On sequential functions

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ON SEQUENTIAL FUNCTIONS

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ABSTRACT :

We introduce SK-domains and define S-functions which generalise the notion of sequential functions, already introduced by Milner, by Vuillemin and by Kahn and Plotkin ; we then show that SK-domains and S-functions constitute a Λ-category.

A PROPOS DES FONCTIONS SEQUENTIELLES

RESUME :

Nous introduisons les SK-domaines et définissons les S-fonctions qui sont une généralisation des fonctions séquentielles déjà introduites par Milner, Vuillemin et par Kahn et Plotkin ; nous montrons que les SK-domaines et les S-fonctions constituent une Λ-catégorie.

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ON SEQUENTIAL FUNCTIONS

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Abstract: We introduce SK-domains and define S-functions which generalise the notion of sequential functions, already introduced by Milner, by Vuillemin and by Kahn andPlotkin; we then show that SK-domains and S-functions constitute a Λ- category .

I) INTRODUCTION

The sequentiality problem is highly related with the full abstraction question for models of λ-calculus based languages.

It was originally raised in its most typical form, and shown difficult by Plotkin in [6]; there, the model derived from Scott's of continuous functions was shown to be non fully abstract for the language PCF (a typed λ-calculus together with arithmetic and boolean operators).

Indeed, Plotkin exhibited two functions which denote two expressions having the same behaviour, but differing on an argument which is undefinable in the language (and thus cannot be supplied to the functions). Such an argument is typically the "parallel or" function, a binary boolean function which yields value true, as soon as one of its arguments is true.

The reason for this failure of full abstraction is essentially an inadequate teatment of sequentiality: it is known that PCF-like languages can be evaluated sequentially; so in order to provide fully abstract models for such languages, just restrict to continuous sequential functions since it is known as well that the continuous functions model is complete.

On this way, Milner and Vuillemin (see[4] and[7]) proposed two different propositions which rely heavily on the product structure of the input space of functions. These propositions work well at the first level but fail at highers.

In order to solve the problem, Kahn and Plotkin[3] proposed a more general definition which is independant of the way that functions are viewed as having arguments.

Their proposition relies on the axiomatisation of the notion of place (which generalise that of argument's place) for a particular class of domains, called concrete domains. Unfortunately, this definition cannot be used to provide the fully abstract model of PCF.

Let us recall that independantly, Milner showed that the functions-based fully abstract model of PCF is unique (up to isomorphism) and is extensional (ie: ordered with the Scott's- like order).

Indeed the concrete domains and sequential functions as defined by Kahn and Plotkin cannot be used to define a cartesian closed categorie. Notice however that substituting the notion of sequential algorithm for that of function and relaxing Scott's order allow to provide PCF with a fully abstract but non extensional model (see[1,2]).

Attacking the problem led Berry to define the notion of stability, an approximation of sequentiality. Roughly speaking, stable functions are continuous functions together with a particular mimimality property (see[1]).

These functions, together with rather standard domains, provide a cartesian closed category (but which cannot be order enriched with Scott's order however). More crucially, Berry left the problem open by exhibiting a function which is stable but not sequential.

The challange of this paper is to get closer the closure of the sequentiality question.
First we introduce SK-domains which are particular bi-ordered structures. The two ordering relations we consider allow to compare the behaviour of individuals both extensionaly and intensionaly as in the bistructures used by Berry. However contrary to Berry's approach our intensional relation does not impose any significant structure on domains; it only allows to generalise the notion of place for a rather wide class of abstract domains. We then use this notion and define S-functions which are continuous functions together with those conditions one intuitively require for sequential behaviours. Finaly we proceed on showing that SK-domains together with S-functions constitute a cartesian closed category order enriched with Scott's order.

II) SK-DOMAINS

This section introduces the spaces and structures we'll be using all along this paper; before, let us precise our terminology and recall the following well known facts:

Let $< D,\leq >$ be a partial order (usually called po and freely denoted D), and let $X\subseteq D$ and $x,y,z\in D$ then we say:

- x is *dominated* -resp. *strictly dominated*- by y (or y *dominates* -resp.*dominates strictly* x) iff: $x\leq y$ -resp. $x<y$ -
- x and y are *comparable* (notation: $x\Diamond y$) iff: $x\leq y$ or $y\leq x$

- z is the *lub (least upper bound)* of X (notation: $\vee X$) iff z dominates any element in X and is dominated by any element in D wich dominates all elements of X.
- z is the *glb (greatest lower bound)* of X (notation: $\wedge X$) iff z is dominated by all elements of X and dominates any element in D wich is dominated by all elements of X.;
- x is the *maximum élément* - resp.*minimum élément* - (notation: $\top$ -resp. $\perp$) of D iff: $x=\vee D$ -resp. $x=\wedge D$ -
- x and y are *compatible* (notation: $x\uparrow y$) iff $\exists\, z \in D$: $x\leq z \geq y$;
-X is *consistent* (notation $X\uparrow$) iff $\exists\, z \in D$: $\forall\, x \in X$, $x\leq z$;
- x is *covered* by y (notation: $x \prec y$) iff $x<y$ & $\forall z, (x\leq z<y) \Rightarrow x=z$

Further, we need the following well known concepts:

2.1 definition: let D be po , X a subset of D, and x an element of D; then:
- X is said to be *directed* iff X is a non empty subset st: $\forall\, \alpha,\beta \in X$, $\exists\, \gamma\in X$: $\alpha\leq\gamma\geq\beta$
- x is a *finite* (or isolated or compact) element of D iff $\forall\, X \subseteq D$, whenever X is directed, then $(x\leq \vee X) \Rightarrow (\exists\, \alpha \in X: x\leq\alpha)$.
- x is a *prime* element of D iff $\forall\, X \subseteq D$, if$\vee X$ exits then $(x \leq \vee X) \Rightarrow (\exists\, \alpha \in X: x\leq\alpha)$ ■

2.2 definition: let D be a po; then D is said to be a *cpo* iff it has a minimum element and a lub for any of its directed subset. A cpo is *algebraic* iff the subset of finite elements dominated by an element x, is directed and has x as lub; it is *coherent* iff any consistent subset has a lub; finally it is *prime algebraic* iff any element is the lub of prime elements it dominates and is said to be *ω-prime-algebraic* iff any element dominates only a denumerable set of primes. ■

Henceforth, we refer to ω-prime-algebraic and coherent cpos as *domains*; notice that a prime element is a finite one; and thus a domain is also an algebraic cpo.

Prime algebraic domains are well known structures; see for example [1] where they allow a nice characterizeation of stable functions; moreover their properties constitute an important part of the theories around the concretes domains as defined by Kahn and Plotkin; see for example [5] where prime algebraic lattices are related to nets and to events structures and also characterized in term of distributivity.

Here we use prime algebraic cpos trying to imagine the least coherent structure to add with prime algebraicity so as to be able to deal with operational as well as abstract behaviours in the same framework. Indeed we use primes to represent the most basic elements of information one can get about the behaviour of individuals and we consider that individuals are nothing but the collections of those basic elements that approximate them. More precisely we intoduce the following:

2.3 definition: *SK-domains* are bi-ordered structures $< D, \leq, \lesssim >$ st:

i) $< D, \leq >$ is a domain

ii) $\lesssim$ is a preorder containing $\leq$ (ie: $\forall$ x,y$\in$D, $x \leq y \Rightarrow x \lesssim y$), and which is inductif over the po $< D, \leq >$ ie: $\forall$ $X \subseteq D$, $\forall$ $y \in D$,

$$(\bigvee X \text{ exists } \& \ \forall x \in X, x \lesssim y) \Rightarrow \bigvee X \lesssim y \quad . \blacksquare$$

From now on, whenever the order and preorder relations will not be significant, an SK-domain $< D, \leq, \lesssim >$ will be freely denoted D. The preorder relation "$\lesssim$" is called *intensional*, and the equivalence relation it induces is noted "$\approx$" . The equivalence class, modulo $\approx$, of an element x is denoted [x] . Further, the partial order "$\leq$" is called *extensional.*

Finally, we refer to the set of prime elements of a domain D as $\mathbf{Pr}$ (D), while $\mathbf{Pr}$ (x) represents the set of primes dominated by x .

Thus an SK-domain is an ω-prime algebraic and coherent cpo together with an intensional preorder such that the extensional comparability of terms be finer than the intensional one.

Of course the intensional relation is introduced in order to take into account parts of the intensional behaviour of terms within their denotational semantics; but notice however that we still consider the extensional parts as the most significant ones.

Actually, the intensional relation is needed only while defining SK-domains' morphisms and it will be always defined via en equivalence relation provided with the set of primes of a domain; the problem will be essentially to find the right equivalence relation so as to get the expected SK-domain, since à priori any equivalence relation on $\mathbf{Pr}$ (D) could be used:

2.4 proposition: let $< D, \leq >$ be a domain provided with an equivalence relation (denoted$\sim$) on $\mathbf{Pr}$ (D); then relation $\lesssim \subseteq D^2$ st: $\forall$ x,y $\in$ D, $x \lesssim y \Leftrightarrow$: $\mathbf{Pr}(x)/\sim \ \subseteq \mathbf{Pr}(y)/\sim$, is a preorder relation which makes $< D, \leq, \lesssim >$ an SK-domain .

proof: "$\lesssim$" is trivially a preorder. It is immediate as well that it is coarser than relation $\leq$ and is inductive on po $< D, \leq >$ since in a domain $x \leq y$ iff $\mathbf{Pr}(x) \subseteq \mathbf{Pr}(y)$ and $\mathbf{Pr}(\bigvee X) = \bigcup \{\mathbf{Pr}(x) / x \in X\}$. $\blacksquare$

The above result only means that each domain can be readily transformed into an SK-domain: you only have to encode its elements as sets of prime elements and then define an equivalence relation on the set of primes.

Now, all the real difficulty brakes down in finding out the adequate equivalence relation, and in showing that the resulting preorder "formalises correctely" the intuition one had.

In our case, we are dealing with the problem of generalizing the notion of "the place of an argument" of a function so as to leave away any reference to the product structure of the input domain. Moreover, since this problem is related to the question of full abstraction of (typed) λ-calculus models, it is clear that our generalization should apply to functionnals first (since λ-calculus allows to treat any individual as function), and eventually restrict to functions. However in the first place, we'll consider two different classes of SK-domains and define two different preorders, depending on whether we are dealing with functions or functionals. Indeed this is only in order to illustrate the relations between the notion of place as used in Kahn-Plotkin's definition of sequentiality and our definition, at least in basic domains (those closest to concrete domains). In a further step, we show that the two preorders collapse in a same one: considering individuals,we get the same preorder as soon as they are viewed as constant functions.

Henceforth, given an SK-domain D, it will be called *basic* if its underlying domain (D) is a flat one or a product of such domains; and it will be called *functional* if its underlying domaine is a domain of functions between domains.

We first examine basic domains and now introduce the intensionnal equivalence relation on the primes of a basic domain.

2.5 definition: let a and b be two primes of a basic domain D; then a is *intensionally equivalent* to b (notation: $a \sim b$) iff: $\forall c \in \mathbf{Pr}(D)$, $(a \neq c \ \&\ a \uparrow c) \Leftrightarrow (b \neq c \ \&\ b \uparrow c)$. ■

Before we analyse and try to understand the intuition behind this relation, let us check that it is actually an equivalence relation:

2.6 proposition: $\sim$ is an equivalence relation over $\mathbf{Pr}(D)$.
proof: reflexivity and symetry are immediate. Let us show transitivity; let a,b, $c \in \mathbf{Pr}(D)$ st: $a \sim b \sim c$; we have to show $a \sim c$. But, $\forall d \in \mathbf{Pr}(D)$, $(a \neq d \ \&\ a \uparrow d) \Leftrightarrow (b \neq d \ \&\ b \uparrow d) \Leftrightarrow (d \neq c \ \&\ d \uparrow c)$; qed . ■

Intuitively $a \sim b$ can be interpreted as follows: considering any piece of information (here represented by c), if it can be used to increase significantly the information contained in a then it can be made so with b and vice versa.

Our attempt is to generalise the notion of place introduced by Kahn and Plotkin. Now since individuals are considered as sets of very basic pieces of information it is natural to set two elements equivalent as soon as they contain the same equivalence classes of prime elements. Further, if a prime element is viewed as filling a place in an element then two elements should be equivalent if they provide the same places to be filled. This idea, when transposed to the very basic level, leads to the above definition.

Let us go further and illustrate on an example the relation between the notion of place and our definition. We want to show here, that our intensional preorder allows to verify wheither or not two sequences of integer are defined on the same components. That is to say that we are able to deal with arguments' places without counting the arguments, at least in the particular case of flat input domains.

Define domain $D=(\mathbb{N})^k$ where $k\in\mathbb{N}$ and where $\mathbb{N}$ is the flat SK-domain of integers: the extensional relation is the discrete one (ie: all elements are prime) and the intensional relation is defined as indicated above. The elements of D are sequences of integers with lengh k, and they are ordered componentwisely, using the product order. It is immediate that D is a domain with as prime elements, those sequences which contain only element $\perp$ except for a unique component. Such primes can thus be represented by pairs (p,v), where $1\leq p\leq k$ et $v\in\mathbb{N}$.

Now we claim: if $a=(p_a,v_a)$ and $b=(p_b,v_b)$ are two prime and equivalent elements, then $p_a=p_b$.

Indeed, letting $c=(p,v)$, be a prime element, is is easily seen that $(a\uparrow c\ \&\ a\neq c)$ iff $(\ (p_a=p) \Rightarrow (v_a=v)\ )\ \&\ (\ (p_a\neq p$ <u>or</u> $v_a\neq v)\ )$; ie: $a\sim b \Rightarrow \forall p,\ p\neq p_a \Leftrightarrow p\neq p_b$ and thus $p_a=p_b$ .That is to say, two elements $x=\langle x_i\rangle_{i\leq k}$, $y=\langle y_i\rangle_{i\leq k}$ in D are defined on the same components iff $\mathbf{Pr}(x)/\!\sim=\mathbf{Pr}(y)/\!\sim$. We can therefore talk about arguments' position in a list of arguments.

Now we can get confidence that our definition introduces a general notion of place in domains which are more abstract than concrete domains. Before we proceed, let us illustrate it on some more examples:

<u>example1</u>: Let D be the integer domain $\mathbb{N}$ ordered s.t $n\leq m$ iff $n=m$ <u>or</u> $n=\perp$ then

$$\forall x,y\quad (x\neq\perp\neq y) \Rightarrow x\approx y.$$

<u>example2</u>: Set $D=\mathbf{T}^2$ orderd with the product order, where **T** is as previous; then:

i) $(\perp,\perp)\lesssim(tt,\perp)\approx(ff,\perp)$.

ii) $(tt,\perp)\not\lesssim(\perp,tt)$, since $(tt,\perp)\neq(\perp,ff)\neq(\perp,tt)\ \&\ (\perp,ff)\uparrow(tt,\perp)$ but $\neg\ (\perp,ff)\uparrow(\perp,tt)$.

finally, notice that our definition illuminates the fact that at the very basic level, the equivalence of domains is actually made of to separate relations: on one hand the intensional equivalence relation and on the other hand the compatibility relation, as shawn in the following

2.7 <u>lemma</u>: if a and b are two prime elements of a basic SK-domain then:

$$a=b \Leftrightarrow (a\uparrow b\ \&\ a\sim b).$$

<u>proof</u>: let $a\sim b$ and suppose $a\neq b$; but since $a\uparrow b$, we deduce the contradiction $b\neq b$. ∎

An easy corollary of the above result is that, in a basic domain no element can dominate more than one equivalence class of prime elements; moreover, two elements which are compatible and intensionally comparable, are extensionally comparable; more precisely:

$$\forall\ x,y,\quad (x \lesssim y\ \&\ x\uparrow y\ ) \Rightarrow x \leq y\ .$$

We now turn to the functional case

2.8 <u>definition</u>: let $< D, \leq_1, \lesssim_1 >$ and $< E, \leq_2, \lesssim_2 >$ be two SK-domains and let φ and ψ be two prime elements of the domain ($[D \rightarrow E]$) constituated from all continuous functions from D to E; then φ and ψ are said to be *intensionally equivalent* (notation $\varphi \sim \psi$) iff $[\varphi(D)] = [\psi(D)]$; ie: ($\forall x \in D, \forall\ \alpha \in \mathbf{Pr}(\varphi(x)), \exists y \in D\ \&\ \exists \beta \in \mathbf{Pr}(\psi(y))$ st: $\alpha \sim \beta$) & ($\forall x \in D, \forall \alpha \in \mathbf{Pr}(\psi(x)), \exists y \in D\ \&\ \exists\ \beta \in\ \mathbf{Pr}(\varphi(y))$ st: $\alpha \sim \beta$). ■

2.9 <u>proposition</u>: if $< D, \leq_1, \lesssim_1 >$ and $< E, \leq_2, \lesssim_2 >$ are two SK-domains, then relation $\sim$ is an equivalence relation and allows to transform $[D \rightarrow E]$ into an SK-domain. ■

Notice that the functional intensional relation is a strict weakening of the pointwise extension of the basic one: if φ and ψ are two prime elements in $[D \rightarrow E]$ s.t $\forall x \in D$, $\varphi(x) \sim \psi(x)$, then $\varphi \sim \psi$; the contrary being false in general.

More significantly, if we now consider φ and ψ as two constant functions with results α and β respectively, then it is clear that $\varphi \sim \psi$ iff $\alpha \sim \beta$. This means that as soon as an intensional equivalence has been defined on the prime elements of the domains providing the constants of the calculus, then our two intensional relations reduce to the functional one; indeed, the basic relation is a specialisation of the functional one.

Henceforth if $X, Y \subseteq D$, we'll write "$X \lesssim Y$" to mean: $\forall\ \alpha \in X,\ \exists \beta \in Y$: $\alpha \lesssim \beta$ (samething with extensional preorder -or its negation- substituated for the intensional one -or its negation-). Further, we'll freely write x, for singleton $\{x\}$, anywhere the context disallows confusions; for example if $x \in D$, then "$x \lesssim X$" could be put for: $\exists\ \alpha \in X\ \&\ x \lesssim \alpha$.

III) <u>S-FUNCTIONS</u>

First off all, let recall that a function f from domain D to domain E is said to be continuous iff: $\forall x \in D, \forall \beta \in E$: β isolated & $\beta \leq f(x)$, $\exists\ \alpha \in D$ st: α isolated & $\alpha \leq x$ & $\beta \leq f(\alpha)$.
Equivalently f is continuous iff it preserves lubs of directed subsets.

Considering our domains, the subset of finite elements dominated by an element x is directed and has x as lub. That is to say that a continuous function is completely characterized by its values on the finite elements. Therefore, while considering continuous functions, we shall restrict to finite elements without any lost of generality.

Now, given two SK-domains D and E, we are interested in those continuous functions from D to E which satisfy the following condition (S):

$$\forall x \in D, \forall \beta \in E, (\beta \not\leq f(x) \ \&\ \exists z \in D \text{ st: } z > x \ \&\ \beta \leq f(z)) \Rightarrow \exists \omega(f,x,\beta) \in D:$$

(S): i) $\omega(f,x,\beta) \not\leq x$

ii) $\forall y \in D, \ (y > x \ \&\ \beta \leq f(y)) \Rightarrow \omega(f,x,\beta) \leq y$ ∎

Intuitively, a function verifies condition (S) iff any increase of information in its result necessitates the increase of an intensionaly predefined information in its argument. Indeed this idea is a generalization of Kahn and Plotkin's definition of sequential functions, in particular while considering basic domains, those where relation $\beta \leq x$ can be interpreted as:" β represents a place in x". By the way, $\omega(f,x,\beta)$ can be viewed as *the index of f in x, for value* β; one thus see, as already noticed by Vuillemin, that this index is not necessarily unique.

From now on, we write $[D -_S\to E]$ for the set of continuous functions from D to E that verify (S) and we call them S-functions.

Before we examine S-functions further, let us verify that SK-domains together with S-functions do actually form a category.

3.1 proposition: let D be an SK-domain then the identity function $1_D : D \to D$ st $\forall x \in D$, $1_D(x) = x$, is an S-function from D to D.

proof: one immediately verifies that β can be put for $\omega(1_D,x,\beta)$. ∎

3.2 proposition: $\forall f \in [D -_S\to E]$, $\forall g \in [E -_S\to F]$, $g \circ f \in [D -_S\to F]$.

proof: let $x \in D$, $\beta \in F$ & $z \in D$ st: $(\beta \not\leq g \circ f(x)$ & $z > x$ & $\beta \leq g \circ f(z))$; then it exists $\omega(g,f(x),\beta) \in E$: $\omega(g,f(x),\beta) \not\leq f(x)$ & $\omega(g,f(x),\beta) \leq f(z)$ since $\beta \leq g \circ f(z)$; thus it exists $\omega(f,x,\omega(g,f(x),\beta)) \in D$, st: $(\forall y \in D$, $y > x$ & $\beta \leq g \circ f(y)$ then $\omega(g,f(x),\beta) \leq f(y)$ and thus $\omega(f,x,\omega(g,f(x),\beta)) \leq y$). qed ∎

The above results allow to speak of the category whose objects are SK-domains and whose arrows are S-functions. We shall call it SKD. Actually, our intent is to show that SKD is a Λ-category. Before, let's examine more closely S-functions and some of their properties.

Fact 1: *S-functions do exist*: an example of S-function is the following f: $\mathbf{T}\rightarrow\mathbf{T}$ st: $f(\perp)=\perp$, $f(tt)=tt = f(ff)$. Indeed, it is easily verified that $\omega(f, \perp, tt)$ can be set to tt.
Another example of the existence of S-functions is given by the following

3.3 proposition: let d and e be two isolated element of an SK-domaine D, then function f denoted $(d \Rightarrow e)$ and such that: f(x)= if $x \geq d$ then e else $\perp$, is an S-fonction from D to E.

proof: function f is continuous since d and e are isolated. Now $\omega(f,x,\beta)$ is definable only if $\beta \leq e$ and if $\neg(x \geq d)$ & $(\exists\, z>x$ s.t $z \geq d)$; but in that case one immediately verifies that $\omega((d\Rightarrow e),x,\beta)$ can be set to d. ■

An immediate corollary of the above result is that any constant function is an S-fonction.

Fact 2: *S-functions are not divided in strict versus constant continuous functions*:
for example, function f: $\mathbf{T}^2\rightarrow\mathbf{T}^2$ st: f(x)= if $x \geq (\perp,tt)$ then (tt,tt) else $(tt,\perp)$, is an S-fonction which neither strict nor constant.

Fact 3: *continuous functions are strictly included in S-functions*, as shown by these examples:

i) here we produce the paradigm of non sequential function: the "parallel-or", which is not an S-function since $\omega(por, \perp, tt)$ does not exist; indeed $(\perp,tt) \not\leq (tt,\perp) \not\leq (\perp,tt)$.

ii) another example of non S-function is due to Berry: define function perm:$\mathbf{T}^3 \rightarrow \mathbf{T}$ to be the least continuous function st: perm(tt,ff, $\perp$)=perm(ff, $\perp$,tt)=perm($\perp$,tt,ff)=tt. Now remark that perm is not an S-function since for example $\omega(perm, (\perp,\perp,\perp),tt)$ does not exist.

Fact 4: *absence of indexes are not nessarily detected in* $\perp$: define f to be the least continuous function from $\mathbf{T}^3$ to T st: f(x)=tt iff $x \geq (tt,ff,\perp)$ or $x \geq (tt,\perp,ff)$; then f is not an S-function (since $\omega(f, (tt,\perp,\perp),tt)$ does not exist) thought $\omega(f, (\perp,\perp,\perp),tt)$ does exist.

T.Ehrhard and A. Bucciarelli, pointed to us that this phenomenum means that an index of a function in one point and for a given value does not in general provide informations about the argument, which would be sufficient to increase, in order to increase the result. In order to show that, consider the following example (due to T.Ehrhard and A. Bucciarelli):
let f be the least continuous function from $\mathbf{T}^3$ to T st: f(x)=tt iff $x \geq (tt,tt,\perp)$ or $x \geq (ff,\perp,tt)$; then f is intuitively sequential (and indeed an S-function) though it cannot be exhibited any element ε which could be put for $\omega(f, (\perp,\perp,\perp),tt)$), and s.t: $tt \leq f(\varepsilon)$.

To synthesize, the preceding fact and remark make the point that formula $\beta \leq f(\omega(f,x,\beta))$ cannot be made part of condition (S), since it cannot be made part of sequentiality definition. But taking into account this point, we can introduce a rather large class of S-functions which we call strongly sequential functions.

3.4 definition: given SK-domains D and E, a continuous function f from D to E is said to be *strongly sequential* iff it verifies the following condition (S_f) :

$$\forall x \in D, \forall \beta \in E, \; (\beta \not\leq f(x) \;\&\; \exists\, z \in D \text{ tq: } z > x \;\&\; \beta \leq f(z)) \Rightarrow \exists\, \Omega(f,x,\beta) \in D :$$

(S_f): $\quad$ i) $\Omega(f,x,\beta) \not\leq x \quad \& \quad \beta \leq f(\Omega(f,x,\beta)) \;\&$

$\quad$ ii) $\forall y \in D, \; (y > x \;\&\; \beta \leq f(y)) \Rightarrow \Omega(f,x,\beta)) \leq y$ ■

It is immediate that S_f-functions constitute a subclass of S-functions; moreover, they can be characterized by the fact that the absence of index can always be detected in $\perp$, as indiquated in the following:

3.5 proposition: a continuous function $f \in [D \to E]$, verifies (S_f), iff it verifies condition:

($S_\perp$): $\forall \beta \in E, \; (\beta \leq f(D)) \Rightarrow \exists\, \omega(f,\beta) \in D$ s.t: $\beta \leq f(\omega(f,\beta)) \;\&\; \forall y \in D, \; \beta \leq f(y) \Rightarrow \omega(f,\beta) \leq y$.

proof: (S_f)$\Rightarrow$($S_\perp$): let $f \in [D \text{-}_{S_f}\!\!\to E]$ and $\beta \in E$, st: $\beta \leq f(D)$; then only two cases are possible:

i) $\beta \leq f(\perp)$; in this case put $\omega(f,\beta)) = \perp$.

ii) $\beta \not\leq f(\perp)$; in this case one immediately verifies that $\omega(f,\beta)$ can be identified with $\Omega(f,\perp,\beta)$.

($S_\perp$)$\Rightarrow$(S_f): let f verify ($S_\perp$) and suppose x and β s.t. premises of (S_f) hold; we claim that $\Omega(f,x,\beta)$ exists and can be set to $\omega(f,\beta)$; indeed $\omega(f,\beta)$ exists by hypothesis; moreover, by definition, $\beta \leq f(\omega(f,\beta)) \;\&\; \forall y \in D, \; \beta \leq f(y) \Rightarrow \omega(f,\beta) \leq y$. finally, $\omega(f,\beta) \leq x$ is impossible since otherwise monotonicity of f would imply $\beta \leq f(x)$, yielding a contradiction. qed ■

To finish with, let us say that we have not been able to exhibit such characterization result for the whole class of S-functions; the best we come with is an approximation result based on the notion of covering relation:

3.6 proposition: let $f \in [D \text{-}_S\!\!\to E]$, then f verifies the following condition (R) :

$$\forall x \in D, \forall \beta \in E \;\; (\beta \not\leq f(x) \;\&\; \exists\, z > x : \beta \leq f(z)) \Rightarrow \left(\forall\, y_1, y_2 \in D \text{ s.t: } x \prec y_1 \;\&\; x \prec y_2, \right.$$
$$\left. (\beta \leq f(y_1) \;\&\; \beta \leq f(y_2) \Rightarrow y_1 = y_2) \right).$$

proof: recall that in a domain, $x \prec y \Leftrightarrow \exists!\, \alpha \in \mathcal{P}r(D)$: $\alpha \in \mathcal{P}r(y) \backslash \mathcal{P}r(x)$.■

We now examine the categorical properties of SKD.

IV) THE Λ- CATEGORY SKD

We first examine S-functions spaces between SK-domains. Here, our intent is to show that this space can be provided with both an extensional partial order and an intensional preorder, so as to make it an SK-domaine.

For the extensional partial order, we use Scott's order which is s.t: $f \leq g \Leftrightarrow \forall x, f(x) \leq g(x)$. The intensional preorder is that generated by the intensional equivalence we already defined on functional prime elements.

A first result of this section is the following:

4.1 theorem: the functions set of S-functions between SK-domains is an SK-domain.

The proof of this result is an easy corollary of propositions 2.4 and 3.7 and of the following result:

4.2 proposition: let D and E be two SK-domains then $< [D \text{-}_S\!\to E], \leq >$ is a domain.

proof: first, if D and E are two domains, then $< [D \to E], \leq >$ is a domaine, with prime elements the S-functions $(d \Rightarrow e)$ s.t: $e \in \mathbf{Pr}(E)$ and d isolated.

Now we show $[D \text{-}_S\!\to E]$ is a coherent cpo. Indeed, given a non empty subset $\{f_i / i \in I\}$ of S-functions, we define function f s.t: $\forall x, f(x) = \wedge f_i(x)$. f is obviously continuous; it verifies (S) since $\beta \not\lesssim (\wedge f_i(x)) \Rightarrow \exists f_j: \beta \not\lesssim f_j(x)$, $\beta \lesssim (\wedge f_i(z)) \Rightarrow \beta \lesssim f_j(z)$ and $\beta \lesssim (\wedge f_i(y)) \Rightarrow \beta \lesssim f_j(y)$; thus it is enough to identify $\omega(f,x,\beta)$ with $\omega(f_j,x,\beta)$, which is known to exist by hypothesis. That is to say, f is the glb of set $\{f_i / i \in I\}$. Noticing that the constant function which yields value $\perp$ is obviously the minimum element, one deduce that $[D \text{-}_S\!\to E]$ is actually a pime-algebraic down-complete semi-lattice and thus a domaine. ■

We now turn to the categorical properties of SKD.

First of all, recall that a Λ-category is a cartesian closed category, which is order-enriched (ie: the homsets are provided with order relations which make composition operation continuous), together with some additional continuity properties.

Berry[1] shaw that in the particular case where one tries to restrict Scott's Λ-category by imposing condition P to objects and condition Q to arrows (this is our case indeed), one has to verify the following conditions:

1) composition closedness: if D,E,F verify P and if $f:D \rightarrow E$ & $g:E \rightarrow F$ verify Q, then $g \circ f:D \rightarrow F$ verify Q; moreover, the identity arrow of objects that verify P, verifies Q.

2) product closedness: if D and E verify P, then $D \times E$ verify P. The projection functions from $D \times E$ to D and to E verify Q. If F verify P, and if $f: F \rightarrow D$ & $g:F \rightarrow E$ verify Q, then the function $\langle f,g \rangle$: $F \rightarrow D \times E$ verifies Q. (idem for the denumerable case).

3) exponentiation closedness: if D and E verify P, then the set $[D -_Q\rightarrow E]$ (of functions from D to E that verify Q) can be provided with an order $\leq$ such that $\langle [D -_Q\rightarrow E], \leq \rangle$ verify P, and such that the function"app" which applies functions to their arguments verify Q and finally such that the curryfied form of a function between objets that verify P, verifies Q, as soon as the function verifies Q.

4) continuity properties: the functions of composition, of pair formation and of curryfication are continuous.

First we shaw the following :

4.3 theorem: SKD is cartesian closed.

proof: let us notice that we have a little bit less to do here since propositions 3.1 and 3.2 constitute a proof of the composition closedness; moreover, since we use the "standard" extensional order, we need not worry about continuity. Indeed we just have to verify product and exponentiation closednesses in order to ensure that SKD is cartesian closed.

1) product closedness:

i):the cartesian product of a finite (or denumerably infinite) family of SK-domains is an SK-domain

This is the case for domains, with prime elements of the product domain those sequences of undefinite elements except one, which is a prime element , and with as partial order, the product of the orders of the domains. Now to get an SK-domain, we define the intensional preorder to be the product of the preorders of the domains.

ii) the projection functions from $D_1 \times D_2$ to D_i for $i \in \{1,2\}$, are S-functions:

it is easy to verify that $\omega(\pi_1,\langle x,y \rangle,\beta) = \langle \beta, \perp \rangle$ and that $\omega(\pi_2,\langle x,y \rangle,\beta) = \langle \perp, \beta \rangle$, for any $\beta \in D_i$, $i \in \{1,2\}$ and any $\langle x,y \rangle \in D_1 \times D_2$

iii) if $f \in [F -_S\rightarrow D]$ and $g \in [F -_S\rightarrow E]$ then $\langle f,g \rangle \in [F -_S\rightarrow D \times E]$.

verify that $\omega(\langle f,g \rangle,x,\beta)$ can be set to:

i) $\omega(f,x,\pi_1(\beta))$ if $\pi_1(\beta) \not\leq f(x)$ & $\pi_2(\beta) \leq g(x)$

ii) $\omega(g,x,\pi_2(\beta))$ if if $\pi_2(\beta) \not\leq g(x)$ & $\pi_1(\beta) \leq f(x)$

iii) to either $\omega(f,x,\pi_1(\beta))$ or $\omega(g,x,\pi_2(\beta))$, if both $\pi_1(\beta) \not\lesssim f(x)$ and $\pi_2(\beta) \not\lesssim g(x)$.

2) <u>exponentiation closedness:</u>

i) function $app:<[D\text{–}_S\rightarrow E] \times D> \rightarrow E$ is an S-fonction
let $<f,x> \in <[D\text{–}_S\rightarrow E] \times D>$, $\beta \in E$ and let $<h,z> \in <[D\text{–}_S\rightarrow E] \times D>$ s.t:
$(\beta \not\lesssim f(x))$ & $(<f,x> < <h,z>)$ & $(\beta \lesssim h(z))$; let us exhibit $\omega(app,<f,x>,\beta)$ s.t:

- $\omega(app,<f,x>,\beta) \in <[D\text{–}_S\rightarrow E] \times D>$ & $\omega(app,<f,x>,\beta) \not\lesssim <f,x>$ &
- $\forall <g,y> \in <[D\text{–}_S\rightarrow E] \times D>$, $(<f,x> < <g,y>$ & $\beta \lesssim g(y)) \Rightarrow \omega(app,<f,x>,\beta) \lesssim <g,y>$.

We claim $<\varphi, \perp>$, with $\varphi = (x \Rightarrow \beta)$, is a possible value.
Inded $\beta \not\lesssim f(x) \Rightarrow (x\Rightarrow\beta) \not\lesssim f$; and $\beta \lesssim g(y) \Rightarrow \exists\, \alpha \in E$ s.t: $\beta \approx \alpha$ & $\alpha \leq g(y)$; ie: $(y \Rightarrow \alpha) \leq g$ and thus since $(x\Rightarrow\beta) \approx (y \Rightarrow \alpha)$, then $<\varphi, \perp> \lesssim <g,y>$.

ii) function $curry :[D\times E \text{-}_S\rightarrow F] \rightarrow [D \rightarrow [E \rightarrow F]$ is an S-isomorphisme of SK-domains: consider an S-function f from $D\times E$ to F:

a) we show that the function f_a, from E to F, s.t:$\forall\, b \in E$, $f_a(b) = f(a,b)$, verifies (S).
Let $x, z \in E$ & $\beta \in F$ and suppose $z>x$, $\beta \not\lesssim f_a(x)$ & $\beta \lesssim f_a(z)$, then we have:
$(a,x), (a,z) \in D$ & $\beta \in F$ & $(a,z)>(a,x)$ & $\beta \not\lesssim f(a,x)$ & $\beta \lesssim f(a,z)$; thus $\exists\, \omega(f,(a,x),\beta) \in (D\times E)$, s.t $\omega(f,(a,x),\beta) \not\lesssim (a,x)$ & $\forall (a,y)>(a,x)$, $\beta \lesssim f(a,y) \Rightarrow \omega(f,(a,x),\beta) \lesssim (a,y)$. That is to say, we can set $\omega(f_a,x,\beta) \approx \pi_2(\omega(f,(a,x),\beta))$.

b) we show that the function $c(f)$ from D to $[E \text{–}_S\rightarrow F]$ and s.t:$\forall\, x \in D$, $c(f)(x)=f_x$, is an S-function.
Let $x, z \in D$ & $\beta \in [E \text{–}_S\rightarrow F]$ be s.t: $z>x$ & $\beta \not\lesssim c(f)(x)$ & $\beta \lesssim c(f)(z)$, then:
$\exists\, (x,e) \in D\times E$: $\beta(e) \not\lesssim f(x,e)$, & $\exists\, (z,e) \in D\times E$ st: $\beta(e) \lesssim f(z,e)$. That is to say $\omega(f,(x,e),\beta(e))$ exists; we then just set: $\omega(c(f),x,\beta) \approx \pi_1(\omega(f,(x,e),\beta(e))))$.

c) let us examine function curry .
we have to prove: $\forall\, f,h \in [D\times E \text{–}_S\rightarrow F]$, $\forall\, \beta \in [D \text{–}_S \rightarrow [E \text{–}_S\rightarrow F]]$, if $f<h$ & $\beta \not\lesssim c(f)$ & $\beta \lesssim c(h)$, then $\exists\, \omega(curry,f,\beta) \in [D\times E \text{–}_S\rightarrow F]$ s.t: $\omega(curry,f,\beta) > f$ & $\forall\, g>f$, $\beta \lesssim c(g) \Rightarrow \omega(curry,f,\beta) \lesssim g$.
Indeed, one verifies that $\omega(curry,f,\beta) \approx ((\delta,\varepsilon) \Rightarrow \phi) / (\delta \Rightarrow (\varepsilon \Rightarrow \phi) \leq \beta$.
As a corollary, we conclude that curry is an S-isomophism of SK-domains. ■

We now introduce our most significant result:

4.4 theorem: SKD is an order extensional Λ-category.

proof: corollary of results 4.1 & 4.3. ■

V) DISCUSSION

By way of conclusion, we examine here the relation between S-functions and Berry's stable functions.

In [1] it is shawn that the functions of Milner's syntactic model are stable. Of course it would be very nice to shaw that S-functions are stable; however it is clear as well that this result cannot hold with our extensional order on domains since Berry already shaw that the exponentiation operation is not stable when Scott's order is used.

We conjecture that some kind of intensional stability can subsist however; now the question is to exhibit this form of stability since in general it cannot be reduced to: $\forall f \in [D\text{-}_S \rightarrow E]$, $\forall x \in D, \forall \beta \in E, \beta \lesssim f(x) \Rightarrow \exists\ m(f,x,\beta) \in D$ s.t: $(m(f,x,\beta) \lesssim x)$ & $(\forall y \lesssim x, \beta \lesssim f(y) \Leftrightarrow m(f,x,\beta) \lesssim y)$.

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