problem posed on a more general open $\widetilde{\Omega}$ with boundary a smooth Jordan curve $\widetilde{\Gamma}$ can be transformed in a problem posed on Ω (or Ω_i : the unit disk) by using a convenient conformal mapping.
- In a recent work, see [12], Ruotsalainen and Wendland considered the same problem as above (with a slightly stronger hypothesis on the nonlinearity : $\frac{\partial \beta}{\partial u}(x,u) \geq \lambda > 0$) weakening the regularity assumptions on β and f, but it is not our purpose here.

Let us, now, recall the definition and main properties of the exterior capacity operator (for more details and proofs, we refer to Benilan, ch. II, § 5 in [8]).

We denote by $\mathcal{C}_b(\overline{\Omega})$ (resp. $\mathcal{C}(\Gamma)$) the space of bounded continuous functions on $\overline{\Omega}$ (resp. Γ) equipped with the uniform norm

$$\|u\|_\infty = \max_{x \in \overline{\Omega}} |u(x)| \qquad (\text{resp. } \|u\|_\infty = \max_{x \in \Gamma} |u(x)|).$$

We denote by $\mathcal{C}^1_n(\overline{\Omega})$ the space of functions $v \in \mathcal{C}_b(\overline{\Omega})$ such that the normal derivative $\frac{\partial v}{\partial n}$ is defined and continuous on Γ.

If $\varphi \in \mathcal{C}(\Gamma)$, we denote by $u_e(\varphi)$ the unique solution of the exterior Dirichlet problem

$$(2.4) \qquad \begin{cases} u \in \mathcal{C}_b(\overline{\Omega}) \\ \Delta u = 0 & \text{on } \Omega \\ u = \varphi & \text{on } \Gamma. \end{cases}$$

We then define the exterior capacity operator C_e by

$$(2.5) \qquad \begin{cases} D(C_e) = \{\varphi \in \mathcal{C}(\Gamma);\ u_e(\varphi) \in \mathcal{C}^1_n(\overline{\Omega})\} \\ C_e\,\varphi = \dfrac{\partial u_e(\varphi)}{\partial n} \end{cases}$$

With this definition, problem (2.1) is equivalent to

$$(2.6) \qquad v = u_e(\varphi)$$

where φ is solution of

$$(2.7) \qquad C_e\varphi + \beta(x,\varphi) = f.$$

Remark. The case for interior problem is exactly the same, definition of interior capacity operator being analogous with the interior Dirichlet problem instead of (2.4).

Here are the properties of the exterior capacity operator we need for the following :

$$(2.8) \quad \forall\varphi \in D(C_e), \quad \nabla u_e(\varphi) \in L^2(\Omega) \text{ and } \int_\Omega |\nabla u_e(\varphi)|^2 dx = \int_\Gamma \varphi C_e \varphi d\gamma.$$

Let $\lambda \in \mathcal{C}(\Gamma)$, $\lambda \geq 0$, $\lambda \not\equiv 0$ then :

(2.9) The operator $\lambda I + C_e$ is one-to-one from $D(C_e)$ into $\mathcal{C}(\Gamma)$.

(2.10) The inverse operator is continuous from $\mathcal{C}(\Gamma)$ into itself and its norm is

$$\|(\lambda I + C_e)^{-1}\|_\infty = \|(\lambda I + C_e)^{-1}1\|_\infty$$

$$(2.11) \qquad (\psi \in \mathcal{C}(\Gamma),\ \psi \geq 0) \Longrightarrow (\lambda I + C_e)^{-1}\psi \geq 0.$$

(2.12) Let $\varphi \in \mathcal{C}(\Gamma), \varphi$ non constant and $x_0 \in \Gamma$ (resp. $x_1 \in \Gamma$) such that $\varphi(x_0) = \max_\Gamma \varphi$ (resp. $\varphi(x_1) = \min_\Gamma \varphi$) then $C_e\varphi(x_0) > 0$ (resp. $C_e\varphi(x_1) < 0$).

We will not prove here these properties which can be found in [8]. We proved in [10], using properties of C_e and classical perturbation results that (2.7) has a unique solution φ^*. Our aim, here, is to calculate it, by a fixed point scheme.

(2.13) Let $\gamma(x,u) = \beta(x,u) - \lambda(x)u$ (according to (2.3) γ is monotone in u).

We prove the convergence (in L^∞-norm) of the following fixed point scheme :

Theorem 2.1. *For φ^0 given in $\mathcal{C}(\Gamma)$, the sequence defined by*

$$(m + \lambda(x))\varphi^{n+1} + C_e\varphi^{n+1} = f - \gamma(x, \varphi^n) + m\varphi^n \tag{2.14}$$

converges (in L^∞-norm) to φ^* solution of (2.7) as soon as the relaxation parameter m is great enough, and φ^0 is in a small enough neighbourhood of φ^*.

Proof. Let us denote by F the nonlinear operator :

$$\begin{aligned} F : \mathcal{C}(\Gamma) &\longrightarrow \mathcal{C}(\Gamma) \\ \varphi &\longmapsto f + m\varphi - \gamma(\,.\,, \varphi) \end{aligned} \tag{2.15}$$

and by T the operator : $T = \big((m+\lambda)I + C_e\big)^{-1} \circ F$. To prove that the fixed point scheme $\varphi^{n+1} = T(\varphi^n)$ converges, it is sufficient to prove that

$$\left\| dT_{(\varphi^*)} \right\|_\infty < 1 \tag{2.16}$$

Since $\big((m+\lambda)I + C_e\big)^{-1}$ is a linear operator, we have to estimate separately $\left\| \big((m+\lambda)I + C_e\big)^{-1} 1 \right\|_\infty$ and $\left\| dF_{(\varphi^*)} \right\|_\infty$.

Let $\psi_0 = \big((m+\lambda)I + C_e\big)^{-1} 1$, and $x_0 \in \Gamma$ such that $\psi_0(x_0) = \max_\Gamma \psi_0$. According to (2.11) we have $\psi_0 \geq 0$ and then $\|\psi_0\|_\infty = \psi_0(x_0)$. Now

$$\big(m + \lambda(x_0)\big)\psi_0(x_0) + C_e\psi_0(x_0) = 1.$$

If $\psi_0 = cst$, we obtain immediatly $\lambda = cst > 0$ and $\psi_0 = 1/(m+\lambda)$. Otherwise $C_e\psi_0(x_0) > 0$ and then $\psi_0(x_0) < \frac{1}{m+\lambda(x_0)}$. So in every case, we have :

$$\left\| \big((m+\lambda)I + C_e\big)^{-1} \right\|_\infty < \frac{1}{m}. \tag{2.17}$$

Now $dF_{(\varphi^*)}$ is the linear operator defined on $\mathcal{C}(\Gamma)$ by :

$$dF_{(\varphi^*)}(h) = \Big(\lambda(.) - \frac{\partial\beta}{\partial u}(.\,,\varphi^*) + m\Big)h \tag{2.18}$$

so we have (in the sense of $\mathcal{C}(\Gamma)$-norm) :

$$\|dF_{(\varphi^*)}\| = \Big\|m - \Big(\frac{\partial\beta}{\partial u}(x,\varphi^*(x)) - \lambda(x)\Big)\Big\|_\infty \tag{2.19}$$

Let $M = \max\limits_{x\in\Gamma}\Big(\dfrac{\partial\beta}{\partial u}(x,\varphi^*(x)) - \lambda(x)\Big)$, because of assumption (2.3) we have $\|dF_{\varphi^*)}\| \leq m$ as soon as $m \geq M$, whence theorem (2.1) since

$$\|dT_{(\varphi^*)}\| \leq \big\|\big((m+\lambda)I + C_e\big)^{-1}\big\| < 1/m \cdot m = 1. \tag{2.20}$$

3. The approximate problem.

We denote, as usual, by S_N the (finite dimensional) space of trigonometric polynomials whose degree do not exceed N, defined as :

$$S_N := \text{span}\{\psi_k = e^{ik\theta};\ -N \leq k \leq N\} \tag{3.1}$$

(the set of functions ψ_k is an orthogonal system over the interval $[0,2\pi]$) and by Π_N the projection operator defined from $L^2(\Gamma)$ into S_N and defined for all $v \in L^2(\Gamma)$ by :

$$(v - \Pi_N v, \psi_k) = 0 \qquad \forall \psi_k \in S_N \tag{3.2}$$

where $(.\,,.)$ denotes the usual inner product $(u,v) = \frac{1}{2\pi}\int_0^{2\pi} u(x)\overline{v}(x)dx$ and $\|.\|_2$ the associated norm.

The use of spectral methods for the resolution of (2.7) is first motivated by the fact that, if v is harmonic and bounded at infinity, it can be expanded on Ω like :

$$v(r,\theta) = \sum_{k=-\infty}^{+\infty} \frac{a_k}{r^{|k|}} e^{ik\theta} \tag{3.3}$$

and then by the two following interesting properties of the operator C_e :

(i) In the case of the unit circle, the expression of C_e is simple :
if $\varphi \in D(C_e)$, $\varphi = \sum_{k=-\infty}^{+\infty} \widehat{\varphi_k} e^{ik\theta}$
then we have :

$$C_e\varphi = \sum_{k=-\infty}^{+\infty} |k|\widehat{\varphi_k} e^{ik\theta}. \tag{3.4}$$

(ii) The commutativity of the operator C_e with the operator Π_N, more precisely S_N is invariant for C_e and :

$$\forall \varphi \in D(C_e) \qquad C_e(\Pi_N(\varphi)) = \Pi_N(C_e(\varphi)) \tag{3.5}$$

(this equality is obvious using formulae (3.3) and (3.4)).

Let us state, now, how the problem (2.7) and the fixed point scheme (2.14) are approximated in S_N by a Fourier–Galerkin approach.

We denote by $\gamma_N = \Pi_N o \gamma$ the projection of the nonlinear operator γ (see (2.13)) on S_N, and also, to simplify notation, $\lambda_N = \Pi_N(\lambda)$ and $f_N = \Pi_N(f)$. The problem (2.14) is written in S_N as :

$$\begin{cases} \text{Find } \varphi_N \text{ in } S_N \text{ such that} \\ m\varphi_N + \Pi_N(\lambda_N\varphi_N) + C_e\varphi_N = f_N - \gamma_N(.,\varphi_N) + m\varphi_N \end{cases} \tag{3.6}$$

and the Galerkin method is defined by the set of equations

$$\begin{cases} \varphi_N \in S_N \text{ and } \forall k \in \{-N,\dots,N\} \\ ((m+\lambda_N)\varphi_N + C_e\varphi_N, \psi_k) = (f - \gamma(x,\varphi_N) + m\varphi_N, \psi_k). \end{cases} \tag{3.7}$$

To solve (3.7), we consider the following fixed point scheme :

$$\begin{cases} \varphi_N^0 \text{ given in } S_N, \quad \varphi_N^{n+1} \in S_N \text{ and} \\ (((m+\lambda_N)I + C_e)\varphi_N^{n+1}, \psi_k) = (f - \gamma(x,\varphi_N^n) + m\varphi_N^n, \psi_k) \\ \text{for } -N \le k \le N \end{cases} \tag{3.8}$$

Equivalently, the Fourier coefficients $\widehat{\varphi}_k^{n+1}$ of φ_N^{n+1} are deduced from the Fourier coefficients $\widehat{\lambda}_k$ of λ_N, $\widehat{\varphi}_k^n$ of φ_N^n and $\widehat{g}_k$ of $f - \gamma(.,\varphi_N^n)$ by the set of linear relations:

$$\begin{cases} \varphi_N^0 \text{ given in } S_N \\ \displaystyle\sum_{\ell=-N}^{N} a_{k\ell}\widehat{\varphi}_\ell^{n+1} = \widehat{g}_k + m\widehat{\varphi}_k^n \qquad -N \le k \le N \end{cases} \tag{3.9}$$

where

$$a_{k\ell} = \begin{cases} \widehat{\lambda}_{k-\ell} & \text{if } 0 < |k-\ell| \leq N \\ \widehat{\lambda}_0 + m + |k| & \text{if } \ell = k \\ 0 & \text{if } |k-\ell| > N \end{cases} \tag{3.10}$$

are the coefficients of the matrix of the system (3.9) which we denote by A_N.

Remark : In the case where we can choose $\lambda = cst$, the matrix A_N is diagonal.

Proposition 3.1. *The $(2N+1)$-matrix A_N is hermitian positive definite(and then invertible).*

Proof. Since λ_N is a **real** function defined by $\lambda_N(x) = \sum_{k=-N}^{N} \widehat{\lambda}_k \psi_k(x)$, A_N is clearly hermitian. For each $\varphi \in S_N$, we denote by $\vec{\varphi}$ the vector of spectral coefficients $\vec{\varphi} = {}^T(\widehat{\varphi}_{-N}, \ldots, \widehat{\varphi}_N)$. It follows from the properties of the set $\{\psi_k\}$ that

$$\langle A_N \vec{\varphi}, \vec{\varphi} \rangle = \Big(((m + \lambda_N) I + C_e)\varphi, \varphi \Big) \tag{3.11}$$

where $\langle \cdot, \cdot \rangle$ is the canonical hermitian product on $\mathbf{C}^{2N+1}$. Hence the strict positivity of A_N follows using (2.8) and the fact that φ is a real function.

Then the Galerkin fixed point scheme (3.9) is well defined. We give, now a convergence result of scheme (3.8) analogous to theorem 2.1. Due to the Galerkin approach, it is more convenient to use the L^2-norm, which we will do in the following.

Theorem 3.1. *For N given, N large enough, and m like in theorem 2.1, the problem (3.6) has a unique solution and for φ_N^0 in a small enough neighbourhood of $\Pi_N(\varphi^*)$ the sequence $(\varphi_N^n)_{n\geq 0}$ defined by (3.8) converges as $n \to +\infty$ to the solution φ_N of (3.6).*

We will use in the proof of this theorem as well in the next one the following lemma :

Lemma 3.1. *There exists a strictly positive constant C_0 which depends only on λ such that*

$$\forall v \in H^1_{\mathrm{loc}}(\overline{\Omega}) \qquad C_0 \int_\Gamma v^2 \leq \int_\Gamma \lambda v^2 + \int_\Omega |\nabla v|^2. \tag{3.12}$$

Proof of the lemma. Assume that (3.12) is not true. Then there exists a sequence (v_k) of functions $v_k \in H^1_{\text{loc}}(\overline{\Omega})$ such that

$$(3.13) \qquad \begin{cases} \dfrac{1}{k} \geq \displaystyle\int_\Gamma \lambda v_k^2 + \int_\Omega |\nabla v_k|^2 \\ \displaystyle\int_\Gamma v_k^2 = 1. \end{cases}$$

Let $\omega = B(0,R) \setminus \overline{B}(0,1)$ where $R > 1$. It is classical that there exists a constant C (depending only on R) such that

$$(3.14) \qquad \forall v \in H^1(\omega) \qquad \int_\omega v^2 \leq C\Big[\int_\Gamma v^2 + \int_\omega |\nabla v|^2\Big]$$

see, for instance [8], p. 928.

Applying (3.14) to $v_k\big|_\omega$ we obtain :

$$(3.15) \qquad \int_\omega v_k^2 \leq C\Big(1 + \frac{1}{k}\Big)$$

hence v_k is bounded in $H^1(\omega)$. By Rellich theorem, there exists a subsequence (still denoted by v_k) which converges in $L^2(\omega)$ to v.
Since ∇v_k converges to 0 in $L^2(\omega)$ (according to (3.13)) we obtain $\nabla v = 0$ on ω and $v = cst$ on ω.

Moreover, since v_k converges to $v = cst$ in $H^1(\omega)$, using continuity of the trace application we have $v_k\big|_\Gamma$ converges to $v\big|_\Gamma = cst$ in $L^2(\Gamma)$. We deduce, from (3.13) that $0 = \int_\Gamma \lambda v^2 = cst \int_\Gamma \lambda$. Since $\int_\Gamma \lambda \neq 0$ we have $cst = 0$, but this is in contradiction with $v_n\big|_\Gamma$ converge, in $L^2(\Gamma)$ to 0 and $\int_\Gamma v_n^2 = 1$.

Proof of theorem 3.1. The sequence φ_N^n is defined by

$$\Pi_N[(m + \lambda_N)I_N + C_e]\varphi_N^{n+1} = \Pi_N[f - \gamma(.,\varphi_N^n) + m\varphi_N^n]$$

or equivalently :

$$(\varphi_N^{n+1}, \psi_k) = (T_N(\varphi_N^n), \psi_k) \qquad -N \leq k \leq N$$

where :

$$(3.16) \qquad T_N = \Big[\Pi_N\big((m + \lambda_N)I_N + C_e\big)\Big]^{-1} \circ \Pi_N \circ F$$

(F is the nonlinear operator defined in (2.15) and I_N the identity operator of S_N). To prove theorem 3.1, it is sufficient to prove that, for N large enough, we have :

$$\left\| dT_{N(\Pi_N(\varphi^*))} \right\| < 1 \tag{3.17}$$

Now :

$$dT_{N(\Pi_N(\varphi^*))} = \Big[\Pi_N\big((m+\lambda_N)I_N + C_e\big)\Big]^{-1} \circ \Pi_N \circ dF_{(\Pi_N(\varphi^*))}. \tag{3.18}$$

Let us estimate the L^2-norm of the operator

$$X_N = \Big[\Pi_N\big((m+\lambda_N)I_N + C_e\big)\Big]^{-1}.$$

Let ψ and φ be in S_N such that $(\varphi, \psi_k) = (X_N\psi, \psi_k)$ for $-N \leq k \leq N$. Then we have

$$\big((m+\lambda_N)\varphi + C_e\varphi, \psi_k\big) = (\psi, \psi_k) \qquad \text{for } -N \leq k \leq N \tag{3.19}$$

Multiplying by $\widehat{\varphi}_k$, the k^{th} spectral coefficient of φ, and adding up from $k = -N$ to N yields to (see (2.8)) :

$$m\|\varphi\|_2^2 + \int_\Gamma \lambda_N|\varphi|^2 + \int_\Omega |\nabla\varphi|^2 = \int_\Gamma \psi\overline{\varphi} \leq \|\varphi\|_2\|\psi\|_2. \tag{3.20}$$

For every v in $L^2(\Gamma)$, we set

$$Q_N(v) = v - \Pi_N(v). \tag{3.21}$$

Report (3.21) in (3.20) to obtain :

$$m\|\varphi\|_2^2 + \int_\Gamma \lambda|\varphi|^2 - \int_\Gamma Q_N(\lambda)|\varphi|^2 + \int_\Omega |\nabla\varphi|^2 \leq \|\varphi\|_2\|\psi\|_2. \tag{3.22}$$

Using lemma 3.1 and uniform majoration of $Q_N(\lambda)$ we obtain

$$\big(m + C_0 - \|Q_N(\lambda)\|_\infty\big)\|\varphi\|_2 \leq \|\psi\|_2. \tag{3.23}$$

The regularity assumptions (2.3) on λ imply that we can choose N large enough so that

$$\|Q_N(\lambda)\|_\infty < \frac{C_0}{2} \qquad \text{(see, for instance [5], p. 278).} \tag{3.24}$$

Therefore

$$(3.25) \qquad \left\| \left[\Pi_N\left((m+\lambda_N)I_N + C_e\right)\right]^{-1} \right\|_2 < \frac{1}{m + C_0/2} \,.$$

Now, since $dF_{(\varphi)}(h) = \left(\lambda - \frac{\partial\beta}{\partial u}(.,\varphi) + m\right)h$, we still have

$$\left\|dF_{(\varphi)}\right\|_2 \leq \left\|\lambda - \frac{\partial\beta}{\partial u}(.,\varphi) + m\right\|_\infty$$

and we use continuity of $\frac{\partial\beta}{\partial u}$ to claim that in a neighbourhood of $\varphi*$, we have $\left\|dF_{(\varphi)}\right\|_2 \leq m + C_0/2$ as soon as $m \geq M$ (see (2.20)). So we can choose N in order that $\Pi_N(\varphi*)$ be in this neighbourhood. Since $\left\|\Pi_N\right\|_2 \leq 1$, we finally obtain

$$(3.26) \qquad \begin{aligned} &\left\|dT_{N_{(\Pi_N(\varphi^*))}}\right\|_2 \\ &\leq \left\|\left[\Pi_N\left((m+\lambda_N)I_N + C_e\right)\right]^{-1}\right\|_2 \cdot \|\Pi_N\|_2 \left\|dF_{(\Pi_N(\varphi^*))}\right\|_2 < 1 \end{aligned}$$

which proves, in the same time, convergence of the sequence φ_N^n and existence of a solution for the problem (3.6).

Uniqueness of this solution is easily obtained using the same argument as in 3.30 below, which proves theorem 3.1.

To conclude our study of the approximate problem, it is necessary to give an estimate of the difference between the approximate solution φ_N in S_N limit of the sequence defined in 3.8, and the exact solution φ^*.
We obtain the following result :

Theorem 3.2. *Let φ^* be the solution of (2.7) and φ_N the solution of (3.6). Then, there exists a constant C depending only on λ, β and φ^* such that*

$$\|\varphi_N - \varphi^*\|_2 \leq C\left(\|\lambda - \Pi_N\lambda\|_2 + \|\varphi^* - \Pi_N\varphi^*\|_2\right)$$

Corollary. *Assume that the solution $\varphi^* \in H^s(\Gamma), s \geq 1$ and $\lambda \in H^p(\Gamma)$, $p \geq 1$, then*

$$\|\varphi_N - \varphi^*\|_2 \leq C' N^{-\min(s,p)}\left(\|\varphi^*\|_{H^s(\Gamma)} + \|\lambda\|_{H^p(\Gamma)}\right)$$

Important remark. This theorem show that the accuracy of the approximation essentially depends on the regularity of λ on the one hand and the solution φ^* on the other hand. If we can't act on the regularity of φ^* which is nearly a data of the problem, on the other hand, we can always choose λ verifying (2.3) with a prescribed regularity (for more details, see part. 4).

Note that the regularity of the data f does not appear explicitely in this theorem, but it is implicitely "contained" in the regularity of the solution φ^*.

Proof of theorem 3.2. We begin by estimating the distance between $\Pi_N\varphi^*$ and φ_N.
φ_N is solution of (see (3.6), (3.7))

$$((\lambda_N I_N + C_e)\varphi_N + \gamma(.\,,\varphi_N), \psi_k) = (f, \psi_k) \quad -N \le k \le N \tag{3.27}$$

and by projection of $\lambda\varphi^* + C_e\varphi^* + \gamma(.\,,\varphi^*) = f$ on S_N we obtain

$$(C_e(\Pi_N\varphi^*), \psi_k) = (f - \lambda\varphi^* - \gamma(.\,,\varphi^*), \psi_k) \quad -N \le k \le N$$

so $\Pi_N\varphi^*$ is solution of

$$\begin{aligned} &((\lambda_N I_N + C_e)\Pi_N\varphi^* + \gamma(.\,,\Pi_N\varphi^*), \psi_k) \\ &\quad = (f + \lambda_N\Pi_N\varphi^* - \lambda\varphi^* + \gamma(.\,,\Pi_N\varphi^*) - \gamma(.\,,\varphi^*), \psi_k) \\ &\quad -N \le k \le N \end{aligned} \tag{3.28}$$

Let us set $e_N = \Pi_N\varphi^* - \varphi_N$. Substracting (3.27) from (3.28), we obtain:

$$\begin{aligned} &((\lambda_N I_N + C_e)e_N + \gamma(.\,,\Pi_N\varphi^*) - \gamma(.\,,\varphi_N), \psi_k) \\ &\quad = (\lambda_N\Pi_N\varphi^* - \lambda\varphi^* + \gamma(.\,,\Pi_N\varphi^*) - \gamma(.\,,\varphi^*), \psi_k) \end{aligned} \tag{3.29}$$

Multiplying by the spectral coefficients of e_N and adding up from $-N$ to N yields

$$\begin{aligned} &\int_\Gamma \lambda_N |e_N|^2 + \int_\Gamma C_e e_N \overline{e_N} + \int_\Gamma (\gamma(.\,,\Pi_N\varphi^*) - \gamma(.\,,\varphi^*))\overline{e_N} \\ &\quad = (\lambda_N\Pi_N\varphi^* - \lambda\varphi^* + \gamma(.\,,\Pi_N\varphi^*) - \gamma(.\,,\varphi^*), e_N) \end{aligned} \tag{3.30}$$

or (see (2.8), (2.13))

(3.31)

$$\begin{aligned} &\int_\Gamma \lambda_N |e_N|^2 + \int_\Omega |\nabla e_N|^2 \\ &\quad + \int_\Gamma \left[\beta(.\,,\Pi_N\varphi^*) - \lambda\Pi_N\varphi^* - \beta(.\,,\varphi^*) + \lambda\varphi^*\right]\overline{(\Pi_N\varphi^* - \varphi_N)} \\ &\quad = \int_\Gamma (\lambda_N - \lambda)\Pi_N\varphi^*\overline{e_N} + \int_\Gamma \left[\beta(.\,,\Pi_N\varphi^* - \beta(.\,,\varphi^*)\right]\overline{e_N} \end{aligned}$$

Monotonicity property of $u \mapsto \beta(.,u) - \lambda u$ (see 2.3) and existence of constants C_1, C_2 such that

$$\|\Pi_N \varphi^*\|_\infty \leq C_1 \text{ and } |\beta(.,\Pi_N \varphi^*) - \beta(.,\varphi^*)| \leq C_2 |\Pi_N \varphi^* - \varphi^*|$$

imply

$$(3.32)\quad \int_\Gamma \lambda_N |e_N|^2 + \int_\Omega |\nabla e_N|^2 \leq C_1 \int_\Gamma |\lambda_N - \lambda| \overline{e_N} + C_2 \int_\Gamma |\Pi_N \varphi^* - \varphi^*| \overline{e_N}$$

Using Cauchy–Schwarz inequality for second member of (3.32) yields :

$$(3.33)\quad \int_\Gamma \lambda_N |e_N|^2 + \int_\Omega |\nabla e_N|^2 \leq (C_1 \|Q_N(\lambda)\|_2 + C_2 \|Q_N(\varphi^*)\|_2) \|e_N\|_2$$

We write now $\lambda_N = \lambda - Q_N(\lambda)$ and we use lemma 3.1 together with uniform majoration of $Q_N(\lambda)$ to obtain :

$$(C_0 - \|Q_N(\lambda)\|_\infty) \|e_N\|^2_{L^2(\Gamma)} \leq (C_1 \|Q_N(\lambda)\|_2 + C_2 \|Q_N(\varphi^*)\|_2) \|e_N\|_{L^2(\Gamma)}$$

whence theorem 3.2 using the triangle inequality

$$\|\varphi_N - \varphi^*\|_2 \leq \|Q_N(\varphi^*)\|_2 + \|e_N\|_2$$

and the fact that $\|Q_N(\lambda)\|_\infty$ can be chosen (for great N) as little as wanted.

The proof of corollary follows from estimation of $\|v - \Pi_N v\|_2$ since when $v \in H^m_{(\Omega)}$, we have : (see [5], p. 227)

$$\|v - \Pi_N v\|_2 \leq C N^{-m} \|v\|_{H^m_{(\Omega)}}.$$

4. Numerical results.

We begin by giving some remarks about the algorithm described above. At each iteration of the fixed point scheme, we have to compute the spectral coefficients of the right hand side of (3.8). Due to the nonlinearity β, it seems better to us, and especially cheaper, to use a procedure based on Fast Fourier Transform (F.F.T.) to calculate them. To obtain the approximate solution φ_N, we consider the following algorithm :

(4.1)
- φ^0_N given in S_N
- for $n \geq 0$, then
- **step 1 :** evaluation of the vector of spectral coefficients, $\vec{g}$, of the non-linear term : $f + m\varphi^n_N - \gamma(.,\varphi^n_N)$ by F.F.T.
- **step 2 :** computation of φ^{n+1}_N by solving the system :

$$A_N \overrightarrow{\varphi^{n+1}_N} = \vec{g}$$

The convergence of the algorithm is obtained when

$$\|\overrightarrow{\varphi_N^{n+1}} - \overrightarrow{\varphi_N^{n}}\|_\infty < 10^{-6}.$$

Remarks :

– The error between the exact spectral coefficients and those obtained by F.F.T. is of the same order as the approximation error, (see [5]).

– The evaluation of the spectral coefficients of the nonlinearity needs about $N \log_2 N$ operations

– The resolution of the system in step 2, is done using a Choleski factorization of the band matrix A_N, (unless if λ is constant, of course !).

4.1. First example.

The two numerical examples we give here are intended to illustrate the theoretical error analysis and also to prove the accuracy of our method. They are taken from the paper [12]. We became aware of that paper after developing our method for example 2 next. We chose to apply it to the same examples as in [12] in order to compare their efficiency. We consider the same two types of non-linearities, namely:

$$-\frac{\partial u}{\partial n} = u + \sin u - f \qquad \text{on } \Gamma \tag{4.2}$$

$$-\frac{\partial u}{\partial n} = |u|u^3 - f \qquad \text{on } \Gamma \tag{4.3}$$

Remarks :

– We construct f such that the exact solution is

$$u(r,\theta) = \frac{4r\cos\theta + 2}{4r^2 + 4r\cos\theta + 1} \tag{4.4}$$

(we have to choose the exact solution, not in S_N, otherwise the results would be misleaded, the convergence being very fast. Then, we did not choose $u \equiv 1$ which gives an immediate convergence here, with $N = 1$).

– Although these nonlinearities do not satisfy the condition (2.3) globally, in a neighbourhood of the solution u, we have $\frac{\partial \beta}{\partial u} \geq m = cst > 0$, then our algorithm with $\lambda(x) = 0$ and $m > 0$ converges in all the cases we studied.

So the problem is : find u harmonic in Ω verifying either 4.2 or 4.3. We obtain the following results :

N	8	12	20
L^2 error example 4.2	$6.8\ 10^{-3}$	$2.3\ 10^{-4}$	$4.8\ 10^{-7}$
L^2 error example 4.3	$9.4\ 10^{-4}$	$4.7\ 10^{-5}$	$9.7\ 10^{-7}$

Table 4.1

where N denotes the number of modes (we recall that the number of spectral coefficients computed is, then, $2N+1$) and the L^2 error is the L^2-norm of the difference between the exact solution (4.4) and those obtained by the algorithm (4.1).

We numerically observe that the approximate orders of convergence are exponential, more precisely we have :

$$L^2\text{error} \leq Ce^{-aN} \tag{4.5}$$

with $a \simeq 4.9\,10^{-3}$ in the first case, and $a \simeq 4.3\,10^{-3}$ in the second case. This result is completely in agreement with our theoretical study, because in these examples, the exact solution is **analytic** on Γ and then the distance between u and $\Pi_N u$ decreases exponentially with N.

4.2. A physical example.

The study of a free boundary problem arising in electromagnetic casting leads us to consider the following problem (we refer to [9], [10] for more details) :

$$\begin{cases} \Delta v = 0 & \text{on } \Omega \\ -\dfrac{\partial v}{\partial n} = -g^2 e^{-v} + pe^v - f & \text{on } \Gamma \\ v \quad \text{bounded at infinity} \end{cases} \tag{4.6}$$

where p is a constant and g is a known function, analytic on Γ in our case. The physical interpretation of these equations is the following :

We set $v(x,y) = \ell n|\Phi'(x+iy)|$ where Φ is the conformal mapping from Ω to the exterior of the free boundary, and the nonlinear boundary condition express the equilibrium between the pressure magnetic forces and the superficial tension forces.

Since here

$$\frac{\partial \beta}{\partial v}(x,v) = g^2(x)e^{-v} + pe^v \geq 2\sqrt{p}|g(x)| \tag{4.7}$$

assumption (2.3) is verified with

$$\lambda(x) = 2\sqrt{p}|g(x)| \tag{4.8}$$

Moreover, we can prove, using analyticity of g that the solution v of (4.6) is analytic up to the boundary, see [9]. The term $\|\varphi^* - \Pi_N \varphi^*\|_2$ in theorem 3.2 has therefore an exponential decrease. The accuracy of our approximate solution φ_N depends essentially on the choice of λ. In the case when g does not vanish on Γ, the choice of $\lambda(x) = 2\sqrt{p}|g(x)|$ is quite good, since λ is still analytic. But in the most frequent case where g vanish, this choice is not good, as shown by the table below, and it is much better to choose :

$$\tilde{\lambda}(x) = 2\sqrt{p}\frac{g^2(x)}{\|g\|_\infty} \tag{4.9}$$

which is analytic and less than $2\sqrt{p}|g(x)|$. The results below illustrate, in a convincing way, our theoretical study, and particularly theorem 3.2.

Remark : In our case, we are easily able to choose a function $\tilde{\lambda}(x)$ more regular than the initial function $\lambda(x)$ because λ is the absolute value of a regular function. More generally, one can still make such a choice, using spline functions which vanish at the same points as λ with horizontal tangent at this points. It is always possible to construct such a spline function, positive, less than λ with a prescribed regularity, using polynomials like $(x - x_i)^{2m}$ (where x_i is a zero of λ).

We choose the two following functions g which arise when the magnetic field is created by linear vertical conductors (see [10])

$$g_1(x) = \frac{240 \cos 2x}{257 - 32 \cos 4x} \tag{4.10}$$

$$g_2(x) = 4 + \frac{128 \cos 4x - 1}{257 - 32 \cos 4x} \tag{4.11}$$

and we construct f such that the exact solution is still given by (4.4). In both cases, we are interested in the evolution of the L^2-norm of the error when N increases.

In the first case, where g vanishes, we can observe a significant difference between the choices of λ (given by 4.8) or $\widetilde{\lambda}$ (given by 4.9) due to the regularity of $\widetilde{\lambda}$.

N	8	12	16	20	24
L^2 error, λ chosen	$6.9\ 10^{-3}$	$5.8\ 10^{-3}$	$2.4\ 10^{-3}$	$1.3\ 10^{-3}$	$2.4\ 10^{-4}$
L^2 error, $\tilde{\lambda}$ chosen	$2.4\ 10^{-2}$	$2.1\ 10^{-3}$	$1.7\ 10^{-4}$	$1.3\ 10^{-5}$	$4.6\ 10^{-7}$

Table 4.2

The figures below, show the module of the spectral coefficients of the exact solution and approximates solutions for different values of N in the two cases. We observe that the influence of the aliasing terms due to the FFT is clearly less important in the regular case (choice of $\tilde{\lambda}$) than in the other.

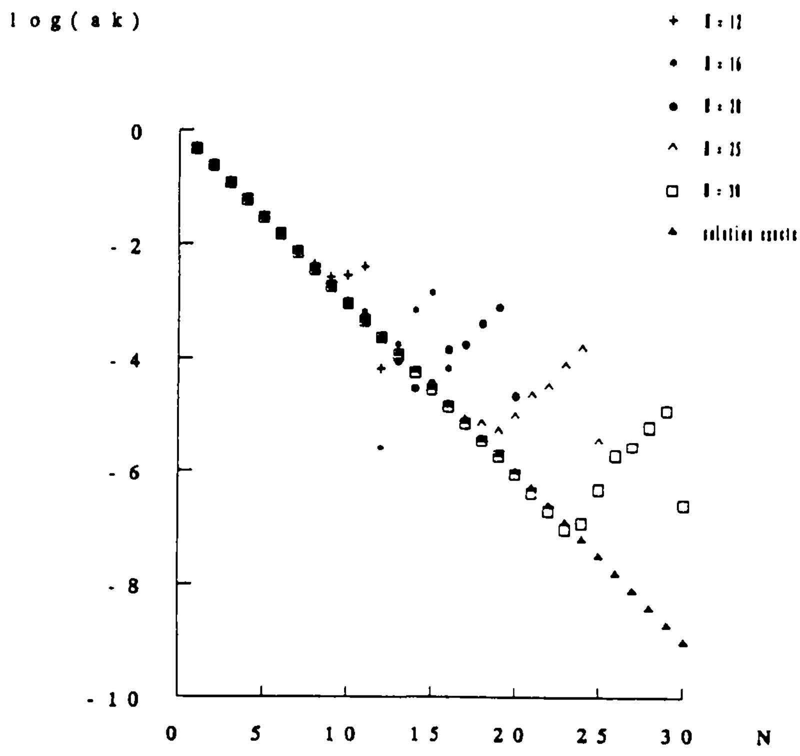

Fig. 4.1 a. Spectral coefficients of the solution, with λ

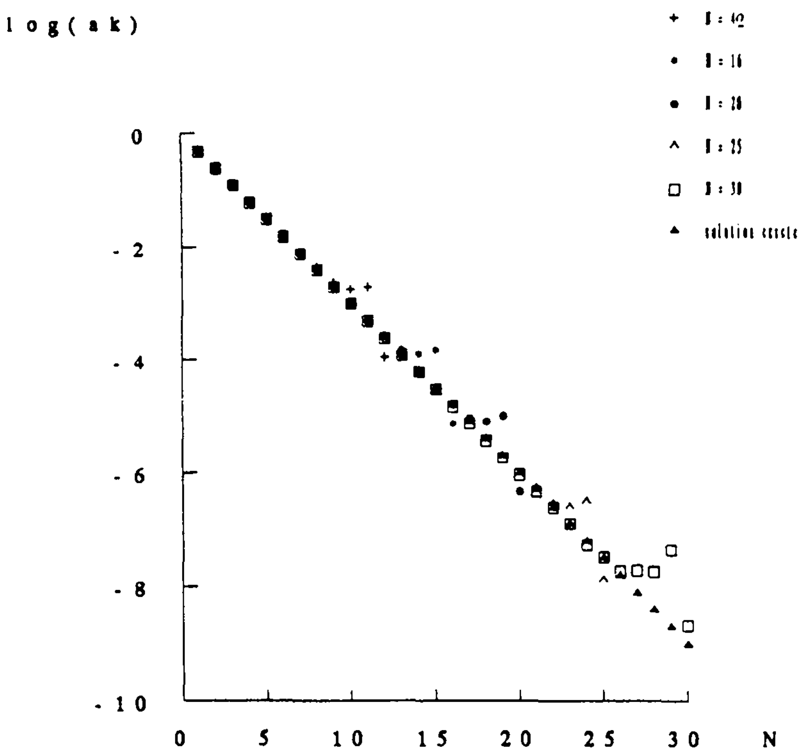

Fig. 4.1 b. Spectral coefficients of the solution, with $\widetilde{\lambda}$

In the case of g_2 (given by 4.11), both functions λ and $\widetilde{\lambda}$ are regular and we do not observe any significant difference in the convergence which is still quite good (see table 4.4 below). We can also choose here $\lambda = cst$ (e.g. $\lambda = 3$) for which algorithm (4.1) is quite faster because the matrix A_N is diagonal.

N	8	12	16	20	24
L^2 error, with λ (given by 4.8)	$4.5\ 10^{-3}$	$3.1\ 10^{-4}$	$2.1\ 10^{-5}$	$1.4\ 10^{-6}$	$2.8\ 10^{-7}$
L^2 error, $\lambda = cst$ chosen	$4.6\ 10^{-3}$	$2.8\ 10^{-4}$	$1.7\ 10^{-5}$	$1.2\ 10^{-6}$	$4.2\ 10^{-7}$

Table 4.4

Since $\widetilde{\lambda}$ is analytic in these two examples, we can observe on tables 4.2 and 4.4 that the convergence is still of exponential order in accordance with our theorem 3.2.

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