Fast solution of discrete Isaacs' inequalities

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Rapports de Recherche

N° 1167

Programme 5
Automatique, Productique,
Traitement du Signal et des Données

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Février 1990

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SOLUTION RAPIDE DES INEQUATIONS D'ISAACS DISCRETES

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+ This paper is included in the activities developed in the frame of the Cooperation Project INRIA - Instituto de Matemática Beppo Levi (coordinators of the project : E. Rofman - R. González).

RESUME

On présente ici un algorithme acceleré pour la résolution numérique du problème de point fixe associé à une discrétisation de l' inéquation d'Isaacs correspondant aux problèmes de jeux différentiels avec temps d'arrêt

ABSTRACT

In this paper we present an accelerated algorithm to solve the fixed point problem related to a special discretization of a differential game problem with stopping times.

1- INTRODUCTION

Frequently, differential games problems with stopping times originate bilateral variational inequalities (see [4] and [7]). When the actualization rate of the original problem is small (see [8]), the numerical resolution (found by finite elements method and relaxation type iterative algorithms) may lead to slowly convergent procedures. In this paper, we present an accelerated algorithm to improve the convergence.

The developed procedure, which has been specially designed for the treatment of this particular kind of games, consists in an extension of the methodology presented in [5] for optimal control problems. It is proved here that the accelerated algorithm converges to the discrete solution in a finite number of steps. The proof is based on the use of a system of quasi-variational inequalities associated with the bilateral inequality (see [4]).

Various stopping rules for the internal loop of the algorithm are presented. They allow us to extend the acceleration procedure to those problems where some of the original hypotheses, are not verified (see [5] pag 7).

Extensions to the general case of differential games and to optimization problems with other contractive operators are considered in [6].

2- DESCRIPTION OF THE DISCRETIZED PROBLEM AND ITS SOLUTION.

2.1 Elements of the problem.

Let B be a square matrix of order n such that

i) $B(i, j) \geq 0 \quad \forall\ i,j = 1,\dots,n,$

$$\text{ii)} \sum_{j} B(i, j) \leq \gamma \quad \forall\ i,\ \gamma < 1, \tag{1}$$

let $\psi_1(j)$, $\psi_2(j)$, $j=1,\dots,n$ be such that

iii) $\psi_2(j) - \psi_1(j) \geq \alpha > 0 \qquad \forall\ j=1,\dots,n,$

and let $f(j) \in \Re$, $j=1,\dots,n$.

Definition:

For $w \in \Re^n$, we define an operator $M : \Re^n \to \Re^n$ by

$(Mw)(j) = P_{\mathfrak{I}}([Bw+f])(j)$,

where $P_{\mathfrak{I}}$ represents the projection over the interval $\mathfrak{I}$, with $\mathfrak{I}=[\psi_1(j), \psi_2(j)]$.

In terms of single operations, operator M is given by the formula:

$$(Mw)(j)=\max\Big(\min\big((Bw+f)(j),\psi_2(j)\big),\psi_1(j)\Big)$$

The acceleration algorithm proposed in this paper, computes the solution of the following problem:

$$\boxed{\textit{Find } \bar{w}\in\Re^n\textit{, such that } M\bar{w}=\bar{w}.} \qquad (2)$$

2.2 Existence and uniqueness of the solution.

Proposition 2.1:

Operator M is contractive and therefore there exists an unique solution for (2).

Proof:

We introduce in $\Re^n$ the norm $\|w\| = \max\{|w(j)|, j=1,\dots,n\}$; due to properties i) and ii), we have:

$$\|Mw - M\hat{w}\| \le \gamma \|w - \hat{w}\| \quad \forall\, w, \hat{w} \in \Re^n. \qquad (3)$$

In fact:

$(Mw)(j)=\max\big(\min((Bw+f)(j),\psi_2(j)),\psi_1(j)\big)$

Let $\xi=\|w-\hat{w}\|$, then $w(i) \le \hat{w}(i) + \xi \quad \forall\, i$,

therefore, $(Bw)(j) \le (B\hat{w})(j) + \gamma\,\xi$, so

$\min((Bw+f)(j), \psi_2(j)) \le \min((B\hat{w}+f)(j) + \gamma\,\xi, \psi_2(j)+\gamma\,\xi)$.

Hence

$$\max\big((\min((Bw+f)(j),\psi_2(j)),\psi_1(j))\big) \le \max\big(\min((B\hat{w}+f)(j)+\gamma\xi,\psi_2(j)+\gamma\xi),\psi_1(j)+\gamma\xi\big)=$$
$$= \max\big(\min((B\hat{w}+f)(j), \psi_2(j)), \psi_1(j)\big) + \gamma\,\xi$$

that is:

$$(Mw)(j) \le (M\hat{w})(j) + \gamma\,\xi$$

Similarly it is proved that:

$$(Mw)(j) \ge (M\hat{w})(j) - \gamma\,\xi$$

which implies:

$$\|Mw - M\hat{w}\| \le \gamma \|w - \hat{w}\| \qquad \square$$

2.3 **Characterization of the solution and its computation.**

Properties:

1°) If $\overline{w} = M\overline{w}$, then

a) $\overline{w}(j) < \psi_2(j) \Rightarrow (B\overline{w}+f)(j) \leq \overline{w}(j)$

b) $\overline{w}(j) > \psi_1(j) \Rightarrow (B\overline{w}+f)(j) \geq \overline{w}(j)$.

2°) The Fixed Point Theorem gives us the following algorithm for the computation of $\overline{w}$:

A0 algorithm:

Step 1: set $w^0 \in \Re^n$, and $\nu=0$.

Step 2: compute $w^{\nu+1}=Mw^{\nu}$

Step 3: if $w^{\nu}(i) = w^{\nu+1}(i) \quad \forall$ i, then stop; else set $\nu=\nu+1$ and go to Step 2.

For the algorithm convergence the following result holds (see [3]):

Theorem 2.1

A0 algorithm produces either a finite sequence w^{ν} whose last element is the exact solution of the problem, or generates an infinite sequence converging to $\overline{w}$. Also, the following bound for the approximation error is valid:

$$\|w^{\nu} - \overline{w}\| \leq \gamma^{\nu} \| w^0 - \overline{w} \|. \tag{4}$$

3- **DESCRIPTION OF THE ACCELERATED ALGORITHM.**

Definitions:

- Given $I(j)$, $j=1,\dots,n$, a family of indexes taking values in {0,1,2}, we define:

$M_I : \Re^n \to \Re^n$, such that:

$v \to M_I v$

where:

$$(M_I v)(j) = \begin{cases} \psi_1(j) & \text{if } I(j)=1 \\ \psi_2(j) & \text{if } I(j)=2 \\ (Bv+f)(j) & \text{if } I(j)=0. \end{cases} \tag{5}$$

• We define:

$$
I(j,w) = \begin{cases} 0 & \text{if } (Mw)(j) \in (\psi_1(j),\ \psi_2(j)) \\ 1 & \text{if } (Mw)(j) = \psi_1(j) \\ 2 & \text{if } (Mw)(j) = \psi_2(j). \end{cases} \tag{6}
$$

• Given the sequence $\{w^{\mu,\nu},\ \mu \geq 0,\ \nu \geq 0\}$, let $R(w^{\mu,o},\ldots,w^{\mu,\nu})$, for all $\mu \geq 0$, be a "stopping rule" with values 0 and 1. We shall assume the following condition is satisfied:

$$
\forall (\mu, w^{\mu,o})\ \exists\ \bar{\nu}(\mu, w^{\mu,o})\ /\ R(w^{\mu,o},\ldots,w^{\mu,\bar{\nu}(\mu)}) = 1 \tag{7}
$$

where $\{w^{\mu,\nu},\ \nu = 0,1,\ldots\}$ is the sequence generated by A0 algorithm, given an initial condition $w^{\mu,o}$.

Description of the A1 algorithm.

A1 Algorithm.

Step 0: set $w^{0,0} \in \Re^n$, set $\mu = 0$, $\nu = 0$ and begin the procedure.

Step 1: compute $w^{\mu,\nu+1} = Mw^{\mu,\nu}$.

Step 2: if $w^{\mu,\nu+1} = w^{\mu,\nu}$, then stop (the discrete solution is $\bar{w} = w^{\mu,\nu}$); else go to Step 3.

Step 3: if $R(w^{\mu,o},\ldots,w^{\mu,\nu}) = 1$, then set $I(j) = I(j, w^{\mu,\nu})$, $\forall j = 1,\ldots,n$ and go to Step 4; else set $\nu = \nu + 1$ and go to Step 1.

Step 4: compute $\hat{w}^{\mu+1}$ as the fixed point of operator M_I, set $w^{\mu+1,o} = \hat{w}^{\mu+1}$, $\nu = 0$, $\mu = \mu + 1$ and go to Step 1.

4- CONVERGENCE OF A1 ALGORITHM.

Theorem 4.1

If condition (7) is satisfied, then A1 algorithm converges in a finite number of steps to the fixed point $\bar{w}$ of operator M.

Previously we shall need some definitions and properties in order to prove this theorem.

Definitions:

1°) Each $w \in \Re^n$ can be associated with pairs of elements (w_1,w_2), $w_i \in \Re^n$, i=1,2 such that $w_2-w_1=w$.

So we define:

$$J_w=\{ (w_1,w_2) \ / \ w_2-w_1=w \}. \tag{8}$$

2°) $\hat{\psi}_2=\psi_2$, $\hat{\psi}_1=-\psi_1$, $\hat{f}_2=\frac{f}{2}$, $\hat{f}_1=-\frac{f}{2}$ (9)

$\hat{\imath}$ =1 if i=2

=2 if i=1.

3°) We define operator $\hat{M}: \Re^{2n} \to \Re^{2n}$ such that:

$$[\hat{M}(w_1,w_2)]_i(j) = \min\{(Bw_{\hat{\imath}}+\hat{f}_{\hat{\imath}})(j)+\hat{\psi}_i(j),\ (Bw_i+\hat{f}_i)(j)\}. \tag{10}$$

Properties of operator $\hat{M}$

a) We say the pair (u_1,u_2) is a subsolution of problem: $(v_1,v_2)=\hat{M}(w_1,w_2)$ if

$u_i \leq u_{\hat{\imath}}+\hat{\psi}_i$

i=1,2

$u_i \leq (Bw_i+\hat{f}_i)$

It is verified $[\hat{M}(w_1,w_2)]_i(j)$ is the maximum element of the set of subsolutions.

b) $$[\hat{M}(w_1,w_2)]_2 - [\hat{M}(w_1,w_2)]_1 = Mw \quad \forall (w_1,w_2) \in J_w. \tag{11}$$

Proof:

If $(Mw)(j)=\psi_1(j) \Rightarrow (Bw+f)(j)\leq\psi_1(j) \Rightarrow (B(w_2-w_1)+f)(j) \leq \psi_1(j) < \psi_2(j)$, $\forall (w_1,w_2) \in J_w \Rightarrow$

$\Rightarrow (Bw_2 - Bw_1 + \frac{f}{2} + \frac{f}{2})(j) \leq \psi_1(j) < \psi_2(j)$.

The last inequality implies the following pair of inequalities:

$(Bw_2+\frac{f}{2})(j) < (\psi_2+Bw_1-\frac{f}{2})(j)$

$(Bw_2+\frac{f}{2}-\psi_1)(j) \leq (Bw_1-\frac{f}{2})(j)$

then by virtue of (10), we obtain:

$\hat{M}_2(j)=(Bw_2+\frac{f}{2})(j)$

$\hat{M}_1(j)=(Bw_2+\frac{f}{2}-\psi_1)(j)$ (12)

Then $(\hat{M}_2-\hat{M}_1)(j)=\psi_1(j)$.

In the same way, when $(Mw)(j)=\psi_2(j)$ we obtain $(\hat{M}_2-\hat{M}_1)(j)=\psi_2(j)$.

If $\psi_1(j)<(Mw)(j)<\psi_2(j) \Rightarrow \psi_1(j)<(Bw+f)(j)<\psi_2(j) \Rightarrow \psi_1(j)<(B(w_2-w_1)+f)(j)<\psi_2(j), \forall (w_1,w_2)\in J_W \Rightarrow$

$$\Rightarrow \psi_1(j) < (Bw_2 - Bw_1 + \frac{f}{2} + \frac{f}{2})(j) < \psi_2(j).$$

This implies

$$(B(w_2)+\frac{f}{2})(j) < (Bw_1-\frac{f}{2}+\psi_2)(j)$$

$$(B(w_1)-\frac{f}{2})(j) < (Bw_2+\frac{f}{2}-\psi_1)(j)$$

then by virtue of (10), we obtain

$$\hat{M}_2(j)=(Bw_2+\frac{f}{2})(j)$$

$$\hat{M}_1(j)=(Bw_1-\frac{f}{2})(j) \tag{13}$$

consequently $(\hat{M}_2-\hat{M}_1)(j)=(Bw+f)(j)$.

□

c) Corollary

$$\text{If } \bar{w}=M\bar{w} \Rightarrow \hat{M}(J_{\bar{w}}) \subset J_{\bar{w}} \tag{14}$$

d) $\hat{M}$ is monotone:

$$\hat{w}_i \geq w_i \Rightarrow [\hat{M}(\hat{w}_1,\hat{w}_2)]_i \geq [\hat{M}(w_1,w_2)]_i,\ i=1,2 \tag{15}$$

Definition:

Let $\hat{M}_I : \Re^{2n} \to \Re^{2n}$ be such that

$$(v_1,v_2)\to\left([\hat{M}_I(v_1,v_2)]_1,[\hat{M}_I(v_1,v_2)]_2\right)$$

where:

$$[\hat{M}_I(v_1,v_2)]_i(j) = (Bv_i+\hat{f}_i)(j) \text{ if } I(j) \neq i$$

$$= ((Bv_{\hat{i}}+\hat{f}_{\hat{i}}) + \hat{\psi}_i)(j) \text{ if } I(j) = i \tag{16}$$

Properties of operator $\hat{M}_I$

a) $[\hat{M}_I(v_1,v_2)]_2-[\hat{M}_I(v_1,v_2)]_1 = [M_I(v)] \quad \forall (v_1,v_2) \in J_V$ (17)

b) $[\hat{M}_I(v_1,v_2)]_i=[\hat{M}(v_1,v_2)]_i \quad \forall (v_1,v_2) \in J_V,\ I=I(v).$ (18)

c) $\hat{M}_I$ is contractive (therefore it has a unique fixed point). (19)

d) $\hat{M}_I$ is monotone:

$$\hat{v}_i \geq v_i \Rightarrow [\hat{M}_I(\hat{v}_1,\hat{v}_2)]_i \geq [\hat{M}_I(v_1,v_2)]_i. \tag{20}$$

e) if $\hat{w}_i$ is the fixed point of $\hat{M}_I \Rightarrow$

$\hat{w}=\hat{w}_2-\hat{w}_1$ is the fixed point of operator M_I. (21)

f) if $\hat{w}$ is the fixed point of operator $M_I \Rightarrow$ there exists a unique $(\hat{w}_1,\hat{w}_2)$ such that $\hat{w}=\hat{w}_2-\hat{w}_1$, with $\hat{w}_i$ the fixed point of $\hat{M}_I$. (22)

Proof of a).

If $I(j)=0$ then $I(j)\neq i$, for $i=1,2$. From definition (16) we have:

$[\hat{M}_I(v_1,v_2)]_1(j)=(Bv_1+\hat{f}_1)(j)$

$[\hat{M}_I(v_1,v_2)]_2(j)=(Bv_2+\hat{f}_2)(j)$

then

$[\hat{M}_I(v_1,v_2)]_2(j)-[\hat{M}_I(v_1,v_2)]_1(j) = (Bv_1+\hat{f}_1 - Bv_2-\hat{f}_2)(j) = (Bv+f)(j)=[M_I(v)](j)$

If $I(j)=1$, also from definition (16) we have:

$[\hat{M}_I(v_1,v_2)]_1(j)=(Bv_2+\hat{f}_2+\hat{\psi}_1)(j)$

$[\hat{M}_I(v_1,v_2)]_2(j)=(Bv_2+\hat{f}_2)(j)$

then

$[\hat{M}_I(v_1,v_2)]_2(j)-[\hat{M}_I(v_1,v_2)]_1(j) = \psi_1(j) =[M_I(v)](j)$

We proceed analogously if $I(j)=2$.

Proof of b).

If $I(v)=0$ then from (13) we have:

$\hat{M}_2(j)=(Bw_2+\frac{f}{2})(j)$

$\hat{M}_1(j)=(Bw_1-\frac{f}{2})(j)$,

by (9) and (16)

$[\hat{M}_I(v_1,v_2)]_1(j) = (Bv_1+\hat{f}_1)(j) = (Bv_1 - \frac{f}{2})(j)$

$[\hat{M}_I(v_1,v_2)]_2(j) = (Bv_2+\hat{f}_2)(j) = (Bv_2 + \frac{f}{2})(j)$

with $[\hat{M}_I(v_1,v_2)]_i(j) = [\hat{M}(v_1,v_2)]_i(j)$.

If $I(v)=1$ we have (see (12)):

$\hat{M}_2(j)=(Bw_2+\frac{f}{2})(j)$

$\hat{M}_1(j)=(Bw_2+\frac{f}{2}-\psi_1)(j)$,

and because of (9) and (16):

$[\hat{M}_I(v_1,v_2)]_1(j)=(Bv_2+\hat{f}_2+\hat{\psi}_1)(j) = (Bv_2 + \frac{f}{2} - \psi_1)(j)$

$[\hat{M}_I(v_1,v_2)]_2(j)=(Bv_2+\hat{f}_2)(j) = (Bv_2 + \frac{f}{2})(j)$

Analogously if $I(v)=2$.

Finally we have proved, $[\hat{M}_I(v_1,v_2)]_i(j)= [\hat{M}(v_1,v_2)]_i(j)$.

□

5- SEQUENCE ASSOCIATED TO $w^{\mu,\nu}$.

Each $w^{\mu,\nu}$ is associated to a pair $(w_1^{\mu,\nu},w_2^{\mu,\nu})$. We shall prove convergence for sequences $w_i^{\mu,\nu}$, $i=1,2$.

We shall denote by $(\rightarrow)$ such association. So

1°) $w^{0,0}\in\Re^n \rightarrow (w^{0,0})_i\in\Re^n$ such that $(w_1^{0,0},w_2^{0,0})\in J_{w^{0,0}}$. (23)

2°) $w^{\mu,\nu+1} \rightarrow \hat{M}(w_1^{\mu,\nu},w_2^{\mu,\nu})$. (24)

3°) $\hat{w}^{\mu+1}$, the fixed point of operator $M_I \rightarrow (\hat{w}^{\mu+1})_i$,

the fixed point of operator $\hat{M}_I$. (25)

4°) $w^{\mu,0} \rightarrow (\hat{w}^{\mu})_i$. (26)

By virtue of (11), (17), (21) and (22) the following diagrams, which synthetize the above association, are commutative:

$$\begin{tikzcd} w\in\Re^n \arrow[r, leftrightarrow] \arrow[d] & (w_1,w_2)\in J_w \arrow[d] \\ Mw\in\Re^n & \hat{M}(w_1,w_2) \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} v\in\Re^n \arrow[r, leftrightarrow] \arrow[d] & (v_1,v_2)\in J_v \arrow[d] \\ M_I v\in\Re^n & \hat{M}_I(v_1,v_2) \arrow[l] \end{tikzcd}$$

Properties of sequences $(w^{\mu,\nu})_i$.

a) $(w^{\mu,\nu+1})_i = \min((Bw_{\hat{i}}^{\mu,\nu}+\hat{f}_{\hat{i}})+\hat{\psi}_i,\ (Bw_i^{\mu,\nu}+\hat{f}_i))$, $\nu\geq 0$ (see (10)). (27)

b) $(\hat{w}^{\mu+1})_i = (B\hat{w}_i^{\mu+1}+\hat{f}_i)$ if $i \neq I$

$= (B\hat{w}_{\hat{i}}^{\mu+1}+\hat{f}_{\hat{i}}) + \hat{\psi}_i$ if $i = I$, (see (16)). (28)

c) $(\hat{w}^{\mu+1})_i$ can be obtained by the following convergent iteration (see (19)):

$(v^{\lambda+1})_i = \hat{M}_I(v_1^{\lambda}, v_2^{\lambda})$, with initial condition $(v^0)_i = (w^{\mu,\overline{\nu}(\mu)})_i$. (29)

Proposition 5.1

$(w^{\mu,\nu})_i$ is a not increasing sequence for $\mu\geq 1$, $\nu\geq 0$, in the following sense:

$$(w^{\mu,\nu+1})_i \leq (w^{\mu,\nu})_i \tag{30}$$

$$(w^{\mu+1,0})_i \leq (w^{\mu,\overline{\nu}(0)})_i \tag{31}$$

In fact, by definition, $w^{\mu,o}$, we have for all $\mu \geq 1$:

$$(w^{\mu,o})_i = (\hat{w}^{\mu})_i$$

Given μ, we know by A1 algorithm the following equality holds:

$$(\hat{w}^{\mu})_i = \begin{cases} (B(\hat{w}^{\mu})_i + \hat{f}_i) & \text{if } i \neq I(w^{\mu-1,\overline{\nu}(\mu-1)}) \\ (B(\hat{w}^{\mu})_{\hat{i}} + \hat{f}_{\hat{i}} + \hat{\psi}_i) & \text{if } i = I(w^{\mu-1,\overline{\nu}(\mu-1)}). \end{cases}$$

With $\overline{\nu}(\mu)$ we mean that value of ν where the test of Step 3 is satisfied, therefore

$$(w^{\mu,1})_i = [\hat{M}(w_1^{\mu,o}, w_2^{\mu,o})]_i = \min \{B(w^{\mu,o})_{\hat{i}} + \hat{f}_{\hat{i}} + \hat{\psi}_i,\ B(w^{\mu,o})_i + \hat{f}_i\} =$$

$$= \min \{B(\hat{w}^{\mu})_{\hat{i}} + \hat{f}_{\hat{i}} + \hat{\psi}_i,\ B(\hat{w}^{\mu})_i + \hat{f}_i\} \leq (\hat{w}^{\mu})_i = (w^{\mu,o})_i$$

Repeating Steps 1 to 3 both sequences $(w^{\mu,\nu})_i$ are obtained, according to the following law:

$$(w^{\mu,\nu+1})_i = [\hat{M}(w_1^{\mu,\nu}, w_2^{\mu,\nu})]_i.$$

Then, by the monotony of operator $\hat{M}$ (see (15)), it is obtained that:

$$(w^{\mu,\nu+1})_i \leq (w^{\mu,\nu})_i, \quad \forall \nu \leq \overline{\nu}(\mu), \quad \forall \mu \geq 1.$$

And consequently (30) is proved.

By definition $(w^{\mu+1,o})_i = (\hat{w}^{\mu+1})_i$.

In Step 4 the element $\hat{w}^{\mu+1}$ is generated as:

$$(\hat{w}^{\mu+1})_i = [\hat{M}_I(\hat{w}_1^{\mu+1}, \hat{w}_2^{\mu+1})]_i$$

Since $(\hat{w}^{\mu+1})_i$ is the fixed point of operator $[\hat{M}_I(w_1,w_2)]_i$, see (19) and (25), we have

$$(\hat{w}^{\mu+1})_i = \lim_{\lambda \to \infty} v^{\lambda},$$

where $(v^{\lambda+1})_i = [\hat{M}_I(\hat{v}_1^{\lambda}, \hat{v}_2^{\lambda})]_i$, with initial condition $(v^o)_i = (w^{\mu,\overline{\nu}(\mu)})_i$.

But

$$(w^{\mu,\overline{\nu}(\mu)})_i = [\hat{M}(w_1^{\mu,\overline{\nu}(\mu)-1}, w_2^{\mu,\overline{\nu}(\mu)-1})]_i,$$

moreover, by (18)

$$[\hat{M}_I(w_1,w_2)]_i = [\hat{M}(w_1,w_2)]_i \quad \forall (w_1,w_2) \in J_w,\ I = I(w).$$

Then

$$[\hat{M}(w_1^{\mu,\overline{\nu}(\mu)-1}, w_2^{\mu,\overline{\nu}(\mu)-1})]_i = [\hat{M}_I(w_1^{\mu,\overline{\nu}(\mu)-1}, w_2^{\mu,\overline{\nu}(\mu)-1})]_i, \quad I = I(w^{\mu,\overline{\nu}(\mu)-1})$$

therefore

$$(v^o)_i = (w^{\mu,\overline{\nu}(\mu)})_i = [\hat{M}_I(w_1^{\mu,\overline{\nu}(\mu)-1}, w_2^{\mu,\overline{\nu}(\mu)-1})]_i \leq (w^{\mu,\overline{\nu}(\mu)-1})_i \quad \text{by (30), so}$$

$$(v^1)_i = [\hat{M}_I(v_1^o, v_2^o)]_i \leq [\hat{M}_I(w_1^{\mu,\overline{\nu}(\mu)-1}, w_2^{\mu,\overline{\nu}(\mu)-1})]_i = (v^o)_i.$$

from the monotony of $\hat{M}_I$ it follows $(v^\lambda)_i$ is not increasing and consequently:

$$(\hat{w}^{\mu+1})_i = \lim_{\lambda\to\infty} (v^\lambda)_i \le (v^o)_i = (w^{\mu,\overline{\nu}(\mu)})_i.$$

and this proves (31).

□

Proof of Theorem 4.1.

By condition (7) Step 3 test is satisfied after a finite number of iterations (on ν) leaving the [1,...,3] loop and generating a new element $\hat{w}^{\mu+1}$. Therefore the theorem will be proved if that sequence is finite.

By proposition 5.1 $(\hat{w}^{\mu})_i$ is not increasing. Considering there is only a finite number of possibilities for $I(j,\hat{w}^{\mu})$, only a finite number of different values for $(\hat{w}^{\mu})_i$ may be generated. Therefore there exists one $\overline{\mu}$ such that $(\hat{w}^{\overline{\mu}})_i=(\hat{w}^{\overline{\mu}+1})_i$
Since

$$(\hat{w}^{\overline{\mu}+1})_i \le (w^{\overline{\mu},\nu+1})_i \le (w^{\overline{\mu},\nu})_i \le (w^{\overline{\mu},o})_i = (\hat{w}^{\overline{\mu}})_i$$

we have $(w^{\overline{\mu},1})_i = (w^{\overline{\mu},o})_i$, then by (11) $w^{\overline{\mu},1} = w^{\overline{\mu},o}$. Hence the test in Step 2 is satisfied, for the pair of indexes $\mu=\overline{\mu}$, $\nu=0$ and therefore the fixed point of operator M is found.

□

6- ANALYSIS OF DIFFERENT STOPPING RULES.

• Rule R1

Let $\Gamma:\mathcal{N}_0 \to \mathcal{N}$, we define

$$R1(w^{\mu,o},\dots,w^{\mu,\nu+1}) = \begin{cases} 1 & \text{if } \nu \geq \Gamma(\mu). \\ 0 & \text{in other case.} \end{cases}$$

It is obvious for this rule that condition (7) is satisfied.

• Rule R2

$$R2(w^{\mu,o},\dots,w^{\mu,\nu+\hat{p}}) = \begin{cases} 1 & \text{if } I(j,w^{\mu,\nu}) = I(j,w^{\mu,\nu+p}),\ \forall j,\ \ 1 \leq p \leq \hat{p}. \\ 0 & \text{in other case.} \end{cases}$$

The A1 algorithm would be modified in the following way:

A1 algorithm

Step 0: set $\mu=0$, $\nu=0$, $w^{\mu,\nu} \in \Re^n$, and $\hat{p} \in \mathcal{N}$ arbitrary.

Step 1: set $p=0$.

Step 2: compute $w^{\mu,\nu+1}=Mw^{\mu,\nu}$.

Step 3: set $\hat{I}(j)=I(j,w^{\mu,\nu})$, $\forall$ j.

Step 4: if $w^{\mu,\nu+1}=w^{\mu,\nu}$, then stop (the discrete solution is $\overline{w}=w^{\mu,\nu}$); else go to Step 5.

Step 5: if $\nu \geq 1$ and $I(\cdot,w^{\mu,\nu})=I(\cdot,w^{\mu,\nu-1})$ then set $p=p+1$, and go to Step 6; else set $p=0$, $\nu=\nu+1$, and go to Step 2.

Step 6: if $p \geq \hat{p}$ then compute $\hat{w}^{\mu+1}$ as the fixed point of operator $M_{\hat{I}}$, set $w^{\mu+1,0}= \hat{w}^{\mu+1}$, $\nu=0$, $\mu=\mu+1$ and go to Step 1; else set $\nu=\nu+1$ and go Step 2.

Proposition 6.1.

If the following hypotheses hold:

$$\psi_2(j)=\overline{w}(j) \Rightarrow (B\overline{w}+f)(j)>\psi_2(j) \tag{32}$$

$$\psi_1(j)=\overline{w}(j) \Rightarrow (B\overline{w}+f)(j)<\psi_1(j) \tag{33}$$

then condition (7) is satisfied for rule R2.

Proof:

We have to prove that for all μ the test of Step 6 is satisfied after a finite number of repetitions of the [1,...,5] loop allowing the generation of a new element $\hat{w}^{\mu+1}$.

In fact, by contradiction we assume $\tilde{\mu}$ exists. For such index the test of Step 6 is never satisfied. Consequently the algorithm would generate an infinite sequence $w^{\tilde{\mu},\nu}$. This sequence, by definition of Step 2, is identical to the one generated by A0 algorithm. In that case, theorem 2.1 assures the sequences converges to $\overline{w}$.

For each index j one of the three following conditions must be verified:

i) $I(j,\overline{w}) = 0$

ii) $I(j,\overline{w}) = 1$

iii) $I(j,\overline{w}) = 2$

In case i) we have $\overline{w}(j) \in (\psi_1(j),\psi_2(j))$, then by (4) we deduce ν_0 exists for all $\nu\geq\nu_0$, $w^{\tilde{\mu},\nu}(j) \in (\psi_1(j),\psi_2(j))$, and $I(j,w^{\tilde{\mu},\nu_0+p})=0$, for all $p\geq0$.

In case ii) it is $\overline{w}(j) = \psi_1(j)$, and by virtue of (4) we have:

$\lim_{\nu} (Bw^{\tilde{\mu},\nu}+ f)(j) = (B\overline{w} + f)(j) < \psi_1(j)$, then for all $\nu\geq \nu_1$ $(Bw^{\tilde{\mu},\nu}+ f)(j) < \psi_1(j)$, so for all $\nu\geq \nu_1$ $I(j,w^{\tilde{\mu},\nu})= 1$.

Case iii) is completly analogous to case ii).

Then after a finite number of iterations, $I(\cdot,w)$ remains constant, and the test of Step 6 is verified; this allows the generation of a new element $\hat{w}^{\mu+1}$.

□

• Rule R3

Let $\epsilon_\nu = K \left\|w^{\mu,\nu}-w^{\mu,\nu-1}\right\|$, $K \geq \dfrac{2\gamma}{1-\gamma}$, $\hat{p}\in\mathcal{N}$ is fixed.

We have:

$$(M_I w^{\mu,\nu})(j) = \begin{cases} \psi_1(j) & \text{if } I(j)=1 \\ \psi_2(j) & \text{if } I(j)=2 \\ (Bw^{\mu,\nu}+f)(j) & \text{if } I(j)=0 \end{cases}$$

we introduce the following notation:

$$I_{\epsilon_\nu}(j,w^{\mu,\nu})=\left\{ i/\exists\ I \text{ satisfying } I(j)=i \text{ with } \mid w^{\mu,\nu+1}(j)-(M_I w^{\mu,\nu})(j)\mid < \epsilon_\nu \right\} \tag{34}$$

So we define:

$$R3(w^{\mu,0},\ldots,w^{\mu,\nu+\hat{p}})=\begin{cases} 1 & \text{if } I_{\epsilon_\nu}(j,w^{\mu,\nu})=I_{\epsilon_\nu}(j,w^{\mu,\nu+p}) \quad \forall\ 1\le p\le\hat{p},\ 1\le j\le n \\ 0 & \text{in other case.}\end{cases}$$

We shall see that condition (7) is verified.

In the same way in the proof of proposition 6-1, we have to prove it is not possible to build an infinite sequence

$$\{w^{\mu,\nu},\ \nu=1,\ldots\}, \text{ with } R3(w^{\mu,0},\ldots,w^{\mu,\nu})=0. \tag{35}$$

Assume an infinite sequence for some $\hat{\mu}$ is generated, then by theorem 2.1, it converges to $\bar{w}$. But hypotheses of proposition 6.1 are valid, so:

$\psi_2(j)=\bar{w}(j) \Rightarrow (B\bar{w}+f)(j)>\psi_2(j)$

$\psi_1(j)=\bar{w}(j) \Rightarrow (B\bar{w}+f)(j)<\psi_1(j),$

and in same way as in the proof of proposition 6.1 we have for each j, that only one of the three following conditions must be true:

i) If $I(j,\bar{w}) = 0$, then $I_{\epsilon_\nu}(j,w^{\mu,\nu+p})=\{0\}$, for all $\nu>\nu_0$, $p\ge 0$.

ii) If $I(j,\bar{w}) = 1$, then $I_{\epsilon_\nu}(j,w^{\mu,\nu+p})=\{1\}$, for all $\nu>\nu_1$, $p\ge 0$.

iii) Similarly for $I(j,\bar{w}) = 2$.

Then it remains to prove (7) is valid for all those cases where hypotheses of proposition 6.1 are not satisfied. That is when:

$\psi_2(j)=\bar{w}(j) \ \wedge\ (B\bar{w}+f)(j)=\psi_2(j)$

$\psi_1(j)=\bar{w}(j) \ \wedge\ (B\bar{w}+f)(j)=\psi_1(j)$

We shall only prove it for the first case, (it is analogous for the second).

By continuity of operator M, for all $\nu\ge\nu_2$, $I_{\epsilon_\nu}(j,w^{\mu,\nu})$ is contained in the set $\{0,2\}$.

Let us see that $2 \in I_{\epsilon_\nu}(j,w^{\mu,\nu})\ \forall\ \nu\ge\nu_2$.

In fact, since M is a contraction:

$$\left\| Mw^{\mu,\nu-1} - M\bar{w} \right\| \le \gamma \left\| w^{\mu,\nu-1} - \bar{w} \right\|$$

By the triangular property:

$$\left\| w^{\mu,\nu} - \bar{w} \right\| \le \gamma \left\| w^{\mu,\nu} - \bar{w} \right\| + \gamma \left\| w^{\mu,\nu-1} - w^{\mu,\nu} \right\|$$

from this inequality:

$$\left(1-\gamma\right) \left\|w^{\mu,\nu} - \bar{w}\right\| \leq \gamma \left\|w^{\mu,\nu-1} - w^{\mu,\nu}\right\| \qquad (36)$$

then

$$|w^{\mu,\nu}(j)-\psi_2(j)| = |w^{\mu,\nu}(j) - \bar{w}(j)| \leq \left\|w^{\mu,\nu} - \bar{w}\right\| \leq \frac{\gamma}{\left(1-\gamma\right)} \left\|w^{\mu,\nu-1} - w^{\mu,\nu}\right\| <$$

$$< K \left\|w^{\mu,\nu-1} - w^{\mu,\nu}\right\| \qquad (37)$$

and consequently, by (34), $2 \in I_{\varepsilon_\nu}(j,w^{\mu,\nu})$ $\forall$ $\nu \geq \nu_2$.

Let us see now that $0 \in I_{\varepsilon_\nu}(j,w^{\mu,\nu})$ $\forall$ $\nu \geq \nu_2$.

$$\left\|Bw^{\mu,\nu}+f-w^{\mu,\nu+1}\right\| = \left\|B(w^{\mu,\nu}-\bar{w}) + \bar{w} - w^{\mu,\nu+1})\right\| \leq$$

$$\leq \gamma \left\|w^{\mu,\nu} - \bar{w}\right\| + \left\|w^{\mu,\nu+1} - \bar{w}\right\| \leq 2\gamma \left\|w^{\mu,\nu} - \bar{w}\right\|$$

From this inequality and (36):

$$|(Bw^{\mu,\nu}+f)(j)-w^{\mu,\nu+1}(j)| \leq \left\|Bw^{\mu,\nu}+f-w^{\mu,\nu+1}\right\| \leq$$

$$\leq \frac{2\gamma^2}{(1-\gamma)} \left\|w^{\mu,\nu-1}-w^{\mu,\nu}\right\| \leq K \left\|w^{\mu,\nu-1}-w^{\mu,\nu}\right\| \qquad (38)$$

and consequently, by (34), $0 \in I_{\varepsilon_\nu}(j,w^{\mu,\nu})$ $\forall$ $\nu \geq \nu_2$.

It is then proved that for every possible case there exist a $\hat{\nu}$ such that the families of indexes $I_{\varepsilon_\nu}(j,w^{\mu,\nu})$ verify $R3(w^{\mu,0},\ldots,w^{\mu,\hat{\nu}+\hat{p}})=1$, for $\nu \geq \hat{\nu}$. This proves the impossibility to build sequence (35).

- Rule R4

Let $\bar{\varepsilon}>0$ and

$$\varepsilon_\nu = \bar{\varepsilon} \left\|w^{\mu,\nu}-w^{\mu,\nu-1}\right\|^r, \; r\in(0,1).$$

We define:

$$R4(w^{\mu,0},\ldots,w^{\mu,\nu+1}) = \begin{cases} 1 & \text{if } \left\|w^{\mu,\nu+1}-w^{\mu,\nu}\right\| \leq \epsilon_\nu \\ 0 & \text{in other case.} \end{cases}$$

Condition (7) is satisfied $\forall\ \nu \geq \overline{\nu}$ with $\overline{\nu}$ / $\left\|w^{\mu,\overline{\nu}}-w^{\mu,\overline{\nu}-1}\right\| < \left(\frac{\epsilon}{\gamma}\right)^{\frac{1}{1-r}}$

- Rule R5

Let $\epsilon_\nu = \left\|w^{\mu,\nu}-w^{\mu,\nu-1}\right\|^{r}$, $r\in(0,1)$,

$$I_{\epsilon_\nu}(j,w^{\mu,\nu}) = \left\{ i \ /\ \exists\ I \text{ satisfying } I(j) = i \text{ with } \mid w^{\mu,\nu+1}(j) - (M_I w^{\mu,\nu})(j) \mid < \epsilon_\nu \right\}$$

$\hat{p}\in\mathcal{N}$ is fixed.
So we define:

$$R5(w^{\mu,0},\ldots,w^{\mu,\nu+\hat{p}}) = \begin{cases} 1 & \text{if } I_{\epsilon_\nu}(j,w^{\mu,\nu})=I_{\epsilon_\nu}(j,w^{\mu,\nu+p}) \text{ with } 1\leq p\leq\hat{p}, \\ 0 & \text{in other case.} \end{cases}$$

As in the previous cases we prove it is not possible to build an infinite sequence:

$\{w^{\mu,\nu},\ \nu=1,\ldots\}$, with R5 $(w^{\mu,0},\ldots,w^{\mu,\nu})=0$.

When hypotheses of proposition 6.1 are verified, the proof is identical to that of rule R3. To prove (7) holds when $\psi_2(j)=\overline{w}(j)\ \wedge\ (B\overline{w}+f)(j)=\psi_2(j)$, (the case $\psi_1(j)=\overline{w}(j)\ \wedge\ (B\overline{w}+f)(j)=\psi_1(j)$, is completely analogous) we follow the same reasoning as in R3. To prove

$$|w^{\mu,\nu}(j)-\psi_2(j)| < \epsilon_\nu \quad \text{and} \quad |(Bw^{\mu,\nu}+f)(j) - w^{\mu,\nu+1}(j)| < \epsilon_\nu$$

(see (37) and (38)).

it is necessary to keep in mind

$$\frac{\gamma}{(1-\gamma)} \leq \left\|w^{\mu,\nu-1} - w^{\mu,\nu}\right\|^{r-1},\ \forall\nu\geq\nu_0.$$

7- EXAMPLES

In the following tables we shall show some results concerning computing times and number of iterations of A0 and A1 algorithms with stopping rule R3. Table 1 corresponds to a problem where $\nu=36$ and table 2 to one where $\nu=64$.

TABLE 1

	times				iterations			
ϵ	10^{-3}		10^{-6}		10^{-3}		10^{-6}	
α	A1	A0	A1	A0	A1	A0	A1	A0
1.0	19″	1′14″	19″	35″	5	22	5	51
0.4	21″	2′56″	21″	1′.23″	6	57	6	124
0.2	25″	5′52″	25″	2′53″	8	123	8	253

TABLE 2

	times		iterations	
ϵ	A1	A0	A1	A0
10^{-3}	17″	1′8″	5	86
10^{-5}	17″	1′38″	5	125
10^{-7}	17″	2′8″	5	165
10^{-9}	17″	2′37″	5	205

8- CONCLUSIONS

When the contraction constant of operator M is close to 1, A0 algorithm may converge very slowly. This has lead us to develope an acceleration procedure.

The algorithm presented here, as it can be observed from results in tables 1 and 2, reduces the computing time in those cases by 14 times. This reduction essentially depends on the problem, i.e. of factor γ, and it is more significant for γ closer to 1. Also, it can be observed that the A1 algorithm produces the approximated ε-solution (in fact the exact solution), in the same time (independently of ε), this fact enables us to choose "greater" values for ε than those used for A0 algorithm, whose computing time is proportional to $-\ln \varepsilon$. Convergence properties of A1 algorithm are independent of the initial point $w^{0,0}$ chosen; however, computing times can decrease with an adequate choice of that point (see [6]).

The algorithm is based on a timely and suitable resolution of a certain system of linear equations (for instance, when there exists the vector I mentioned in Rule 2), determined by successive applications of A0 algorithm. The choice of leaving A0 algorithm is related to the use of various stopping rules, some of which have been presented in Paragraph 6, together with the proof of the convergence property. These rules generalize the "control repetitions" rule used in [5] and allow the resolution of those problems where the restrictive conditions needed in [5] to assure the convergence, are not verified.

The use of these rules and extensions of the algorithms presented here and in [5], are considered in [6] to find the solution of more general fixed point problems.

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Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique