Efficiency of the extended Kalman filter for non linear systems with small noise

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Rapports de Recherche

N° 1068

Programme 5
Automatique, Productique,
Traitement du Signal et des Données

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Août 1989

RR-1068

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EFFICACITE DU FILTRE DE KALMAN ETENDU POUR LES SYSTEMES NON LINEAIRES AVEC PETIT BRUIT

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Résumé. On étudie asymptotiquement le problème de filtrage non linéaire lorsque le bruit tend vers 0. On donne des conditions suffisantes pour que l'erreur du filtre tende vers 0 et sous ces conditions, on montre que le filtre de Kalman étendu est un bon filtre sous-optimal; le problème de lissage est également abordé. Les démonstrations utilisent le calcul stochastique des variations, des estimations concernant la linéarisation des systèmes stochastiques et des changements de probabilité.

Abstract. The problem of nonlinear filtering is studied asymptotically as the noise tends to 0. Sufficients conditions for the filtering error to tend to 0 are derived and under these conditions, it is proved that the extended Kalman filter provides a good suboptimal filter; the smoothing problem is also studied. The proofs use the stochastic calculus of variations, some estimates for the linearization of non linear stochastic systems and some changes of probability.

Key-words. Nonlinear filtering with small noise, Extended Kalman filter, Linearization of stochastic systems, Stochastic calculus of variations.

AMS Subject Classification (1980). 93E11, 93B07, 60G35, 60F05.

were devoted to two properties of the asymptotic filter; firstly it was proved to be nearly Gaussian under some particular assumptions in [29], and with a different method in [19]; secondly the problem of estimating its memory length was tackled in [20]. On the other hand, if h is not one-to-one, it is more difficult to find an asymptotic filter; in [30], sufficient conditions for the error to tend to 0 are obtained for a particular class of systems; in [27] some examples are studied formally and in [6] the one-dimensional case with a piecewise monotone function h is considered.

For the second direction of research (application of the theory of large deviations), the filtering problem which was studied is the system

$$\begin{cases} dX_t = \beta(X_t)dt + \sqrt{\varepsilon}\sigma(X_t)dW_t, \\ dY_t = h(X_t)dt + \sqrt{\varepsilon}dB_t. \end{cases} \tag{0.2}$$

Here, both signal and observation noises are small and they have the same order of magnitude. By means of the so-called robust filter, one can consider the conditional law of X_t as a continuous function of Y; in [11], one studies the asymptotic behaviour of this function taken at some fixed observation path; a large deviations principle is obtained; some further results are also obtained in [12]. Moreover, if one puts $\varepsilon = 0$ in (0.2), one obtains a deterministic system; it is explained in [1] how formal approximations of the non linear filtering problem can lead to observers for the deterministic system.

In this work, we are going to study the generalization

$$\begin{cases} dX_t = \beta(t, X_t)dt + \sqrt{\varepsilon}\sigma(t, X_t)dW_t + \sqrt{\varepsilon}\gamma(t, X_t)dB_t \\ dY_t = h(t, X_t)dt + \sqrt{\varepsilon}dB_t \end{cases} \tag{0.3}$$

of (0.2), but with the aim of finding approximate filters as in [23]. Note that we allow correlation between the signal and the observation noise and that the system is inhomogeneous; moreover we allow the coefficients to be random provided they are observable, and no extra regularity with respect to t will be assumed, so that our results can be applied to controlled systems. The coefficients in (0.3) may also depend on ε under some conditions which will be made precise later. We will first look for conditions ensuring that the filtering error (difference between the signal and the optimal filter) is of order $\sqrt{\varepsilon}$ as $\varepsilon \to 0$; these conditions will include the case where h is one-to-one but will be more general; roughly speaking, they will say that some associated linearized system is detectable; in particular we will have to estimate the probability of large deviations of the non linear system from a neighborhood of the linearized system. We will also estimate the difference between the signal and the extended Kalman filter; we will give conditions under which it is of order $\sqrt{\varepsilon}$, but will also check that it may be quite large if the nonlinearities are too strong. Then, under good conditions, we will be interested in proving a central limit theorem: we will check that the conditional law of X_t given Y_s, $s \leq t$, is asymptotically Gaussian (as in [29], [19]) and that it is approximately given by the extended Kalman filter; when $\gamma = 0$ and h is linear, we will also study the conditional law of the whole trajectory $(X_s, s \leq t)$ by means of some approximate Gaussian smoother.

Let us explain the link between (0.3) and the model (0.1) which was previously studied. By changing the time scale ($t \to t/\varepsilon$) and the order of magnitude of Y_t ($Y \to Y/\varepsilon$), we transform (0.3) into

$$\begin{cases} dX_t = \dfrac{1}{\varepsilon}\beta(t, X_t)dt + \sigma(t, X_t)dW_t + \gamma(t, X_t)dB_t, \\ dY_t = h(t, X_t)dt + \varepsilon dB_t. \end{cases} \tag{0.4}$$

Thus, by proving results on (0.3) which will be uniform for large times, we will deduce results on (0.4); moreover, by using other time scales, we can study systems with other magnitudes of noise as in [21]. Note that (0.4) is a generalization of (0.1); however, the drift coefficient can be of order $1/\varepsilon$, and therefore it cannot be neglected as it was in [23]. On the other hand, if we transpose the results of [23] to (0.3) by a change of time scale, we generally get some estimates only for large times; in order to get also results for bounded times, we will need assumptions on the initial condition. Note also that in [23] we only studied homogeneous systems and that we obtained a filter which is simpler than the extended Kalman filter; here we will take advantage of the particular properties of the extended Kalman filter in order to study inhomogeneous systems.

Let us now set some notational convention. Nearly all the functions and processes which are considered in this work are allowed to depend on the parameter ε; however this dependence will generally not be emphasized by some sub- or superscript; we will use the expression 'family of functions, processes, ...' in order to say that the functions, processes, ... are indexed by ε. A family of functions will be said to be bounded if it is bounded by some constant number which does not depend on ε; the same convention will be applied for some other properties such as Lipschitz continuity or ellipticity for matrix-valued functions. If f is a function, its derivative or more generally its Jacobian matrix will be denoted f'; in particular, if f is defined on $\mathbb{R}^n$ and real-valued, f' will be a line vector. If A is a matrix, its transpose will be denoted $A^\star$. In $\mathbb{R}^n$ we will use the Euclidean norm, and for matrices $|A|$ will be the supremum of $|Ax|$ over unit vectors x. The constant numbers involved in the calculations will be denoted by c or C and will vary from line to line; the dependence on some parameter will be emphasized by a subscript. The following assertions will be assumed in all this work:

The framework. *We fix a family of probability spaces $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, we let n, p, d be fixed positive integers and we consider four families of functions β, σ, γ and h defined on $[0,\infty) \times \Omega \times \mathbb{R}^n$ and with values respectively in $\mathbb{R}^n$, $\mathbb{R}^n \otimes \mathbb{R}^p$, $\mathbb{R}^n \otimes \mathbb{R}^d$ and $\mathbb{R}^d$; we suppose that they are $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathbb{R}^n)$ measurable, where $\mathcal{P}(\mathcal{F})$ denotes the σ-field of $\mathcal{F}_t$ predictable events, and that they are locally bounded: the suprema of their values over compact subsets of $\mathbb{R}_+ \times \mathbb{R}^n$ are almost surely finite. We consider two families of independent standard $\mathcal{F}_t$ Brownian motions W_t and B_t with values respectively in $\mathbb{R}^p$ and $\mathbb{R}^d$ and we suppose that (X_t, Y_t) is solution of (0.3). Moreover, letting $\mathcal{Y}_t$ be the filtration generated by Y_t, we assume that the coefficients β, σ, γ and h are $\mathcal{P}(\mathcal{Y}) \otimes \mathcal{B}(\mathbb{R}^n)$ measurable: we will say that they are observable and more generally, an observable process will be a $\mathcal{Y}_t$ predictable process. The conditional mean of X_t given $\mathcal{Y}_T$ will be denoted $\widehat{X}_t$. We will also use the notation $a = \sigma\sigma^\star$.*

Other definitions which will be used in this work will be given in §1.1. The paper is organized as follows. In §1, after some basic definitions, we prove some preliminary lemmas concerning the stability of stochastic systems; as an application, in §2 we find conditions under which $X_t - \widehat{X}_t$ is of order $\sqrt{\varepsilon}$ and also conditions under which the extended Kalman filter satisfies the same estimate. In §3, we estimate the difference between the optimal filter and the Gaussian law provided by the extended Kalman filter; in §4, using an approximate Gaussian smoother, the same study is worked out for the smoothing problem under some particular conditions; in §§3 and 4, we need some results from the stochastic calculus of variations (see [28]); these results are obtained by perturbing the initial state (§3) or a Brownian motion (§4); since they do not appear exactly in the literature, we prove them (actually in our particular framework, we do not need the best possible results); as a consequence, no previous knowledge of this theory will be assumed. Finally, in §5, we consider some stochastic differential equations driven by X_t and prove that they can be approximated by the similar equations driven by the extended Kalman filter; this will provide examples of filters involving two time scales.

1. Definitions and preliminary results

In this section, we first recall the equations of the extended Kalman filter and give some other definitions in §1.1. Then we obtain some basic results about estimation of processes in §1.2, and we study the stability of linear stochastic systems in §1.3 and §1.4.

§1.1 Basic definitions

We first recall the construction of the extended Kalman filter (see for instance [8] for the uncorrelated case $\gamma = 0$); the aim of this procedure is to find an easily computable approximation of the conditional law of X_t given $\mathcal{Y}_t$. To this end, one replaces the system (0.3) by an approximate linear system; more precisely, one replaces β and h by linear functions, σ and γ by constant functions (with respect to x) such that the values of β, β', σ, γ, h and h' for (0.3) and the approximate system coincide at some M_t which is an observable process; one also replaces the law of X_0 by some Gaussian law. Since the approximate system is conditionally Gaussian, one can easily compute the approximate conditional mean of X_t given $\mathcal{Y}_t$ (see [17]); thus we construct with this procedure an application which relates any observable process M_t to the corresponding approximate conditional mean, and the extended Kalman filter is by definition the conditional law obtained for a fixed point of this application. In this work, we will not make a precise choice for the initial value of the filter so that it will not be unique (actually the properties of the initial value will be crucial in several results, so we want to keep some freedom for choosing it). By writing precisely the equations, we obtain the

Definition 1.1.1. *Let M_t and P_t be observable processes with values respectively in $\mathbb{R}^n$ and in definite positive symmetric matrices of order n. Consider the process with values in probability measures, the value of which is at each time t the Gaussian law with mean M_t and covariance εP_t. We will say that it is an extended Kalman filter for (0.3) if M_t is solution of*

$$M_t = M_0 + \int_0^t \beta(s, M_s)ds + \int_0^t G_s(dY_s - h(s, M_s)ds), \tag{1.1.1}$$

the gain G_t is given by

$$G_t = \gamma(t, M_t) + P_t h'^{\star}(t, M_t) \tag{1.1.2}$$

and P_t is solution of the Riccati equation

$$\dot{P}_t = -P_t h'^{\star} h'(t, M_t) P_t + (\beta' - \gamma h')(t, M_t) P_t + P_t (\beta' - \gamma h')^{\star}(t, M_t) + \sigma\sigma^{\star}(t, M_t). \tag{1.1.3}$$

If this holds, we will also say that (M_t, P_t), or simply M_t is an extended Kalman filter. If M_t is solution of (1.1.1) with another process G_t, we will say that it is a Kalman-like filter with gain G_t.

Note that P_t is the covariance matrix divided by ε; with this normalization indeed, the parameter ε does not appear explicitely in equations (1.1.1) to (1.1.3) (however the coefficients may depend on it). We now define some terms which will be frequently used.

Definition 1.1.2. *Let $f(t,x)$ be a family of observable functions with values in some Euclidean space. It will be said to be almost linear if there exists a family of matrix-valued observable processes F_t such that*

$$\left|f(t,x) - f(t,m) - F_t(x-m)\right| \leq \mu_\varepsilon |x-m| \tag{1.1.4}$$

for some family of numbers μ_ε converging to 0; the process F_t will be called an almost derivative of f. The function f will be said to be strongly injective if

$$|f(t,x) - f(t,m)| \geq c|x-m| \tag{1.1.5}$$

for some $c > 0$.

When f is C^1, the almost linearity means that the oscillation of f' tends to 0. Note also that if f is almost linear with an almost derivative F_t, then the strong injectivity of f is equivalent to the uniform ellipticity of $F_t^\star F_t$ (at least after restricting to small enough ε). We now explain how we will measure the performance of our approximations. The basic definition is the

Definition 1.1.3. *A family ξ_t of processes is said to be bounded in $L^{\infty-}$ if for any $q < \infty$ there exists $\varepsilon_q > 0$ such that $\|\xi_t\|_q$ is bounded uniformly in $t \geq 0$ and $0 < \varepsilon < \varepsilon_q$. If α is some real constant number, the family ξ_t is said to be of order ε^α if $\varepsilon^{-\alpha}\xi_t$ is bounded in $L^{\infty-}$. More generally, if k_t is a family of deterministic functions with positive values, we will write $\xi_t = O(k_t)$ if $k_t^{-1}\xi_t$ is bounded in $L^{\infty-}$.*

Our last definition is concerned with the stability of linear systems; it will be discussed in §1.3 and will be fundamental in all the estimation of the filtering error.

Definition 1.1.4. *Let A_t be a family of measurable locally bounded processes with values in square matrices of order n, let ζ_t be the matrix-valued solution of*

$$\dot{\zeta}_t = A_t\zeta_t, \qquad \zeta_0 = I, \tag{1.1.6}$$

and put $\zeta_{s,t} = \zeta_t\zeta_s^{-1}$. We will say that A_t is exponentially stable if there exist some constant numbers C and $c > 0$ such that for $s \leq t$,

$$\left|\zeta_{s,t}\right| \leq C \exp -c(t-s). \tag{1.1.7}$$

Moreover consider a family Q_t of absolutely continuous adapted processes with values in symmetric definite positive matrices of order n and a family k_t of locally bounded deterministic functions with positive values. We will say that A_t is (Q_t, k_t) stable if

$$\dot{Q}_t \geq A_tQ_t + Q_tA_t^* + k_tQ_t. \tag{1.1.8}$$

For instance, if A_t is some constant matrix, it is exponentially stable if and only if its eigenvalues have negative real part; if A_t is symmetric, bounded and uniformly definite negative, it is also exponentially stable. As it will be clear later, the general (Q_t, k_t) stability is a generalization of the exponential stability; the matrix Q_t enables a change of space scale, whereas the function k_t is related to the time scale. Note that the notion of (Q_t, k_t) stability is invariant if one multiplies Q_t by a family of scalar numbers.

§ *1.2 Estimation of processes*

This subsection is devoted to the proof of two lemmas linked with definition 1.1.3. This definition involves the estimation of $\|\xi_t\|_q$ uniformly in t. Can we deduce something about $\sup_t |\xi_t|$? This is the aim of the

Lemma 1.2.1. *Let*

$$\xi_t = \xi_0 + \int_0^t f_s ds + \int_0^t g_s dw_s \tag{1.2.1}$$

be a family of semimartingales, where w_t is a Brownian motion, and f_t, g_t are families of locally bounded adapted processes. Suppose that for some real α_0, α_1, ξ_t is of order ε^{α_0} and f_t, g_t are of order ε^{α_1} in $L^{\infty-}$. Then for any $\alpha_2 < \alpha_0$ and any $\alpha_3 \leq 0$, the supremum of $|\xi_t|$ on the time interval $[0, \varepsilon^{\alpha_3}]$ is of order ε^{α_2} in $L^{\infty-}$.

Proof. Put $\alpha_4 = 2(\alpha_0 - \alpha_1)^+$ and consider the subdivision $\tau_i = i\varepsilon^{\alpha_4}$; from classical inequalities, one can check that uniformly in i,

$$\sup_{\tau_i \leq t \leq \tau_{i+1}} \left|\xi_t - \xi_{\tau_i}\right| = O(\varepsilon^{\alpha_0}). \tag{1.2.2}$$

Since we have assumed $\xi_t = O(\varepsilon^{\alpha_0})$, we deduce that uniformly in i, ξ_{τ_i} is also of order ε^{α_0} so

$$\sup_{\tau_i \le t \le \tau_{i+1}} |\xi_t| = O(\varepsilon^{\alpha_0}). \tag{1.2.3}$$

The number of points of the subdivision which are in the interval $[0, \varepsilon^{\alpha_3}]$ is equal to the integer value of $\varepsilon^{\alpha_3 - \alpha_4} + 1$, so for any $1 \le q < \infty$,

$$\mathbb{E} \sup_{0 \le t \le \varepsilon^{\alpha_3}} |\xi_t|^q \le \left(\varepsilon^{\alpha_3 - \alpha_4} + 1\right) \sup_i \mathbb{E} \sup_{\tau_i \le t \le \tau_{i+1}} |\xi_t|^q \le C_q \varepsilon^{q\alpha_0 + \alpha_3 - \alpha_4}. \tag{1.2.4}$$

Thus the supremum of $|\xi_t|$ over $[0, \varepsilon^{\alpha_3}]$ is of order ε^{α_2} in L^q as soon as

$$q \ge (\alpha_4 - \alpha_3)/(\alpha_0 - \alpha_2). \tag{1.2.5}$$

In particular, it is of order ε^{α_2} in $L^{\infty -}$. $\square$

Note however that the estimation of lemma 1.2.1 cannot be extended to the whole time interval $[0, \infty)$ because the paths of the process are generally unbounded as $t \to \infty$. With some stronger assumptions, we can also obtain a result which will imply exponential estimates on the probability of large deviations.

Lemma 1.2.2. *Let ξ_t be the semimartingale defined by (1.2.1) and suppose that for some $c_0 > 0$, α_0, one has*

$$\sup_{t, \varepsilon} \mathbb{E} \exp c_0 \varepsilon^{-\alpha_0} |\xi_t|^2 < \infty. \tag{1.2.6}$$

Suppose also that for some $\alpha_1 \le 0$, the processes f_t and g_t divided by ε^{α_1} are uniformly bounded. Then for some positive constant numbers c_1, c_2 and C, one has

$$\mathbb{P}\left[\exists t \le \exp \frac{c_2}{\varepsilon^{\alpha_0}}, \quad |\xi_t| > 1\right] \le C \exp - \frac{c_1}{\varepsilon^{\alpha_0}}. \tag{1.2.7}$$

Proof. Consider the subdivision $\tau_i = i\varepsilon^{\alpha_4}$ with $\alpha_4 = (\alpha_0 - 2\alpha_1)^+$; for $\tau_i \le t \le \tau_{i+1}$, one has

$$|\xi_t - \xi_{\tau_i}| \le C\varepsilon^{\alpha_4 + \alpha_1} + \left| \int_{\tau_i}^t g_s dw_s \right| \tag{1.2.8}$$

so that

$$\frac{1}{\varepsilon^{\alpha_0}} \left|\xi_t - \xi_{\tau_i}\right|^2 \le C + \frac{2}{\varepsilon^{\alpha_0}} \left| \int_{\tau_i}^t g_s dw_s \right|^2. \tag{1.2.9}$$

Moreover one can prove from our assumption on g_s (see lemma 5.7.2 of [13]) that for some positive K and C,

$$\mathbb{E} \sup_{\tau_i \le t \le \tau_{i+1}} \exp \frac{K}{\varepsilon^{\alpha_0}} \left| \int_{\tau_i}^t g_s dw_s \right|^2 \le C. \tag{1.2.10}$$

We deduce from (1.2.9) and (1.2.10) an estimate on the exponential moments of $|\xi_t - \xi_{\tau_i}|^2$, and by using also (1.2.6) at time τ_i, we obtain the existence of a $c_3 > 0$ such that

$$\mathbb{E} \sup_{\tau_i \le t \le \tau_{i+1}} \exp \frac{c_3}{\varepsilon^{\alpha_0}} |\xi_t|^2 \le C \tag{1.2.11}$$

and therefore

$$\mathbb{P}\left[\sup_{\tau_i \le t \le \tau_{i+1}} |\xi_t| > 1\right] \le C \exp -\frac{c_3}{\varepsilon^{\alpha_0}}. \tag{1.2.12}$$

By summing this inequality over i, we easily obtain (1.2.7) for $c_2 < c_3$ and $c_1 < c_3 - c_2$. $\square$

§1.3 Stable systems

In this subsection, we study the stability of linear stochastic systems. We first discuss the links between exponential stability and (Q_t, k_t) stability of definition 1.1.4.

Lemma 1.3.1. *Consider a family A_t of locally bounded processes with values in square matrices of order n. Suppose that there exists a uniformly bounded and elliptic family Q_t and a constant number $k > 0$ such that A_t is (Q_t, k) stable; then A_t is exponentially stable (the estimate (1.1.7) is satisfied with $c = k/2$). Conversely, if A_t is uniformly bounded and exponentially stable (so that (1.1.7) holds for some $c > 0$), then for any $k < 2c$, there exists a family of uniformly bounded and elliptic processes Q_t which are adapted to the filtration of A_t and are such that A_t is (Q_t, k) stable.*

Proof. First assume that A_t is (Q_t, k) stable for a bounded and elliptic Q_t. Then $V_t = Q_t^{-1}$ is bounded, uniformly elliptic and

$$\dot{V}_t \le -V_t A_t - A_t^\star V_t - k V_t. \tag{1.3.1}$$

We immediately deduce that, with the notation of definition 1.1.4,

$$\frac{d}{dt} \zeta_{s,t}^\star V_t \zeta_{s,t} \le -k \zeta_{s,t}^\star V_t \zeta_{s,t} \tag{1.3.2}$$

so that

$$\operatorname{trace}\big(\zeta_{s,t}^\star V_t \zeta_{s,t}\big) \le e^{-k(t-s)} \operatorname{trace}(V_s). \tag{1.3.3}$$

Since V_t is uniformly bounded and elliptic, we deduce (1.1.7) with $c = k/2$. Conversely, suppose that (1.1.7) holds, that A_t is bounded and choose $k < 2c$. Let Q_t be the solution of

$$\dot{Q}_t = A_t Q_t + Q_t A_t^\star + k Q_t + I, \qquad Q_0 = I. \tag{1.3.4}$$

Then (1.1.8) is satisfied, Q_t is symmetric definite positive and is given by

$$Q_t = \zeta_t\zeta_t^\star e^{kt} + \int_0^t \zeta_{s,t}\zeta_{s,t}^\star e^{k(t-s)}ds, \tag{1.3.5}$$

so we deduce from (1.1.7) that it is bounded. Moreover, $V_t = Q_t^{-1}$ is solution of

$$\dot{V}_t = -A_t^\star V_t - V_t A_t - kV_t - V_t^2 \tag{1.3.6}$$

and therefore, by looking at its trace and using the inequalities

$$\left|\operatorname{trace}(V_t A_t)\right| \le |A_t| \operatorname{trace} V_t, \qquad \left(\operatorname{trace} V_t\right)^2 \le n \operatorname{trace} V_t^2 \tag{1.3.7}$$

(see [25] for the first one), we obtain

$$\frac{d}{dt}\operatorname{trace} V_t \le (2|A_t| - k)\operatorname{trace} V_t - \frac{1}{n}\left(\operatorname{trace} V_t\right)^2. \tag{1.3.8}$$

Since A_t is bounded, the trace of V_t and therefore V_t itself is bounded so Q_t is uniformly elliptic. □

We now explain how the stability can be used in stochastic systems. As in the first part of previous proof, we will use the quadratic function $x^\star Q_t^{-1} x$ as a Lyapunov function in order to estimate the state. Another type of equations will be studied in §1.4.

Lemma 1.3.2. *Let w_t be a $\mathcal{F}_t$ Brownian motion with values in $\mathbb{R}^r$, let A_t be a family of $\mathcal{F}_t$ adapted (Q_t, k_t) stable processes where Q_t is $\mathcal{F}_t$ adapted; we suppose that Z_t is a family of $\mathbb{R}^n$ valued semimartingales satisfying*

$$dZ_t = A_t Z_t dt + f_t dt + g_t dw_t \tag{1.3.9}$$

where f_t and g_t are predictable processes satisfying

$$\begin{cases} Z_t^\star Q_t^{-1} f_t \le \alpha k_t Z_t^\star Q_t^{-1} Z_t + O(k_t), \\ g_t^\star Q_t^{-1} g_t = O(k_t) \end{cases} \tag{1.3.10}$$

for some $\alpha < 1/2$. Then for any $c < 1 - 2\alpha$ and ε small enough,

$$\left\|Z_t^\star Q_t^{-1} Z_t\right\|_q \le C_q + C_q \left\|Z_0^\star Q_0^{-1} Z_0\right\|_q \exp -c\int_0^t k_s ds. \tag{1.3.11}$$

In particular, if $Z_0^\star Q_0^{-1} Z_0$ is bounded in $L^{\infty-}$, then the process $Z_t^\star Q_t^{-1} Z_t$ is also bounded in $L^{\infty-}$.

Remark 1. The process f_t will generally contain the nonlinearities of the system so that (1.3.10) means that the nonlinearities are not too strong. Our assumptions will in most cases imply that

$$f_t^\star Q_t^{-1} f_t \leq \alpha_\varepsilon k_t^2 Z_t^\star Q_t^{-1} Z_t \tag{1.3.12}$$

for some family α_ε; from the Cauchy-Schwarz inequality, this implies that

$$Z_t^\star Q_t^{-1} f_t \leq \alpha_\varepsilon^{1/2} k_t Z_t^\star Q_t^{-1} Z_t \tag{1.3.13}$$

so the first part of (1.3.10) holds for ε small as soon as $\limsup \alpha_\varepsilon < 1/4$.

Remark 2. As it was announced after definition 1.1.4, the role of Q_t and k_t becomes clear in this lemma.

Proof. Denoting $V_t = Q_t^{-1}$, we deduce from Itô's formula that

$$d(Z_t^\star V_t Z_t) = Z_t^\star(\dot{V}_t + V_t A_t + A_t^\star V_t) Z_t dt + 2 Z_t^\star V_t f_t dt + \operatorname{trace}(g_t^\star V_t g_t) dt + 2 Z_t^\star V_t g_t dw_t. \tag{1.3.14}$$

Using the Cauchy-Schwarz inequality to estimate $Z_t^\star V_t g_t$, the assumption (1.3.10), the (Q_t, k_t) stability of A_t, and denoting $\lambda_t = Z_t^\star V_t Z_t$, we can write this equation in the form

$$d\lambda_t = -(1-2\alpha) k_t \lambda_t dt + k_t \mu_t dt + k_t^{1/2} \lambda_t^{1/2} \nu_t^\star dw_t \tag{1.3.15}$$

where μ_t and ν_t are processes with values respectively in $\mathbb{R}$ and $\mathbb{R}^r$, such that μ_t^+ and ν_t are bounded in $L^{\infty-}$. From this equation and Gronwall's lemma, we can deduce that all the moments of λ_t are finite. Then if q is a positive integer, we can again apply Itô's formula in order to decompose λ_t^q and if $\lambda_t^{(q)}$ denotes its mean, we obtain

$$\frac{d}{dt}\lambda_t^{(q)} = -(1-2\alpha) q k_t \lambda_t^{(q)} + q k_t \mathbb{E}\big[\lambda_t^{q-1}\mu_t\big] + \frac{q(q-1)}{2} k_t \mathbb{E}\big[\lambda_t^{q-1}|\nu_t|^2\big]. \tag{1.3.16}$$

Moreover, an elementary analysis shows that for any sequence $c_q > 0$, one can find a sequence C_q such that

$$x^{q-1} y \leq c_q x^q + C_q y^q \tag{1.3.17}$$

for x and $y \geq 0$ so if $c < 1 - 2\alpha$, there exists a positive C_q such that

$$\frac{d}{dt}\lambda_t^{(q)} \leq -cqk_t\lambda_t^{(q)} + C_q k_t \mathbb{E}\Big[|\mu_t^+|^q + |\nu_t|^{2q}\Big]. \tag{1.3.18}$$

Since μ_t^+ and ν_t are bounded in $L^{\infty-}$, we deduce (1.3.11). $\square$

In the framework of lemma 1.3.2, we can also apply lemma 1.2.1 and therefore estimate the supremum of solutions of (1.3.9). We now give two results about the exponential moments which will enable the application of lemma 1.2.2 and will be useful in some other estimations.

Lemma 1.3.3. *Assume the conditions of lemma 1.3.2 and suppose moreover that the '$O(k_t)$' terms in (1.3.10) are actually of order k_t in L^∞. Suppose also that some exponential moment of $Z_0^\star Q_0^{-1} Z_0$ is bounded. Then*

$$\mathbb{E} \exp c Z_t^\star Q_t^{-1} Z_t \leq C \tag{1.3.19}$$

and

$$\mathbb{E} \exp c \int_s^t k_u Z_u^\star Q_u^{-1} Z_u du \leq C \exp C \int_s^t k_u du \tag{1.3.20}$$

for some positive c and C.

Proof. We use the notation of lemma 1.3.2. Under our assumptions, μ_t^+ and ν_t are uniformly bounded; if C_0 is a constant number which dominates μ_t^+ and $|\nu_t|^2/2$, we deduce from (1.3.16) that

$$\frac{d}{dt}\lambda_t^{(q)} \leq q k_t \Big(-(1-2\alpha)\lambda_t^{(q)} + C_0 q \lambda_t^{(q-1)} \Big). \tag{1.3.21}$$

By putting $C_1 = C_0/(1-2\alpha)$,

$$\begin{aligned} \lambda_t^{(q)} &\leq \lambda_0^{(q)} + C_1 q \sup_{s \leq t} \lambda_s^{(q-1)} \\ &\leq q! \sum_{i=0}^{q} C_1^{q-i} \frac{\lambda_0^{(i)}}{i!} \\ &\leq q! C_1^q \mathbb{E} \exp C_1^{-1} Z_0^\star Q_0^{-1} Z_0 . \end{aligned} \tag{1.3.22}$$

Thus if C_0 is chosen large enough so that the exponential moment of order C_1^{-1} of $Z_0^\star Q_0^{-1} Z_0$ is bounded, then (1.3.19) holds for any $c < C_1^{-1}$. Let us now prove (1.3.20) for some constant number c. If $c < (1-2\alpha)^2/(4C_0)$, we have

$$c \int_s^t k_u \lambda_u du \leq 2c \int_s^t k_u \lambda_u du - \frac{2c^2}{(1-2\alpha)^2} \int_s^t k_u \lambda_u |\nu_u|^2 du. \tag{1.3.23}$$

If we express the first integral of the right-hand side by means of (1.3.15), we deduce

$$c\int_s^t k_u\lambda_u du \leq \frac{2c}{1-2\alpha}\lambda_0 + C\int_s^t k_u du + \frac{2c}{1-2\alpha}\int_s^t k_u^{1/2}\lambda_u^{1/2}\nu_u^* dw_u - \frac{2c^2}{(1-2\alpha)^2}\int_s^t k_u\lambda_u|\nu_u|^2 du \tag{1.3.24}$$

for some C depending on c, α and the bound for μ_t^+. By taking the exponential, a local exponential martingale (which is also a supermartingale) appears so that

$$\mathbb{E}\Big[\exp c\int_s^t k_u\lambda_u du \;\Big|\; \mathcal{F}_0\Big] \leq \exp\Big(C\int_s^t k_u du + \frac{2c}{1-2\alpha}\lambda_0\Big). \tag{1.3.25}$$

From the assumption about Z_0, the mean of the right-hand side is bounded by the right-hand side of (1.3.20) if c is small enough. $\square$

§*1.4 Bilinear stable equations*

We now consider stochastic differential equations where both drift and diffusion coefficients are linear; these equations appear in particular when one differentiates a diffusion process with respect to its initial condition (see [15]) or with respect to perturbations on the driving Brownian motion (see [28]), and such differentiations will be an important tool in §§3 and 4. This subsection is devoted to the proof of the

Lemma 1.4.1. *Suppose that Z_t is a matrix-valued process satisfying*

$$Z_t = I + \int_0^t A_s Z_s ds + \int_0^t f_s Z_s ds + \mu_\varepsilon \int_0^t g_s^i Z_s dw_s^i \tag{1.4.1}$$

where w_t is a $\mathcal{F}_t$ semimartingale and a $\mathcal{G}_t$ Brownian motion for some filtration $\mathcal{G}_t \subset \mathcal{F}_t$, μ_ε is a family of positive numbers converging to 0, A_t is a family of $\mathcal{F}_t$ adapted bounded exponentially stable processes, f_t is a family of matrix-valued $\mathcal{F}_t$ adapted processes satisfying

$$\mathbb{E}\exp K\int_s^t |f_u| du \leq C_K \exp C_K \mu'_\varepsilon (t-s) \tag{1.4.2}$$

for any $K > 0$ and some family $\mu'_\varepsilon \to 0$, and g_t^i are families of matrix-valued uniformly bounded processes which are $\mathcal{G}_t$ adapted. Then the process $Z_t Z_s^{-1}$, $s \leq t$, is of order $e^{-c(t-s)}$ in $L^{\infty-}$.

This result is not surprising: since f_t and $\mu_\epsilon g_t$ are small, (1.4.1) is close to $\dot{Z}_t = A_t Z_t$. Note however that there is a technical difficulty due to the fact that A_t and f_t are not supposed to be adapted to the filtration with respect to which w_t is a Brownian motion. In the proof of this lemma, we will need as a preliminary result the

Lemma 1.4.2. *Under the assumptions of lemma 1.4.1, let ζ_t be the solution of*

$$\zeta_t = I + \mu_\varepsilon \int_0^t g_s^i \zeta_s dw_s^i. \tag{1.4.3}$$

For some fixed $\delta > 0$, let τ_j be the stopping times with values in $[0, +\infty]$ defined by induction by $\tau_0 = 0$ and

$$\tau_{j+1} = \inf\Big\{ t \geq \tau_j; \quad \big|\zeta_t \zeta_{\tau_j}^{-1} - I\big| \geq \delta \Big\}. \tag{1.4.4}$$

Put also

$$N_t = \inf\{j; \quad \tau_{j+1} \geq t\}. \tag{1.4.5}$$

Then for any $c > 0$ there exists a C such that

$$\mathbb{E} e^{cN_t} \leq C e^{C\mu_\varepsilon^2 t}. \tag{1.4.6}$$

Proof. First suppose that $\mu_\varepsilon = 1$; for $\tau_j \leq t \leq \tau_{j+1}$, on $\{\tau_j < \infty\}$, the process $\zeta_t \zeta_{\tau_j}^{-1} - I$ is equal to the stochastic integral with respect to w_t of a process which is uniformly bounded; since each component is a time-changed Brownian motion, we deduce that conditionally on $\mathcal{G}_{\tau_j}$, the variable $\tau_{j+1} - \tau_j$ is greater than the first time at which a standard Brownian motion is greater than some constant number; thus

$$\mathbb{E}\big[\exp -K(\tau_{j+1} - \tau_j) \mid \mathcal{G}_{\tau_j}\big] \leq e^{-C_K} \tag{1.4.7}$$

where $C_K \to \infty$ as $K \to \infty$, and therefore

$$\mathbb{E} \exp -K\tau_j \leq e^{-C_K j}. \tag{1.4.8}$$

When $\mu_\varepsilon \neq 1$, it acts as a change of time in (1.4.3) so

$$\mathbb{E} \exp -K\mu_\varepsilon^2 \tau_j \leq e^{-C_K j}. \tag{1.4.9}$$

On the other hand, fix some $c > 0$; let us compute the exponential moment of order c of N_t; since N_t is almost surely finite, we have

$$\begin{aligned} \mathbb{E} e^{cN_t} &= \sum_{j=0}^{\infty} e^{cj} \mathbb{P}[N_t = j] \\ &= \sum_{j=0}^{\infty} e^{cj} \Big(\mathbb{P}[N_t > j-1] - \mathbb{P}[N_t > j] \Big) \\ &= 1 + (e^c - 1) \sum_{j=0}^{\infty} e^{cj} \mathbb{P}[N_t > j]. \end{aligned} \tag{1.4.10}$$

The event $\{N_t > j\}$ is equal to $\{\tau_{j+1} < t\}$, so from the Bienaymé-Chebychev inequality,

$$\mathbb{E}e^{cN_t} \leq 1 + (e^c - 1)e^{K\mu_\epsilon^2 t}\sum_{j=0}^{\infty} e^{cj}\mathbb{E}e^{-K\mu_\epsilon^2\tau_{j+1}} \tag{1.4.11}$$

for any $K > 0$. By choosing K large enough so that $C_K > c$, we deduce (1.4.6) from (1.4.9). $\square$

Proof of lemma 1.4.1. The idea for proving this lemma is to decompose (1.4.1) into its absolutely continuous and martingale parts (see [15] for general decompositions theorems). So let ζ_t and $\overline{\zeta}_t$ be the solutions of (1.4.3) and

$$\overline{\zeta}_t = I + \int_0^t \zeta_s^{-1}A_s\zeta_s\overline{\zeta}_s ds + \int_0^t \zeta_s^{-1}f_s\zeta_s\overline{\zeta}_s ds. \tag{1.4.12}$$

Then the process Z_t is equal to the product $\zeta_t\overline{\zeta}_t$. For some $\delta > 0$ which will be chosen later, let us also consider the stopping times τ_j defined in lemma 1.4.2 and let us estimate $\overline{\zeta}_t$ on $\{t \leq \tau_1\}$. On this event, we can write (1.4.12) in the form

$$\overline{\zeta}_t = I + \int_0^t A_s\overline{\zeta}_s ds + \int_0^t \phi_s\overline{\zeta}_s ds \tag{1.4.13}$$

where

$$|\phi_s| \leq C_\delta(|f_s| + 1) + |f_s| \tag{1.4.14}$$

and $C_\delta \to 0$ as $\delta \to 0$. On the other hand, since A_t is bounded and exponentially stable, we deduce from lemma 1.3.1 that there exists a bounded and elliptic Q_t and a $k > 0$ such that A_t is (Q_t, k) stable; by looking at the equation satisfied by $\overline{\zeta}_t^\star Q_t^{-1}\overline{\zeta}_t$, we easily check from (1.4.13) that

$$|\overline{\zeta}_t| \leq C\exp\Big\{-ct + C\int_0^t |\phi_s|ds\Big\} \tag{1.4.15}$$

so that if δ is chosen small enough,

$$|\overline{\zeta}_t| \leq C\exp\Big\{-ct + C\int_0^t |f_s|ds\Big\} \tag{1.4.16}$$

on $\{t \leq \tau_1\}$ for some positive c and C; since ζ_t is uniformly bounded on $\{t \leq \tau_1\}$, it follows that Z_t also satisfies an estimate of type (1.4.16). Similarly, on $\{\tau_j \leq t \leq \tau_{j+1}\}$, one has

$$|Z_t Z_{\tau_j}^{-1}| \leq C\exp\Big\{-c(t - \tau_j) + C\int_{\tau_j}^t |f_s|ds\Big\}. \tag{1.4.17}$$

Then by means of the decomposition

$$Z_t = Z_t Z_{\tau_{N_t}}^{-1} \prod_{j=0}^{N_t-1} Z_{\tau_{j+1}} Z_{\tau_j}^{-1}, \tag{1.4.18}$$

we deduce

$$|Z_t| \leq C^{N_t} \exp\Big\{-ct + C\int_0^t |f_s| ds\Big\}. \tag{1.4.19}$$

Thus lemma 1.4.2 and (1.4.2) imply that Z_t is of order e^{-ct} for some $c > 0$. More generally, the process $Z_t Z_s^{-1}$ is studied in a similar way. □

2. Observability of the system

We are going to apply the results of §1; we estimate the filtering error in §2.1. In §2.2, we study the extended Kalman filter; we give conditions under which it has a good performance but we also give in §2.3 two counterexamples showing that it can be quite far from the optimal filter.

§2.1 Upper bound for the filtering error

Results concerning upper bounds for the filtering error have been obtained in several papers (see [3], [9], [4]) and here, we want to obtain asymptotic results for (0.3); more precisely, we want to find conditions on the coefficients of our filtering problem that ensure that $X_t - \widehat{X}_t$ is of order $\sqrt{\varepsilon}$. We will first consider the simplest case: h almost linear and strongly injective (theorem 2.1.1). Then we will study the general case; we will not suppose that h is injective but will see how the detectability and the observability of dynamical systems can be applied to our non linear stochastic system. More precisely we will prove two results; in the first one we will assume a uniform detectability condition (theorem 2.1.2); in the second one we will only assume a local detectability condition but will also suppose that the initial error is small and that the time interval is not too long so that large deviations from the linearized system are rare enough (theorem 2.1.3). The results of this subsection are linked with the problem of finding observers for the case $\varepsilon = 0$ (see [1] for a construction of observers based on filters with small noise). As usually, the estimation of $X_t - \widehat{X}_t$ is obtained by considering some Kalman-like filter M_t and by estimating $X_t - M_t$; in further sections, we will limit ourselves to the study of the extended Kalman filter and will not use the results of this subsection; however, it is also important to study more general Kalman-like filters; it will indeed appear that the extended Kalman filter has not always a good behaviour. Note also that if we do not want to prove that $X_t - \widehat{X}_t$ is of order $\sqrt{\varepsilon}$, but only that it is small, then more general conditions can be found: see [27] for a formal study of such a situation.

Theorem 2.1.1. *Assume that*
(i) the variable $X_0 - m_0$ *is of order* v_ε *in* $L^{\infty -}$, *for some families* $m_0 \in \mathbb{R}^n$ *and* $v_\varepsilon \geq 0$;
(ii) there exists a real constant number Γ_β *such that*

$$\big(\beta(t,x) - \beta(t,m)\big)^\star (x-m) \leq \Gamma_\beta |x-m|^2; \tag{2.1.1}$$

(iii) the processes $\sigma(t,X_t)$ *and* $\gamma(t,X_t)$ *are bounded in* $L^{\infty -}$;
(iv) the function h *is strongly injective, uniformly Lipschitz and almost linear.*
Then $X_t - \widehat{X}_t$ *is of order* $\sqrt{\varepsilon}$ *in* $L^{\infty -}$ *on the time interval* $[t_0, \infty)$ *for any fixed* $t_0 > 0$; *if* $v_\varepsilon = O(\sqrt{\varepsilon})$, *one can take* $t_0 = 0$. *More precisely, one has*

$$\left\|X_t - \widehat{X}_t\right\|_q^2 \leq C_q \varepsilon \left(1 - \left(1 - c_0 \frac{\varepsilon}{v_\varepsilon^2}\right)^+ e^{-c_1 t}\right)^{-1} \tag{2.1.2}$$

for some $c_0,\ c_1 > 0$.

Remark 1. The variable X_t is not necessarily integrable (one should add an assumption on $\beta(t,0)$). However, there exist observable variables M_t such that $X_t - M_t$ is integrable (one can take the process M_t used below in the proof), so that the conditional mean of $|X_t|$ given $\mathcal{Y}_t$ is almost surely finite; thus $\widehat{X}_t$ is well defined.

Remark 2. It follows from (2.1.2) that $X_t - \widehat{X}_t$ is small as soon as $t \gg \varepsilon$; thus the initial layer is very short (as in the linear case) and theorem 2.1.1 is an improvement of previous results (compare with [23] after a time change).

Remark 3. If h is not almost linear, we can try to change the signal process so that $h(t,X_t)$ becomes an almost linear function of the new signal; if $\phi(t,x)$ is a $C^{1,2}$ function such that $x = \psi(t, \phi(t,x))$ for some function ψ, then $\overline{X}_t = \phi(t,X_t)$ and Y_t satisfy a system of type (0.3); if theorem 2.1.1 can be applied to this new system and if ψ is Lipschitz with respect to x, we can again conlude that $X_t - \widehat{X}_t$ is of order $\sqrt{\varepsilon}$. For instance, if h itself is $C^{1,2}$, we can try to choose $\phi = h$; in this case the observation function of the new system is $x \mapsto x$. However this procedure cannot be always applied; for the other cases, see theorems 2.1.2 and 2.1.3.

Proof. Let h'_t be an observable almost derivative of h. The suboptimal filter that we are going to use is the Kalman-like filter with initial condition $M_0 = m_0$ and gain G_t solution of

$$\begin{cases} G_t = k_t\big(h_t'^{*}h_t'\big)^{-1}h_t'^{*}, \\ \dot{k}_t = -k_t^2 + 2\Gamma_\beta k_t + 1, \\ k_0 = \dfrac{v_\varepsilon^2}{\varepsilon} \vee \Big(-\Gamma_\beta + \sqrt{\Gamma_\beta^2+1}\Big). \end{cases} \tag{2.1.3}$$

Since h is Lipschitz and β satisfies (2.1.1), the equation (1.1.1) with this gain has a unique solution (apply [10]). We have

$$\begin{aligned} X_t - M_t =& X_0 - M_0 + \int_0^t \big(\beta(s,X_s) - \beta(s,M_s)\big)ds - \int_0^t G_s\big(h(s,X_s) - h(s,M_s)\big)ds \\ &+ \sqrt{\varepsilon}\int_0^t \sigma(s,X_s)dW_s + \sqrt{\varepsilon}\int_0^t \big(\gamma(s,X_s) - G_s\big)dB_s. \end{aligned} \tag{2.1.4}$$

By considering

$$Z_t = X_t - M_t, \qquad w_t = \begin{pmatrix} W_t \\ B_t \end{pmatrix}, \qquad Q_t = \varepsilon k_t I, \tag{2.1.5}$$

we can write (2.1.4) in the form (1.3.9) with

$$A_t = \big(\Gamma_\beta - k_t\big)I, \tag{2.1.6}$$

$$f_t = \beta(t,X_t) - \beta(t,M_t) - \Gamma_\beta(X_t - M_t) - k_t\big(h_t'^{*}h_t'\big)^{-1}h_t'^{*}\big(h(t,X_t) - h(t,M_t) - h_t'(X_t - M_t)\big), \tag{2.1.7}$$

$$g_t = \sqrt{\varepsilon}\big(\sigma(t,X_t) \qquad \gamma(t,X_t) - G_t\big) \tag{2.1.8}$$

and it is easily verified that A_t is (Q_t, k_t) stable. We want to verify (1.3.10). We define

$$\Gamma_+ = -\Gamma_\beta + \sqrt{\Gamma_\beta^2+1}, \qquad \Gamma_- = \Gamma_\beta + \sqrt{\Gamma_\beta^2+1}, \tag{2.1.9}$$

so that Γ_+ and $-\Gamma_-$ are the roots of the Riccati equation (2.1.3); since $k_0 \geq \Gamma_+$, we have $k_t \geq \Gamma_+$ for any t. On the other hand, one gets from (2.1.1) and the almost linearity of h that

$$Z_t^{*}Q_t^{-1}f_t \leq C\frac{\mu_\varepsilon}{\varepsilon}|Z_t|^2 \leq C\mu_\varepsilon k_t Z_t^{*}Q_t^{-1}Z_t \tag{2.1.10}$$

for $\mu_\varepsilon \to 0$, so the first part of (1.3.10) is checked; moreover $g_t^{*}Q_t^{-1}g_t$ is of order $k_t + k_t^{-1}$, so since $k_t \geq \Gamma_+$, it is also of order k_t. Thus we can apply lemma 1.3.2 and obtain that

$|Z_t|^2$ is of order εk_t. In order to estimate k_t, we solve the Riccati equation and obtain

$$\begin{aligned} k_t &= (\Gamma_- + \Gamma_+)\Big(1 - \Big(1 - \frac{\Gamma_+ + \Gamma_-}{k_0 + \Gamma_-}\Big)e^{-(\Gamma_- + \Gamma_+)t}\Big)^{-1} - \Gamma_- \\ &\le (\Gamma_- + \Gamma_+)\Big(1 - \Big(1 - \frac{\Gamma_+}{k_0}\Big)e^{-(\Gamma_- + \Gamma_+)t}\Big)^{-1}. \end{aligned} \tag{2.1.11}$$

Thus we have proved (2.1.2) with M_t in place of $\widehat{X}_t$ and $c_0 = \Gamma_+$, $c_1 = \Gamma_+ + \Gamma_-$. Finally

$$\|X_t - \widehat{X}_t\|_q \le \|X_t - M_t\|_q + \|\widehat{X}_t - M_t\|_q \le 2\|X_t - M_t\|_q \tag{2.1.12}$$

since the conditional expectation (applied here to $X_t - M_t$) is a contraction in L^q. □

In the linear filtering theory, it is well known that the filtering error may remain bounded as the time goes to infinity even if the rank of the observation matrix is less than the dimension of the signal: an assumption on the detectability of (β', h') is sufficient. We are going to prove that a similar assumption also implies an estimate on the error in our framework; since we do not assume here that the system is almost linear, we need a uniform detectability condition (see theorem 2.1.3 for a local condition).

Theorem 2.1.2. *Assume that*
(i) the processes $\sigma(t, X_t)$ and $\gamma(t, X_t)$ are bounded in $L^{\infty-}$;
(ii) the functions β and h are C^1 with bounded derivatives;
(iii) there exists a bounded observable process G_t with values in $\mathbb{R}^n \otimes \mathbb{R}^d$ such that for any family of $\mathcal{F}_t$ adapted processes ξ_t, the process

$$A_t = \beta'(t, \xi_t) - G_t h'(t, \xi_t) \tag{2.1.13}$$

is exponentially stable; more precisely it satisfies (1.1.7) for some constant $c > 0$.
Fix some $c_0 < c$. Then if X_0 is integrable (so that $\widehat{X}_0$ exists),

$$\left\|X_t - \widehat{X}_t\right\|_q \le C_q\sqrt{\varepsilon} + C_q\left\|X_0 - \widehat{X}_0\right\|_q e^{-c_0 t}. \tag{2.1.14}$$

Remark 1. The initial layer is longer than in the injective case. This cannot be avoided: consider the extreme case $h = 0$ and $\beta'(t, \xi_t)$ exponentially stable.

Remark 2. With reference to the theory of linear systems, assumption (iii) can be viewed as a detectability assumption; we can also say that the system is observable if c (and therefore c_0) can be chosen arbitrarily large.

Proof. Let M_t be the Kalman-like filter with gain G_t and initial condition $M_0 = \widehat{X}_0$. There exists a $\mathcal{F}_t$ adapted process ξ_t such that

$$\beta(t, X_t) - \beta(t, M_t) - G_t\big(h(t, X_t) - h(t, M_t)\big) = A_t(X_t - M_t) \tag{2.1.15}$$

with A_t defined by (2.1.13). By defining Z_t and w_t as in (2.1.5) we are in the framework of lemma 1.3.2 with $f_t = 0$ (so that $\alpha = 0$) and g_t as in (2.1.8). For any $k < 2c$, we know from lemma 1.3.1 that A_t is $(\varepsilon Q_t, k)$ stable for some bounded and elliptic Q_t. Then lemma 1.3.2 implies the estimation on $Z = X - M$ and therefore the theorem. □

We now explain what can be said when the detectability condition is only satisfied locally at some Kalman-like filter.

Theorem 2.1.3. *Assume that*
(i) the process X_t is of order ε^{α_1} for some fixed real number α_1;
(ii) the functions σ and γ are uniformly bounded;
(iii) the functions β and h are C^1 with respect to x, and their derivatives are bounded and uniformly continuous, that is

$$\big|\beta'(t,x) - \beta'(t,m)\big| + \big|h'(t,x) - h'(t,m)\big| \le \rho\big(|x - m|\big) \tag{2.1.16}$$

for some fixed function ρ converging to 0 at 0;
(iv) there exists a Kalman-like filter M_t with a bounded gain G_t such that

$$\mathbb{E} \exp \frac{c_0}{\varepsilon} |X_0 - M_0|^2 \le C \tag{2.1.17}$$

for some positive c_0 and C, and

$$A_t = \beta'(t, M_t) - G_t h'(t, M_t) \tag{2.1.18}$$

is exponentially stable.
Then $X_t - \widehat{X}_t$ is of order $\sqrt{\varepsilon}$ on the time interval $[0, \exp(c/\varepsilon)]$ for some $c > 0$.

Remark. If $h'^* h'$ is uniformly elliptic, then it is easy to find gains which make A_t exponentially stable. Note however that $X_t - \widehat{X}_t$ cannot in general be estimated on the whole time interval $[0, \infty)$; for instance when $h(x) = (\cos x, \sin x)$, we generally know with a good precision X_t modulo 2π, but even if X_0 is well known, large deviations phenomena can cause incertitude on the value of X_t after a long time.

Proof. As in the proof of previous theorem, we are in the framework of lemma 1.3.2 with A_t given by (2.1.18), g_t defined in (2.1.8) and f_t defined by

$$f_t = \beta(t,X_t)-\beta(t,M_t)-\beta'(t,M_t)(X_t-M_t)-G_t\big(h(t,X_t)-h(t,M_t)-h'(t,M_t)(X_t-M_t)\big). \tag{2.1.19}$$

However, f_t cannot be estimated by (1.3.10), except when the oscillations of β' and h' are small enough; in the general case, we can only deduce from the uniform continuity of β' and h' that for any c_1, there exists C_1 such that

$$|X_t - M_t| \leq C_1 \quad \Longrightarrow \quad |f_t| \leq c_1 |X_t - M_t|. \tag{2.1.20}$$

Thus, if one considers the stopping time

$$\tau = \inf\Big\{t \geq 0; \quad |X_t - M_t| \geq C_1\Big\} \tag{2.1.21}$$

and the solution of

$$d\overline{Z}_t = A_t \overline{Z}_t dt + f_t 1_{\{t\leq\tau\}} dt + g_t dw_t, \qquad \overline{Z}_0 = X_0 - M_0, \tag{2.1.22}$$

one can apply lemma 1.3.2 to $\overline{Z}_t$ provided c_1 be chosen small enough. Thus $(X_t - M_t)1_{\{t\leq\tau\}}$ is of order $\sqrt{\varepsilon}$. Moreover, an application of lemma 1.3.3 shows that an exponential moment of $|\overline{Z}_t|^2/\varepsilon$ is bounded, so from lemma 1.2.2,

$$\mathbb{P}\Big[\exists t \leq e^{c/\varepsilon}, \qquad |\overline{Z}_t| \geq C_1\Big] = O(e^{-c_2/\varepsilon}) \tag{2.1.23}$$

for some c and c_2, or equivalently

$$\mathbb{P}\big[\tau \leq e^{c/\varepsilon}\big] = O(e^{-c_2/\varepsilon}). \tag{2.1.24}$$

Let $\overline{M}_t$ be equal to M_t if $|M_t|$ is less than ε^{α_2} for some fixed $\alpha_2 < \alpha_1 \wedge 1/2$, to 0 otherwise. On the event $\{t \leq \tau\}$, the processes $M_t - X_t$ and X_t are respectively of order $\sqrt{\varepsilon}$ and ε^{α_1} which are both negligible with respect to ε^{α_2}; by adding these two estimates, we deduce that

$$\mathbb{P}\big[\overline{M}_t \neq M_t,\, t \leq \tau\big] = O(\varepsilon^{\alpha}) \tag{2.1.25}$$

for any α. Then write

$$X_t - \overline{M}_t = (X_t - M_t)1_{\{t\leq\tau\}}1_{\{M_t=\overline{M}_t\}} + (X_t - \overline{M}_t)1_{\{t\leq\tau\}}1_{\{M_t\neq\overline{M}_t\}} + (X_t - \overline{M}_t)1_{\{t>\tau\}}. \tag{2.1.26}$$

The first term is of order $\sqrt{\varepsilon}$ and for $t \leq e^{c/\varepsilon}$, the two other ones are very small from (2.1.24) and (2.1.25). Thus $X_t - \overline{M}_t$ is of order $\sqrt{\varepsilon}$. Since $\overline{M}_t$ is observable, we can conclude. □

If one writes the semimartingale decomposition of $\widehat{X}_t$ (see [16] or [13]) and if one applies lemma 1.2.1, one immediately deduces the

Corollary 2.1.4. *Assume the conditions of theorems 2.1.1, 2.1.2 or 2.1.3 with β and h Lipschitz, σ and γ bounded, and $X_0 - M_0 = O(\sqrt{\varepsilon})$. Then for any real α_1 and any $\alpha_0 < 1/2$, the supremum of $|X_t - \widehat{X}_t|$ on the time interval $[0, \varepsilon^{\alpha_1}]$ is of order ε^{α_0} in $L^{\infty-}$.*

§2.2 The extended Kalman filter

We now study the extended Kalman filter which was introduced in §1.1; the point that we want to consider is to know whether $P_t^{-1/2}(X_t - M_t)$ is of order $\sqrt{\varepsilon}$; in the linear case and with the optimal filter, this variable has indeed a Gaussian law with mean 0 and covariance εI, and if the extended Kalman filter is efficient, it is reasonable to think that this property also holds approximately in the non linear case. The Gaussian structure of this variable will be studied more precisely in §3, and we now estimate its order.

As in §2.1, we are going to apply lemma 1.3.2; the equation for $X_t - M_t$ has indeed the form (1.3.9) with

$$A_t = (\beta' - \gamma h')(t, M_t) - P_t h'^{\star} h'(t, M_t), \tag{2.2.1}$$

$$\begin{aligned} f_t = &\beta(t, X_t) - \beta(t, M_t) - \beta'(t, M_t)(X_t - M_t) \\ &- \big(\gamma(t, M_t) + P_t h'^{\star}(t, M_t)\big)\big(h(t, X_t) - h(t, M_t) - h'(t, M_t)(X_t - M_t)\big) \end{aligned} \tag{2.2.2}$$

and

$$g_t = \sqrt{\varepsilon}\big(\sigma(t, X_t) \qquad \gamma(t, X_t) - \gamma(t, M_t) - P_t h'^{\star}(t, M_t)\big). \tag{2.2.3}$$

Moreover, for any function k_t, saying that A_t is (P_t, k_t) stable is equivalent to

$$P_t h'^{\star} h'(t, M_t) P_t + a(t, M_t) \geq k_t P_t \tag{2.2.4}$$

where we recall the notation $a = \sigma\sigma^{\star}$. From these equations, we get conditions under which we can apply lemma 1.3.2. We first consider the almost linear case.

Theorem 2.2.1. *Let (M_t, P_t) be an extended Kalman filter such that*
(i) the variable $P_0^{-1/2}(X_0 - M_0)$ is of order $\sqrt{\varepsilon}$;
(ii) the functions σ and γ are bounded;
(iii) the functions β and h are differentiable, almost linear and $\beta'(t, M_t)$, $h'(t, M_t)$ are bounded almost derivatives;
(iv) the inequality (2.2.4) holds for some family of deterministic functions k_t such that

$$P_t + P_t^{-1} \leq C k_t I. \tag{2.2.5}$$

Then $P_t^{-1/2}(X_t - M_t)$ is of order $\sqrt{\varepsilon}$. Moreover, if

$$\mathbb{E}\exp\frac{c_0}{\varepsilon}(X_0 - M_0)^\star P_0^{-1}(X_0 - M_0) \le C_0 \tag{2.2.6}$$

for some positive c_0, C_0, then

$$\mathbb{E}\exp\frac{c}{\varepsilon}(X_t - M_t)^\star P_t^{-1}(X_t - M_t) \le C \tag{2.2.7}$$

and

$$\mathbb{E}\exp\frac{c}{\varepsilon}\int_s^t \left|X_u - M_u\right|^2 du \le C\exp C\int_s^t k_u du \tag{2.2.8}$$

for any $s \le t$ and some c, C.

Proof. The two results are respectively corollaries of lemmas 1.3.2 and 1.3.3 applied with $Q_t = \varepsilon P_t$; in order to verify the first part of (1.3.10), note that since β and h are almost linear, there exists a family $\mu_\varepsilon \to 0$ such that

$$\begin{aligned} f_t^\star P_t^{-1} f_t &\le \mu_\varepsilon\big(1 + |P_t| + |P_t^{-1}|\big)|X_t - M_t|^2 \\ &\le C\mu_\varepsilon k_t |X_t - M_t|^2 \\ &\le C\mu_\varepsilon k_t^2 (X_t - M_t)^\star P_t^{-1}(X_t - M_t), \end{aligned} \tag{2.2.9}$$

so that (1.3.12) is satisfied for some $\alpha_\varepsilon \to 0$. For (2.2.8), in order to apply (1.3.20), note that (2.2.5) implies that $k_t P_t^{-1}$ is uniformly elliptic. □

We now describe a situation for which we can check condition (iv); this situation is the most pleasant one: it enables the conclusion for any initial condition X_0 provided P_0 be chosen large enough. Actually, one can also find more precise sufficient conditions if one has some other information about the boundedness or the ellipticity of P_t.

Corollary 2.2.2. *Let (M_t, P_t) be an extended Kalman filter satisfying conditions (i) to (iii) of theorem 2.2.1. Suppose also that h is strongly injective, a is uniformly elliptic and that the quotient between the largest and smallest eigenvalues of P_0 is bounded. Then $P_t^{-1/2}(X_t - M_t)$ is of order $\sqrt{\varepsilon}$.*

Proof. One has

$$P_t h'^\star h'(t, M_t)P_t + a(t, M_t) \ge c(P_t^2 + I) \ge c(P_t + P_t^{-1})P_t. \tag{2.2.10}$$

If there exists a family of deterministic positive functions p_t such that $p_t^{-1}P_t$ is uniformly bounded and elliptic, then the condition (iv) of theorem 2.2.1 will be satisfied with k_t proportional to $p_t + p_t^{-1}$; thus we only have to prove the existence of such a family. Let p_0 be the trace of P_0; since β', a are bounded, $h'^{*}h'$ is elliptic, by means of (1.3.7) one can find positive numbers c_1, c_2 such that the trace of P_t is less than the solution of

$$\dot{p}_t = -c_1 p_t^2 + c_2 \tag{2.2.11}$$

with initial condition p_0. Similarly, if p_0' is the trace of P_0^{-1}, writing the equation for P_t^{-1}, since h' is bounded, a is elliptic, one can find c_1', c_2' such that the trace of P_t^{-1} is less than the solution of

$$\dot{p}_t' = -c_1' {p_t'}^2 + c_2'. \tag{2.2.12}$$

Then P_t is between ${p_t'}^{-1}I$ and $p_t I$, so we only have to prove that ${p_t'}^{-1} \geq c p_t$, or equivalently, that $p_t p_t'$ is bounded. We have

$$\begin{aligned} \frac{d}{dt}(p_t p_t') &= -(c_1 p_t + c_1' p_t') p_t p_t' + c_2' p_t + c_2 p_t' \\ &\leq -c_3 (p_t + p_t') p_t p_t' + c_4 (p_t + p_t') \\ &\leq c_3 (p_t + p_t')(c_5 - p_t p_t') \end{aligned} \tag{2.2.13}$$

for some c_3, c_4, c_5; thus the derivative of $p_t p_t'$ is negative as soon as $p_t p_t'$ is greater than c_5; moreover from our assumption about the eigenvalues of P_0, $p_0 p_0'$ is bounded; thus $p_t p_t'$ is bounded. □

We can also deduce from lemma 1.2.1 the

Corollary 2.2.3. *Under the assumptions of theorem 2.2.1, if k_t is of order ε^{α_0} for some real α_0, then for any fixed $\alpha \leq 0$ and any $\delta > 0$, the supremum of $P_t^{-1/2}(X_t - M_t)$ over the time interval $[0, \varepsilon^{\alpha}]$ is of order $\varepsilon^{1/2-\delta}$.*

When the functions β and h are not almost linear, we can deduce from the proofs of theorems 2.1.2 and 2.1.3 the

Theorem 2.2.4. *Let M_t be an extended Kalman filter with gain G_t given by (1.1.2).*
(a) If the conditions of theorem 2.1.2 are satisfied for the gain G_t, then

$$\|X_t - M_t\|_q \leq C_q \sqrt{\varepsilon} + C_q \|X_0 - M_0\|_q e^{-c_0 t}. \tag{2.2.14}$$

(b) If the conditions of theorem 2.1.3 are satisfied for M_t, then there exists a family of measurable sets N such that

$$\mathbb{P}[N] \leq C \exp -\frac{c}{\varepsilon} \tag{2.2.15}$$

and $P_t^{-1/2}(X_t - M_t)1_{N^c}$ is of order $\sqrt{\varepsilon}$ on the time interval $[0, e^{c/\varepsilon}]$.

In case (b), when P_t is uniformly bounded and elliptic, a sufficient condition for the exponential stability of A_t is (2.2.4) for $k_t = k > 0$; however this condition is not necessary; for instance there may be bounded time intervals where $a = h = 0$; note also that we can deduce that $X_t - M_t = O(\sqrt{\varepsilon})$ on $[0, e^{c/\varepsilon}]$ if we have a priori estimates on the moments of M_t. Properties similar to (2.2.7) and (2.2.8) can also sometimes be checked; in case (a), when some exponential moment of $\varepsilon^{-1}|X_0 - M_0|^2$ is bounded, then

$$\mathbb{E} \exp \frac{c}{\varepsilon} \int_s^t |X_u - M_u|^2 du \leq C e^{C(t-s)} \tag{2.2.16}$$

for some c and C.

§*2.3 Two counterexamples*

In theorem 2.2.4(b), large deviations phenomena prevent us from deriving estimates for very large times; moreover when X_0 is badly known at time 0 (P_0 not bounded), the extended Kalman filter may provide wrong values for a nonlinear function h even if it is strongly injective. We now explain why on two examples corresponding respectively to non linear β and h. In these two cases, the fact is that the extended Kalman filter may have more than one stable equilibrium.

Example 1. Suppose that X_t and Y_t are one-dimensional and solution of

$$\begin{cases} dX_t = \big(2 \arctan X_t - X_t\big) dt + \sqrt{\varepsilon} dW_t, \\ dY_t = H X_t dt + \sqrt{\varepsilon} dB_t \end{cases} \tag{2.3.1}$$

for some positive number H. The deterministic dynamical system associated with X_t (obtained for $\varepsilon = 0$) has two stable equilibrium points $x_0 > 0$ and $-x_0$, and 0 is an unstable equilibrium; we suppose that $X_0 = x_0$. We consider the extended Kalman filter (M_t, P_t) with initial value $(M_0, P_0) = (x_0, p_0)$ for some bounded family p_0; we can apply theorem 2.2.4(b) to this system but we are going to see that for some values of H, $X_t - M_t$ does not converge to 0 as $\varepsilon \to 0$ uniformly with respect to $t \in [0, \infty)$. This can be done by applying some results of the theory of large deviations (see for instance [7], we will not give all the details); first, we can check that

$$\alpha_0 = \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E} \inf\{t > 0;\ X_t < 0\} \tag{2.3.2}$$

exists and is finite. We fix some $\alpha_1 > \alpha_0$ and we are going to study the behaviour of the system at time $t_1 = \exp(\alpha_1/\varepsilon)$. The probability that X_t becomes negative before time t_1 tends to 1, so using the symmetry of the equation of X and the strong Markov property,

$$\lim_{\varepsilon \to 0} \mathbb{P}\big[X_{t_1} < 0\big] = 1/2. \tag{2.3.3}$$

Moreover, one can prove from the theory of large deviations that if K is large enough,

$$\lim_{\varepsilon \to 0} \mathbb{P}\Big[\sup_{t \le t_1} |X_t| \le K\Big] = 1. \tag{2.3.4}$$

If Ω_1 denotes the set inside the bracket, we are going to study on the time interval $[0, t_1]$ the process (M_t, P_t) conditioned on Ω_1. The process P_t is solution of

$$\dot{P}_t = -H^2 P_t^2 + \Big(\frac{4}{1 + M_t^2} - 2\Big) P_t + 1 \tag{2.3.5}$$

so if we choose δ small enough so that

$$\beta_1 = \frac{2}{1 + (x_0 - \delta)^2} - 1 < 0 \tag{2.3.6}$$

and if we put

$$\tau = \inf\{t > 0; \quad M_t < x_0 - \delta\}, \tag{2.3.7}$$

then

$$P_t \le H^{-2}\big(\beta_1 + \sqrt{\beta_1^2 + H^2}\big) \vee p_0 < C \tag{2.3.8}$$

on $\{t \le \tau\}$, where the constant C does not depend on H. On the other hand, the process M_t satisfies

$$dM_t = \big(2 \arctan M_t - M_t\big)\, dt + P_t H^2 (X_t - M_t) dt + \sqrt{\varepsilon} P_t H \, dB_t. \tag{2.3.9}$$

Consider the deterministic equation obtained from (2.3.9) by putting $\varepsilon = 0$ and by fixing X_t at some trajectory bounded by K; if H is small enough, the flow of this equation sends the interval $[x_0-\delta,\infty)$ into itself; now in (2.3.9) we have added a noise of order $\sqrt{\varepsilon}H$ so that by using upper estimates for large deviations, one can prove that for some $c>0$,

$$\lim_{\varepsilon H^2\to 0}\mathbb{P}\Big[\tau\le\exp\frac{c}{\varepsilon H^2}\;\Big|\;X\Big]=0 \tag{2.3.10}$$

uniformly on Ω_1. Thus by choosing again H small enough, one can manage so that

$$\lim_{\varepsilon\to 0}\mathbb{P}\big[\tau\le t_1\;\big|\;\Omega_1\big]=0. \tag{2.3.11}$$

We deduce from (2.3.4) that, with such an H, one has $M_{t_1}>x_0-\delta$ with a probability converging to 1; on the other hand, one has $X_{t_1}<0$ with a probability converging to 1/2, so $X_{t_1}-M_{t_1}$ does not converge to 0 in probability. This example shows that the extended Kalman filter cannot be used in order to detect phase transitions on the signal; this is due to the fact that the gain is not large enough: in this situation, the filter of theorem 2.1.1 has a better behaviour.

Example 2. We suppose here that X_t is one-dimensional and Gaussian: it satisfies $dX_t=\sqrt{\varepsilon}dW_t$ and its initial law is the Gaussian law with mean -2 and variance 1; we suppose that the observation is two-dimensional and that the observation function is a smooth non linear function such that $h^1(x)=x$ for any x and $h^2(x)=2|x|$ for $|x|>1$. Consider the extended Kalman filter with $M_0=-2$ and $P_0=1/\varepsilon$ (which are the mean and normalized covariance of X_0); we are going to estimate $X-M$ at some fixed time $t_0>0$. Until the first time at which $X_t<1$ or $M_t>-1$, the filter satisfies

$$\begin{cases} dM_t = P_t(-3X_t-5M_t)dt+\sqrt{\varepsilon}P_t(dB_t^1-2dB_t^2), \\ \dot{P}_t = -5P_t^2+1. \end{cases} \tag{2.3.12}$$

Let $(\overline{M}_t,\overline{P}_t)$ be the solution of this equation for any t, with $(\overline{M}_0,\overline{P}_0)=(M_0,P_0)$; note in particular that

$$\varepsilon\int_0^t\overline{P}_s^2ds=\frac{\varepsilon}{5}(t+P_0-\overline{P}_t)\le\frac{\varepsilon t}{5}+\frac{1}{5}. \tag{2.3.13}$$

We have

$$3X_t+5\overline{M}_t=3X_0-10-5\int_0^t\overline{P}_s(3X_s+5\overline{M}_s)ds+\sqrt{\varepsilon}\int_0^t\sqrt{125\overline{P}_s^2+9}\,d\widetilde{W}_s \tag{2.3.14}$$

for some standard Brownian motion $\widetilde{W}_t$, from which we deduce

$$\mathbb{P}\Big[\sup_{t\le t_0}|3X_t+5\overline{M}_t|\le 1\;\Big|\;|3X_0-10|\le 1/2\Big]\ge\mathbb{P}\Big[\sup_{t\le t_0}\Big|\int_0^t\sqrt{\varepsilon(125\overline{P}_s^2+9)}\,d\widetilde{W}_s\Big|\le 1/2\Big]. \tag{2.3.15}$$

The stochastic integral inside the right-hand side is a Gaussian martingale, the variance of which is bounded from (2.3.13); by writing this martingale as a time-changed Brownian motion, we deduce a lower bound for the right-hand side and therefore

$$\mathbb{P}\Big[\sup_{t\le t_0}|3X_t+5\overline{M}_t|\le 1\ \Big|\ |3X_0-10|\le 1/2\Big]\ge c_0 \tag{2.3.16}$$

for some positive c_0. We also have

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big[\inf_{t\le t_0}X_t\ge 2\ \Big|\ |3X_0-10|\le 1/2\Big]=1. \tag{2.3.17}$$

Thus

$$\liminf_{\varepsilon\to 0}\mathbb{P}\Big[\inf_{t\le t_0}X_t\ge 1,\ \sup_{t\le t_0}M_t\le -1\ \Big|\ |3X_0-10|\le 1/2\Big]>0 \tag{2.3.18}$$

since the event in (2.3.18) contains the intersection of the events in (2.3.16) and (2.3.17). Moreover the probability for $|3X_0-10|$ to be less than $1/2$ is some constant positive number, so we deduce that $X_{t_0}-M_{t_0}$ does not converge in probability to 0; since P_{t_0} is bounded as $\varepsilon\to 0$, the variable $P_{t_0}^{-1/2}(X_{t_0}-M_{t_0})$ is not of order $\sqrt{\varepsilon}$, though this property holds at time 0; this is due to the too strong nonlinearity of the observation function and to the unboundedness of P_t (which prevents us from applying theorem 2.2.4). This example shows that one has to be cautious in applying the extended Kalman filter when the initial condition is badly known; of course, in this example, if one drops the second component of the observation, the problem becomes linear and is still nearly observable.

3. Asymptotic filtering

In §2, we have proved that under some conditions, X_t-M_t is of order $\sqrt{\varepsilon}$ when M_t is the extended Kalman filter. In this section, we prove that $\widehat{X}_t-M_t$ is of order ε; the result is stated in §3.1; the different steps of the proof are detailed in §§3.2 to 3.4. In §3.5, we prove that the conditional law of X_t given $\mathcal{Y}_t$ is asymptotically Gaussian.

§3.1 The main result

We now state the main result which will be proved subsequently in several steps.

Theorem 3.1.1. *Let (M_t,P_t) be an extended Kalman filter. Assume that*
(i) the functions σ and γ are C_b^1 with uniformly Lipschitz derivatives;
(ii) the functions β and h are C^1 and their derivatives are uniformly bounded and Lipschitz;

for any fixed ε and $T > 0$, the suprema over $\{t \leq T\}$ of $|\beta(t,0)|$ and $|h(t,0)|$ are variables of L^∞;

(iii) the law of X_0 has a density p_0 with respect to the Lebesgue measure, the derivative of p_0 in the distribution sense is a function p_0' and there exists a family of positive numbers v_ε such that

$$(X_0 - M_0)^* P_0^{-1} + \varepsilon \frac{p_0'}{p_0}(X_0) = O(v_\varepsilon); \tag{3.1.1}$$

for each ε fixed, an exponential moment of $|X_0|^2$ is finite;

(iv) the process P_t is uniformly elliptic; for any fixed ε and $T > 0$, the supremum over $\{t \leq T\}$ of $|P_t|$ is a variable of L^∞; moreover the process

$$A_t = -P_t^{-1} a(t, M_t) - (\beta' - \gamma h')^*(t, M_t) \tag{3.1.2}$$

is exponentially stable;

(v) the filter satisfies

$$|X_t - M_t|^2 = O\Big(\varepsilon + v_\varepsilon' e^{-c_0 t}\Big) \tag{3.1.3}$$

for a family v_ε' and for any $K > 0$ and $t \geq s$,

$$\mathbb{E} \exp K \int_s^t |X_u - M_u| du \leq C_K \exp C_K \sqrt{\varepsilon}(t-s). \tag{3.1.4}$$

Then

$$P_t^{-1}(\widehat{X}_t - M_t) = O\Big(\varepsilon + (v_\varepsilon + v_\varepsilon')e^{-ct}\Big). \tag{3.1.5}$$

In the end of this subsection, we discuss the assumptions of this theorem.

Remark on condition (iii). The left-hand side of (3.1.1) is zero as soon as the law of X_0 is the Gaussian law with mean M_0 and variance εP_0. Thus the distance (in some sense) between the law of X_0 and this Gaussian law enters the initial layer of (3.1.5). The basic use of the variable $p_0'/p_0(X_0)$ involved in (3.1.1) is the following integration by parts formula: if ϕ is a smooth function with compact support then

$$\mathbb{E}\Big[\phi'(X_0) + \frac{p_0'}{p_0}(X_0)\phi(X_0)\Big] = 0. \tag{3.1.6}$$

Actually a converse statement also holds; if

$$\mathbb{E}\big[\phi'(X_0) + \nu_0 \phi(X_0)\big] = 0 \tag{3.1.7}$$

for a variable ν_0 of $L^{\infty-}$ and any ϕ, then following the basic lemma of Malliavin's calculus, this implies that the law of X_0 is absolutely continuous with respect to the Lebesgue measure and ν_0 is $p_0'/p_0(X_0)$ (see lemma 3.2 of [26]).

Remark on condition (iv). Suppose that P_t is uniformly bounded and elliptic, and that

$$P_t h'^* h'(t, M_t) P_t + a(t, M_t) \geq cI \tag{3.1.8}$$

for some $c > 0$. Then one can check that the process A_t is (P_t^{-1}, k) stable for some $k > 0$, so that from lemma 1.3.1, it is exponentially stable and (iv) holds. If A_t is bounded, its exponential stability is also preserved when the above properties do not hold on a bounded time interval.

Remark on condition (v). We have obtained in §2.2 sufficient conditions implying (3.1.3); note that $X_0 - M_0$ is not necessarily small but its magnitude enters the initial layer of the result. In the framework of theorem 2.2.1 and if k_t is bounded for $t \geq t_0$, we can also deduce from (2.2.8) that

$$\mathbb{E} \exp \frac{c}{\varepsilon} \int_s^t |X_u - M_u|^2 du \leq C e^{C(t-s)} \tag{3.1.9}$$

for $t \geq s \geq t_0$ and some c and C (see also (2.2.16)). From Jensen's inequality, this implies that for any K,

$$\mathbb{E} \exp \frac{K}{\sqrt{\varepsilon}} \int_s^t |X_u - M_u|^2 du \leq C e^{CK\sqrt{\varepsilon}(t-s)} \tag{3.1.10}$$

for ε small enough. Then we can deduce (3.1.4) for $t \geq s \geq t_0$ by means of

$$2 \int_s^t |X_u - M_u| du \leq \sqrt{\varepsilon}(t-s) + \frac{1}{\sqrt{\varepsilon}} \int_s^t |X_u - M_u|^2 du. \tag{3.1.11}$$

If the exponential moments of X_t and M_t are bounded on $[0, t_0]$, we can then deduce (3.1.4) for $t \geq s \geq 0$.

Theorem 3.1.1 can be viewed as a generalization of previous results such as [21], [2]; in [22], [23], we did not assume the existence of a density for X_0; here we have preferred to use the initial law because it yields more precise results about the initial layer; however in §4, we will also derive some results without this assumption on X_0. As a corollary, we can also deduce from lemma 1.2.1 an estimate on the supremum of $|\widehat{X}_t - M_t|$ over some time intervals. The proof of theorem 3.1.1 will be divided into three steps. First, we will introduce an absolutely continuous change of probability; then we will deduce an expression for X_t which will involve some derivatives with respect to the initial condition; finally, an extension of the integration by parts formula (3.1.6) will yield the theorem.

§3.2 Change of probability measure

We use an absolutely continuous change of probability which affects both the signal and observation noises. A classical change is the one which leads to the famous reference probability; here, we also modify the probability of the signal in order to focus it on the interesting events. So we fix some family of deterministic terminal times T (however, the estimates will not depend on T) and we now define a new probability on $\mathcal{F}_T$. Let Λ_t be the process defined by

$$\Lambda_t^{-1} = \exp\Big\{ -\frac{1}{\sqrt{\varepsilon}}\int_0^t h^\star(s,X_s)dB_s + \frac{1}{\sqrt{\varepsilon}}\int_0^t (X_s - M_s)^\star P_s^{-1}\sigma(s,X_s)dW_s$$
$$-\frac{1}{2\varepsilon}\int_0^t \Big(\big|h(s,X_s)\big|^2 + \big|\sigma^\star(s,X_s)P_s^{-1}(X_s - M_s)\big|^2\Big)ds\Big\}. \tag{3.2.1}$$

For each ε, the process (X_t, M_t) is solution of a stochastic differential equation, the diffusion and drift coefficients of which are respectively bounded and of linear growth as $x \to \infty$; thus (see lemma 5.7.2 of [13]) some exponential moment of the supremum over $0 \leq t \leq T$ of $|X_t|^2$ and $|M_t|^2$ is finite. It follows from this remark that the local martingale defined in (3.2.1) is actually a martingale, and we can therefore define $\widetilde{\mathbb{P}}$ to be the probability with density Λ_t^{-1} with respect to $\mathbb{P}$ on $\mathcal{F}_t$. Applying the Girsanov theorem, if we define

$$\widetilde{W}_t = W_t - \frac{1}{\sqrt{\varepsilon}}\int_0^t \sigma^\star(s,X_s)P_s^{-1}(X_s - M_s)ds, \tag{3.2.2}$$

then $\widetilde{W}_t$ and $Y_t/\sqrt{\varepsilon}$ are independent $\widetilde{\mathbb{P}}$ standard Brownian motions. The equation for X_t can be written in the form

$$dX_t = a(t,X_t)P_t^{-1}(X_t - M_t)dt + \widetilde{\beta}(t,X_t)dt + \sqrt{\varepsilon}\sigma(t,X_t)d\widetilde{W}_t + \gamma(t,X_t)dY_t \tag{3.2.3}$$

with the notation $\widetilde{\beta} = \beta - \gamma h$. Note also that

$$\Lambda_t = \exp\frac{1}{\varepsilon}\Big\{\int_0^t h^\star(s,X_s)dY_s - \frac{1}{2}\int_0^t \big|h(s,X_s)\big|^2 ds$$
$$-\int_0^t (X_s - M_s)^\star P_s^{-1}\big(dX_s - \widetilde{\beta}(s,X_s)ds - \gamma(s,X_s)dY_s\big)$$
$$+\frac{1}{2}\int_0^t \big|\sigma^\star(s,X_s)P_s^{-1}(X_s - M_s)\big|^2 ds\Big\}. \tag{3.2.4}$$

We are now going to compare Λ_t with the "Gaussian-like density"

$$\exp -\frac{1}{2\varepsilon}(X_t - M_t)^\star P_t^{-1}(X_t - M_t).$$

More precisely, we have from Itô's formula

$$
\begin{aligned}
(X_t - M_t)^\star P_t^{-1}(X_t - M_t) &= (X_0 - M_0)^\star P_0^{-1}(X_0 - M_0) \\
&+ 2\int_0^t (X_s - M_s)^\star P_s^{-1} dX_s - 2\int_0^t (X_s - M_s)^\star P_s^{-1} dM_s \\
&+ \int_0^t (X_s - M_s)^\star \frac{d}{ds} P_s^{-1}(X_s - M_s) ds + \varepsilon \operatorname{trace} \int_0^t a(s, X_s) P_s^{-1} ds \\
&+ \varepsilon \int_0^t \left| P_s^{-1/2}\big(\gamma(s, X_s) - \gamma(s, M_s) - P_s h'^\star(s, M_s)\big)\right|_2^2 ds \qquad (3.2.5)
\end{aligned}
$$

with the notation $|A|_2^2 = \operatorname{trace}(AA^\star)$; from (3.2.4) and (3.2.5), we can verify that

$$
\begin{aligned}
(X_t - M_t)^\star P_t^{-1}(X_t - M_t) + 2\varepsilon \log \Lambda_t &= (X_0 - M_0)^\star P_0^{-1}(X_0 - M_0) \\
&+ 2\int_0^t (X_s - M_s)^\star P_s^{-1}\Big[\big(\beta(s, X_s) - \beta(s, M_s)\big) - \gamma(s, X_s)\big(h(s, X_s) - h(s, M_s)\big)\Big] ds \\
&+ 2\int_0^t (X_s - M_s)^\star P_s^{-1}\big(\gamma(s, X_s) - \gamma(s, M_s)\big)\big(dY_s - h(s, M_s) ds\big) \\
&+ 2\int_0^t \big(h(s, X_s) - h'(s, M_s)(X_s - M_s)\big)^\star \big(dY_s - h(s, M_s) ds\big) \\
&- \int_0^t \Big[\big|h(s, X_s) - h(s, M_s)\big|^2 - \big|h'(s, M_s)(X_s - M_s)\big|^2\Big] ds + \int_0^t |h(s, M_s)|^2 ds \\
&+ \int_0^t (X_s - M_s)^\star \Big[\frac{d}{ds} P_s^{-1} + P_s^{-1} a(s, X_s) P_s^{-1} - h'^\star h'(s, M_s)\Big](X_s - M_s) ds \\
&+ \varepsilon \int_0^t \left| P_s^{-1/2}\big(\gamma(s, X_s) - \gamma(s, M_s) - P_s h'^\star(s, M_s)\big)\right|_2^2 ds \\
&+ \varepsilon \operatorname{trace} \int_0^t a(s, X_s) P_s^{-1} ds. \qquad (3.2.6)
\end{aligned}
$$

Our aim is to estimate in some sense this expression. From the equation satisfied by P_t^{-1}, we can write it in the form

$$
\begin{aligned}
(X_t - M_t)^\star P_t^{-1}(X_t - M_t) + 2\varepsilon \log \Lambda_t &= (X_0 - M_0)^\star P_0^{-1}(X_0 - M_0) \\
&+ 2\int_0^t \psi_1^\star(s, X_s)\big(dY_s - h(s, M_s) ds\big) + \int_0^t \psi_2(s, X_s) ds + \psi_3(t) \qquad (3.2.7)
\end{aligned}
$$

where the functions ψ_i are observable, defined by

$$
\psi_1(t, x) = h(t, x) - h'(t, M_t)(x - M_t) + \big(\gamma(t, x) - \gamma(t, M_t)\big)^\star P_t^{-1}(x - M_t), \qquad (3.2.8)
$$

$$
\begin{aligned}
\psi_2(t,x) =& 2(x-M_t)^\star P_t^{-1}\big(\beta(t,x)-\beta(t,M_t)-\beta'(t,M_t)(x-M_t)\big)\\
&-2(x-M_t)^\star P_t^{-1}\big(\gamma(t,x)-\gamma(t,M_t)\big)\big(h(t,x)-h(t,M_t)\big)\\
&-2(x-M_t)^\star P_t^{-1}\gamma(t,M_t)\big(h(t,x)-h(t,M_t)-h'(t,M_t)(x-M_t)\big)\\
&+(x-M_t)^\star P_t^{-1}\big(a(t,x)-a(t,M_t)\big)P_t^{-1}(x-M_t)\\
&+\big|h(t,x)-h(t,M_t)\big|^2-\big|h'(t,M_t)(x-M_t)\big|^2\\
&+\varepsilon\Big|P_t^{-1/2}\big(\gamma(t,x)-\gamma(t,M_t)-P_t h'^\star(t,M_t)\big)\Big|_2^2\\
&+\varepsilon\,\mathrm{trace}\big(a(t,x)P_t^{-1}\big),
\end{aligned}
\tag{3.2.9}
$$

and the remaining term is put in ψ_3.

§3.3 Differentiation with respect to the initial condition

Since equation (3.2.3) has a unique strong solution, the processes X_t and Λ_t are measurable with respect to $(X_0, \widetilde{W}, Y)$, so that they can be viewed as functionals defined on the canonical space $\mathbb{R}^n \times C_T^p \times C_T^d$, where C_T^k is the space of continuous functions from $[0,\infty)$ into $\mathbb{R}^k$; more precisely endow this product space with the product of the Lebesgue measure on $\mathbb{R}^n$ and the standard Wiener measures on C_T^p and C_T^d, and consider the solution Φ_t of (3.2.3) with $(X_0, \widetilde{W}, Y)$ replaced by the canonical process of $\mathbb{R}^n \times C_T^p \times C_T^d$; then, since the law of X_0 is absolutely continuous with respect to the Lebesgue measure, X_t is almost surely equal to the composition $\Phi_t \circ (X_0, \widetilde{W}, Y)$. We adopt the following notational convention: the same letter will be used to denote a functional Φ on $\mathbb{R}^n \times C_T^p \times C_T^d$ and the variable $\Phi(X_0, \widetilde{W}, Y)$ on Ω; note however that since the Lebesgue measure is not necessarily absolutely continuous with respect to the law of X_0, the extension of a variable on Ω to a variable on $\mathbb{R}^n \times C_T^p \times C_T^d$ is not necessarily unique; nevertheless the variables which will be considered will be given by stochastic differential equations, so that one can apply the procedure that we have described for X_t and obtain a canonical extension. Since only X_0 will be perturbed in this section, we will sometimes simply write $\Phi(X_0)$ and omit the dependence on $(\widetilde{W}, Y)$. For any vector μ in $\mathbb{R}^n$, the law of $(X_0+\mu, \widetilde{W}, Y)$ is also absolutely continuous with respect to the product of Lebesgue and Wiener measures so if Φ is a functional defined almost everywhere, the notation $\Phi(X_0+\mu, \widetilde{W}, Y)$ defines a unique variable up to almost sure equality.

Definition 3.3.1. *If μ is a unit vector, a measurable variable $\Phi(X_0, \widetilde{W}, Y)$ will be said to be continuous in probability in the direction μ with respect to the initial condition if*

$\Phi(X_0+\delta\mu)$ *converges in probability to* $\Phi(X_0)$ *as* $\delta \to 0$*; it will be said to be differentiable in probability in the direction* μ *with derivative* $\nabla_0^\mu \Phi$ *if*

$$\nabla_0^\mu \Phi = \lim_{\delta\to 0} \frac{1}{\delta}\big(\Phi(X_0+\delta\mu, \widetilde{W}, Y) - \Phi(X_0, \widetilde{W}, Y)\big), \tag{3.3.1}$$

where the limit holds in $\mathbb{P}$ *probability; it will be said to be differentiable with derivative* $\nabla_0\Phi$ *if it is differentiable in all the directions and* $\nabla_0^\mu\Phi = \nabla_0\Phi\,\mu$*. We will also say that the functional* Φ *is continuous or differentiable in* L^q *when* $\Phi(X_0+\delta\mu)$ *is in* L^q *for* δ *small enough and the limits hold in* L^q*.*

The theory of stochastic flows uses a stronger definition of differentiability, since one assumes the existence of a version $\Phi(x, \widetilde{W}, Y)$ differentiable with respect to x; actually our results can also be proved by using flows and the technique of [5]. However definition 3.3.1 will appear to be sufficient for our purpose and is closer to the derivability of the stochastic calculus of variations of [28]. Since $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are mutually absolutely continuous, the limits in probability coincide for these two probabilities; but for L^q, one has to make precise the probability; except otherwise stated, it will be $\mathbb{P}$. Our aim is to prove the

Lemma 3.3.2. *Let* Z_t *be the matrix-valued solution of*

$$\begin{aligned} dZ_t =& a(t,X_t)P_t^{-1}Z_t dt + \sigma(t,X_t)\big((\sigma^\star)'(t,X_t), P_t^{-1}(X_t-M_t)\big)Z_t dt + (\beta'-\gamma h')(t,X_t)Z_t dt \\ &+ \sqrt{\varepsilon}\sigma_j'(t,X_t)Z_t dW_t^j + \sqrt{\varepsilon}\gamma_j'(t,X_t)Z_t dB_t^j \end{aligned} \tag{3.3.2}$$

with $Z_0 = I$*; in this equation* σ_j *and* γ_j *are the* j*th columns of* σ *and* γ*, and if* u *is a vector, if* $A(t,x)$ *is a matrix-valued function,* $(A'(t,x),u)$ *denotes the matrix, the* j*th column of which is*

$$\big(A'(t,x),u\big)_j = \frac{\partial A}{\partial x_j}(t,x)u. \tag{3.3.3}$$

Then under the conditions of theorem 3.1.1, X_t *and* $\log\Lambda_t$ *are differentiable in* $L^{\infty-}$ *with respect to the initial condition, the derivative of* X_t *is* Z_t *and*

$$(X_t-M_t)^\star P_t^{-1} = \Big(-\varepsilon\nabla_0\log\Lambda_t + (X_0-M_0)^\star P_0^{-1}\Big)Z_t^{-1} + O\big(\varepsilon + v_\varepsilon' e^{-ct}\big). \tag{3.3.4}$$

Remark. The '$O(.)$' in (3.3.4) must be understood in $L^{\infty-}(\mathbb{P})$. Actually, the change of probabilities $\mathbb{P}\to\widetilde{\mathbb{P}}$ can be viewed as a change of variables used in order to compute the derivatives, but when making estimates, we generally return to the basic probability $\mathbb{P}$; except otherwise stated, the L^q norms will be computed under $\mathbb{P}$.

Proof. Fix ε. The main difficulty is that the coefficients of (3.2.3) are locally Lipschitz, but generally not globally. However, if μ is some vector of $\mathbb{R}^n$ and if X_t^μ denotes the solution of (3.2.3) with initial value $X_0 + \mu$, we have

$$\begin{aligned}
d(X_t^\mu - X_t) =& a(t, X_t^\mu)P_t^{-1}(X_t^\mu - X_t)dt + \big(a(t, X_t^\mu) - a(t, X_t)\big)P_t^{-1}(X_t - M_t)dt \\
&+ \big(\beta(t, X_t^\mu) - \beta(t, X_t)\big)dt - \gamma(t, X_t^\mu)\big(h(t, X_t^\mu) - h(t, X_t)\big)dt \\
&+ \sqrt{\varepsilon}\big(\sigma(t, X_t^\mu) - \sigma(t, X_t)\big)d\widetilde{W}_t + \big(\gamma(t, X_t^\mu) - \gamma(t, X_t)\big)\big(dY_t - h(t, X_t)dt\big) \\
=& a(t, X_t^\mu)P_t^{-1}(X_t^\mu - X_t)dt + \sigma(t, X_t^\mu)\big(\sigma(t, X_t^\mu) - \sigma(t, X_t)\big)^* P_t^{-1}(X_t - M_t)dt \\
&+ \big(\beta(t, X_t^\mu) - \beta(t, X_t)\big)dt - \gamma(t, X_t^\mu)\big(h(t, X_t^\mu) - h(t, X_t)\big)dt \\
&+ \sqrt{\varepsilon}\big(\sigma(t, X_t^\mu) - \sigma(t, X_t)\big)dW_t + \sqrt{\varepsilon}\big(\gamma(t, X_t^\mu) - \gamma(t, X_t)\big)dB_t. \qquad (3.3.5)
\end{aligned}$$

Since the exponential moments of $\int_0^t |X_s - M_s|ds$ are finite under $\mathbb{P}$, we can prove from this equation that for ε fixed and $\mu \to 0$, $X_t^\mu - X_t$ is of order $|\mu|$ in $L^{\infty-}$. Then by writing the equation satisfied by the process $X_t^\mu - X_t - Z_t\mu$, one checks similarly that this process is of order $|\mu|^2$, so in particular, X_t is differentiable with derivative Z_t. One then verifies from (3.2.7) that Λ_t is differentiable and that

$$\begin{aligned}
\varepsilon\nabla_0 \log \Lambda_t =& -(X_t - M_t)^* P_t^{-1} Z_t + (X_0 - M_0)^* P_0^{-1} \\
&+ \int_0^t (dY_s - h(s, M_s)ds)^* \psi_1'(s, X_s)Z_s + \frac{1}{2}\int_0^t \psi_2'(s, X_s)Z_s ds. \qquad (3.3.6)
\end{aligned}$$

Now suppose that ε is no more fixed but tends to 0; the lemma will be checked if we prove

$$\Big(\int_0^t (dY_s - h(s, M_s)ds)^* \psi_1'(s, X_s)Z_s + \frac{1}{2}\int_0^t \psi_2'(s, X_s)Z_s ds\Big)Z_t^{-1} = O(\varepsilon + v_\varepsilon' e^{-ct}). \qquad (3.3.7)$$

Fix some time $t_1 > 0$ and consider the subdivision of $[0, \infty)$ consisting of times $t_j = jt_1$; the estimate (3.3.7) will be implied by

$$\begin{aligned}
\sum_{j,\, t_j < t} |Z_{t_j} Z_t^{-1}| \sup_{t_j \le s \le t_{j+1}} \Big| \int_{t_j}^s \Big(\big(h(u, X_u) - h(u, M_u)\big)^* \psi_1'(u, X_u) + \frac{1}{2}\psi_2'(u, X_u)\Big) Z_u Z_{t_j}^{-1} du \\
+ \sqrt{\varepsilon}\int_{t_j}^s dB_u^* \psi_1'(u, X_u) Z_u Z_{t_j}^{-1}\Big| = O(\varepsilon + v_\varepsilon' e^{-ct}). \qquad (3.3.8)
\end{aligned}$$

Note that from (3.2.8), (3.2.9) and the boundedness of P_t^{-1}, we have

$$\begin{cases} |\psi_1'(t, X_t)| \le C|X_t - M_t|, \\ |\psi_2'(t, X_t)| \le C\big(\varepsilon + |X_t - M_t|^2\big), \end{cases} \qquad (3.3.9)$$

so from (3.1.3), $|\psi_1'(t,X_t)|^2$, $|X_t - M_t|\,|\psi_1'(t,X_t)|$ and $\psi_2'(t,X_t)$ are of order $\varepsilon + v_\varepsilon' e^{-c_0 t}$; we can also deduce from classical estimates applied to (3.3.2) and from (3.1.4) that $Z_u Z_{t_j}^{-1}$ is of order 1 for $t_j \le u \le t_{j+1}$; thus the suprema in (3.3.8) are uniformly of order $\varepsilon + v_\varepsilon' e^{-c_0 t_j}$; the L^q norm of the left-hand side of (3.3.8) is therefore dominated by

$$C_q \sum_{j,\,t_j<t} \left(\varepsilon + v_\varepsilon' e^{-c_0 t_j}\right) \left\| Z_{t_j} Z_t^{-1} \right\|_{2q}.$$

Thus (3.3.8) will be proved provided that

$$\left\| Z_s Z_t^{-1} \right\|_q = O(e^{-c(t-s)}) \tag{3.3.10}$$

for $s \le t$ and some $c > 0$. But if A_t is defined by (3.1.2), the inverse of Z_t satisfies

$$\begin{aligned} dZ_t^{-1} = & Z_t^{-1} A_t^\star dt - Z_t^{-1}\Big((\beta' - \gamma h')(t,X_t) - (\beta' - \gamma h')(t,M_t)\Big)dt \\ & - Z_t^{-1}\big(a(t,X_t) - a(t,M_t)\big)P_t^{-1}dt - Z_t^{-1}\sigma(t,X_t)\big((\sigma^\star)'(t,X_t), P_t^{-1}(X_t - M_t)\big)dt \\ & + \varepsilon Z_t^{-1}\Big(\sum_j \sigma_j'\sigma_j' + \sum_j \gamma_j'\gamma_j'\Big)(t,X_t)dt \\ & - \sqrt{\varepsilon} Z_t^{-1}\sigma_j'(t,X_t)dW_t^j - \sqrt{\varepsilon} Z_t^{-1}\gamma_j'(t,X_t)dB_t^j. \end{aligned} \tag{3.3.11}$$

By considering the transpose of this equation, since A_t is exponentially stable and (3.1.4) holds, we can deduce (3.3.10) from lemma 1.4.1 (applied here with $\mathcal{G}_t = \mathcal{F}_t$). □

§3.4 Proof of the main result

Since the conditional expectation is a contraction in the spaces $L^q(\mathbb{P})$, we can take the $(\mathcal{Y}_t, \mathbb{P})$ conditional expectation in (3.3.4) and obtain

$$(\widehat{X}_t - M_t)^\star P_t^{-1} = \mathbb{E}\Big[-\varepsilon \nabla_0 \log \Lambda_t Z_t^{-1} + (X_0 - M_0)^\star P_0^{-1} Z_t^{-1} \;\Big|\; \mathcal{Y}_t\Big] + O\big(\varepsilon + v_\varepsilon' e^{-ct}\big). \tag{3.4.1}$$

In order to prove theorem 3.1.1, we have to estimate the right-hand side. We will use an extension of the integration by parts formula (3.1.6). Since we will consider functionals $\Phi(x, \widetilde{w}, y)$ which are not pathwise smooth with respect to x, we will have to mollify them; to this end, put

$$\psi(x) = \exp -(1 - x^2)^{-1} \Big/ \int_{-1}^{1} \exp -(1 - u^2)^{-1} du \tag{3.4.2}$$

if $|x| < 1$, 0 otherwise, and consider the sequence of mollifiers

$$\psi_k(x) = k\psi(kx). \tag{3.4.3}$$

For some fixed unit vector μ, define also the sequence of variables

$$\Phi^{(k)}(x,\widetilde{w},y)=\int_{-\infty}^{+\infty}\Phi(x+\delta\mu,\widetilde{w},y)\psi_k(\delta)d\delta. \tag{3.4.4}$$

It is easily seen that $\Phi^{(k)}$ is pathwise differentiable with respect to x in the direction μ and

$$\frac{\partial\Phi^{(k)}}{\partial\delta}(x+\delta\mu,\widetilde{w},y)\Big|_{\delta=0}=-\int\Phi(x+\delta\mu,\widetilde{w},y)\psi_k'(\delta)d\delta. \tag{3.4.5}$$

By taking the above equality at $(X_0,\widetilde{W},Y)$, we obtain $\nabla_0^\mu\Phi^{(k)}$. With this definition, we first prove the

Lemma 3.4.1. *Fix $\varepsilon>0$, $t\le T$ and some unit vector μ; let $\Phi=\Phi_1\Phi_2$ be the product of two functionals which are measurable with respect to σ-field generated by $(x,\widetilde{w}_s,y_s;\,s\le t)$; we suppose that for some $q>1$, Φ_1 is differentiable in $L^q(\widetilde{\mathbb{P}})$ with respect to X_0 in the direction μ and that Φ_2 is continuous with respect to X_0 in the direction μ in $L^{\infty-}(\widetilde{\mathbb{P}})$. Then*

$$\widetilde{\mathbb{E}}\Big[\Phi\frac{p_0'}{p_0}(X_0)\mu+\Phi_2\nabla_0^\mu\Phi_1\;\Big|\;\mathcal{Y}_t\Big]=-\lim_{k\to\infty}\widetilde{\mathbb{E}}\big[\Phi_1\nabla_0^\mu\Phi_2^{(k)}\mid\mathcal{Y}_t\big] \tag{3.4.6}$$

in probability.

Proof. Take the scalar product of (3.1.6) with μ; we first note that the resulting equality can be extended to measurable functions ϕ which are C^1 in the direction μ and such that

$$\phi'(X_0)\mu+\phi(X_0)\frac{p_0'}{p_0}(X_0)\mu$$

and $\phi(X_0)$ are integrable; in this case indeed, for almost every x, the functions $\delta\mapsto(\phi p_0)(x+\delta\mu)$ and $(\phi p_0)'(x+\delta\mu)\mu$ are integrable with respect to the Lebesgue measure, so the integral of the latter function is necessarily 0; we conclude by integrating with respect to x on an hyperplane orthogonal to μ. Thus for any integers (k,k') we can apply (3.1.6) in the direction μ to the functional $\Phi_1^{(k)}\Phi_2^{(k')}$ for each $(\widetilde{w},y)$ fixed; by integrating with respect to the Wiener measure on $\widetilde{w}$ we obtain

$$\widetilde{\mathbb{E}}\Big[\Phi_1^{(k)}\Phi_2^{(k')}\frac{p_0'}{p_0}(X_0)\mu+\Phi_1^{(k)}\nabla_0^\mu\Phi_2^{(k')}+\Phi_2^{(k')}\nabla_0^\mu\Phi_1^{(k)}\;\Big|\;\mathcal{Y}_t\Big]=0. \tag{3.4.7}$$

We first study the convergence of $\Phi_1^{(k)}$ as $k\to\infty$; we have

$$\Phi_1^{(k)}-\Phi_1=\int\big(\Phi_1(X_0+\delta\mu)-\Phi_1(X_0)\big)\psi_k(\delta)d\delta \tag{3.4.8}$$

so that

$$\left\|\Phi_1^{(k)} - \Phi_1\right\|_q^{\widetilde{\mathbb{P}}} \leq \sup_{|\delta|\leq 1/k} \left\|\Phi_1(X_0+\delta\mu) - \Phi_1(X_0)\right\|_q^{\widetilde{\mathbb{P}}}. \tag{3.4.9}$$

From (3.4.5), using the fact the integral of ψ_k' is 0 and the integral of $x\psi_k'(x)$ is -1, we also have

$$\nabla_0^\mu \Phi_1^{(k)} - \nabla_0^\mu \Phi_1 = -\int \Big(\frac{\Phi_1(X_0+\delta u) - \Phi_1(X_0)}{\delta} - \nabla_0^\mu \Phi_1\Big)\delta\psi_k'(\delta)d\delta \tag{3.4.10}$$

so that

$$\left\|\nabla_0^\mu \Phi_1^{(k)} - \nabla_0^\mu \Phi_1\right\|_q^{\widetilde{\mathbb{P}}} \leq \sup_{|\delta|\leq 1/k} \left\|\frac{\Phi_1(X_0+\delta\mu) - \Phi_1(X_0)}{\delta} - \nabla_0^\mu \Phi_1\right\|_q^{\widetilde{\mathbb{P}}}. \tag{3.4.11}$$

We deduce from our assumptions that for some $q > 1$, the terms of (3.4.9) and (3.4.11) converge to 0; we can check similarly that $\Phi_2^{(k')}$ is bounded in $L^{\infty-}(\widetilde{\mathbb{P}})$ and converges to Φ_2 as $k' \to \infty$. Now by taking the limit in (3.4.7), firstly as $k \to \infty$, secondly as $k' \to \infty$, we obtain (3.4.6). $\square$

If now we return to the probability $\mathbb{P}$, we have the

Lemma 3.4.2. *Fix $\varepsilon > 0$, $t \leq T$ and some unit vector μ; let $\Phi = \Phi_1\Phi_2$ be the product of two functionals which are measurable with respect to σ-field generated by $(x, \widetilde{w}_s, y_s; s \leq t)$; we suppose that Φ_1 (resp. Φ_2) is differentiable (resp. continuous) in the direction μ with respect to X_0 in $L^{\infty-}$. Then*

$$\mathbb{E}\Big[\Phi\nabla_0^\mu \log\Lambda_t + \Phi\frac{p_0'}{p_0}(X_0)\mu + \Phi_2\nabla_0^\mu\Phi_1 \;\Big|\; \mathcal{Y}_t\Big] = -\lim_{k\to\infty} \mathbb{E}\big[\Phi_1\nabla_0^\mu\Phi_2^{(k)} \mid \mathcal{Y}_t\big] \tag{3.4.12}$$

in probability.

Proof. One can check that the variable Λ_T^{-1} is in $L^q(\mathbb{P})$ for some $q > 1$; this implies that variables which are bounded (for ε fixed) in $L^{\infty-}(\mathbb{P})$ are also bounded in $L^{\infty-}(\widetilde{\mathbb{P}})$. One can also check that (for ε fixed) the variables $\Lambda_t(X_0+\delta\mu)$ are bounded in $L^q(\widetilde{\mathbb{P}})$ for some $q > 1$; this fact and the differentiability of $\log\Lambda_t$ in $L^{\infty-}(\widetilde{\mathbb{P}})$ imply that Λ_t is differentiable in $L^q(\widetilde{\mathbb{P}})$ for some $q > 1$. We can deduce that $\Phi_1\Lambda_t$ and Φ_2 satisfy the assumptions of lemma 3.4.1; in this case (3.4.6) becomes

$$\widetilde{\mathbb{E}}\Big[\Phi\nabla_0^\mu\Lambda_t + \Phi\Lambda_t\frac{p_0'}{p_0}(X_0)\mu + \Phi_2\Lambda_t\nabla_0^\mu\Phi_1 \;\Big|\; \mathcal{Y}_t\Big] = -\lim_{k\to\infty} \widetilde{\mathbb{E}}\big[\Lambda_t\Phi_1\nabla_0^\mu\Phi_2^{(k)} \mid \mathcal{Y}_t\big]. \tag{3.4.13}$$

By dividing by $\widetilde{\mathbb{E}}[\Lambda_t|\mathcal{Y}_t]$ we obtain a formula for $\mathbb{P}$ conditional expectations which is exactly (3.4.12). $\square$

Remark. In the applications, we will estimate the right-hand side of (3.4.13) by means of

$$\nabla_0^\mu \Phi_2^{(k)} = -\int \frac{\Phi_2(X_0 + \delta\mu) - \Phi_2(X_0)}{\delta} \delta\psi_k'(\delta)d\delta \tag{3.4.14}$$

where we note that $-\delta\psi_k'(\delta)d\delta$ is a probability measure on $[-1/k, 1/k]$.

Proof of theorem 3.1.1. Let (e_i) be the canonical basis of $\mathbb{R}^n$; for each i, we apply lemma 3.4.2 with $\Phi_1 = 1$, $\mu = e_i$ and Φ_2 equal to the ith line $Z_{t,0}^i$ of Z_t^{-1}. By applying (3.4.14) and summing over i, we obtain

$$\left\|\mathbb{E}\left[\left(\nabla_0 \log \Lambda_t + p_0^{-1}p_0'(X_0)\right)Z_t^{-1} \mid \mathcal{Y}_t\right]\right\|_q \leq \limsup_{\delta\to 0}\left\|\sum_i \frac{1}{\delta}\left(Z_{t,0}^i(X_0+\delta e_i) - Z_{t,0}^i(X_0)\right)\right\|_q. \tag{3.4.15}$$

Now, for any δ, let Ψ_t be the matrix, the ith line of which is the term inside the sum of (3.4.15); by studying the equation (3.3.11) of Z_t^{-1}, since the coefficients involved in this equation are Lipschitz, we can write Ψ_t in the form

$$\Psi_t = \left(\int_0^t dS_s Z_s\right) Z_t^{-1} \tag{3.4.16}$$

where S_t is a matrix-valued semimartingale, the ith line of of which has the form

$$\begin{aligned} S_t^i = &\sum_j \int_0^t Z_{s,0}^i(X_0+\delta e_i) f_s^j(\delta) \frac{X_s^j(X_0+\delta e_i) - X_s^j(X_0)}{\delta} ds \\ &+ \sqrt{\varepsilon}\sum_{j,k}\int_0^t Z_{s,0}^i(X_0+\delta e_i) g_s^{kj}(\delta)\frac{X_s^j(X_0+\delta e_i) - X_s^j(X_0)}{\delta} dw_s^k \end{aligned} \tag{3.4.17}$$

where w_t consists of components of W_t and B_t ands f_s^j and g_s^{kj} are matrix-valued processes satisfying

$$|f_s^j(\delta)| \leq C(1 + |X_s - M_s|), \qquad |g_s^{kj}(\delta)| \leq C. \tag{3.4.18}$$

For each fixed ε, as $\delta \to 0$, the process S_t^i is contiguous in $L^{\infty-}$ to the process

$$\overline{S}_t^i = \sum_j \int_0^t Z_{s,0}^i f_s^j(\delta) Z_s^{ji} ds + \sqrt{\varepsilon}\sum_{j,k}\int_0^t Z_{s,0}^i g_s^{kj}(\delta) Z_s^{ji} dw_s^k. \tag{3.4.19}$$

Since $Z_{s,0}$ is the inverse matrix of Z_s, we deduce that

$$\sum_i \overline{S}_t^i = \sum_i \int_0^t f_s^{ii}(\delta)ds + \sqrt{\varepsilon}\sum_{i,k}\int_0^t g_s^{kii}(\delta)dw_s^k \tag{3.4.20}$$

where f_s^{ii} and g_s^{kii} are the ith lines of f_s^i and g_s^{ki}. Then we deduce from (3.4.18) that $\sum_i \overline{S}_t^i$, and therefore also $\sum_i S_t^i$, are asymptotically of order 1 as $\delta \to 0$, uniformly in ε. Thus the right-hand side of (3.4.15) is of order 1, so that the theorem follows from (3.4.1), (3.1.1) and (3.3.10). □

§3.5 Asymptotic normality of the conditional law

In this subsection, as a complement to theorem 3.1.1, we check that the conditional law of X_t is almost Gaussian after the initial layer. This type of property has been studied under particular assumptions in [29] and [19] (in [19] the convergence is checked for a topology which is stronger than ours).

Theorem 3.5.1. *Assume the conditions of theorem 3.1.1. Let $f(t,x)$ be a family of real-valued observable C^1 functions defined on $\mathbb{R}^n$ with at most polynomial growth as well as their derivatives; define*

$$\xi_t = (\varepsilon P_t)^{-1/2}(X_t - M_t). \tag{3.5.1}$$

If $f(t,\xi_t)$ is bounded in $L^{\infty-}$, then

$$\mathbb{E}\Big[f(t,\xi_t)\xi_t^\star - f'(t,\xi_t) \;\Big|\; \mathcal{Y}_t\Big]P_t^{-1/2} = O\Big(\sqrt{\varepsilon} + \frac{v_\varepsilon + v'_\varepsilon}{\sqrt{\varepsilon}}e^{-ct}\Big). \tag{3.5.2}$$

Proof. We deduce from (3.3.4) that

$$f(t,\xi_t)\xi_t^\star P_t^{-1/2} = \frac{1}{\sqrt{\varepsilon}}f(t,\xi_t)\Big[-\varepsilon\nabla_0 \log\Lambda_t + (X_0 - M_0)^\star P_0^{-1}\Big]Z_t^{-1} + O\Big(\sqrt{\varepsilon} + \frac{v'_\varepsilon}{\sqrt{\varepsilon}}e^{-ct}\Big). \tag{3.5.3}$$

By using also (3.1.1), we only have to prove that

$$\mathbb{E}\Big[\varepsilon f(t,\xi_t)\big(\nabla_0\log\Lambda_t + p_0^{-1}p_0'(X_0)\big)Z_t^{-1} + \sqrt{\varepsilon}f'(t,\xi_t)P_t^{-1/2} \;\Big|\; \mathcal{Y}_t\Big] = O\big(\varepsilon + (v_\varepsilon + v'_\varepsilon)e^{-ct}\big). \tag{3.5.4}$$

Let us apply lemma 3.4.2 with $\Phi_1 = f(t,\xi_t)$, $\mu = e_i$ and Φ_2 equal to the ith line of Z_t^{-1}; by estimating the approximate derivative of Z_t^{-1} as in the proof of theorem 3.1.1, we can deduce that

$$\mathbb{E}\Big[f(t,\xi_t)\big(\nabla_0\log\Lambda_t + p_0^{-1}p_0'(X_0)\big)Z_t^{-1} + \nabla_0 f(t,\xi_t)Z_t^{-1} \;\Big|\; \mathcal{Y}_t\Big] = O(1). \tag{3.5.5}$$

On the other hand,

$$\nabla_0 f(t,\xi_t) = \frac{1}{\sqrt{\varepsilon}}f'(t,\xi_t)P_t^{-1/2}Z_t \tag{3.5.6}$$

so (3.5.4) is proved. □

Corollary 3.5.2. *Under the assumptions of theorem 3.1.1, let τ be a family of deterministic times such that P_τ is bounded in $L^{\infty-}$ and $(v_\varepsilon + v'_\varepsilon)e^{-c\tau}$ is of order ε for any c. Then the conditional law of the variable ξ_τ given $\mathcal{Y}_\tau$ converges in probability to the standard Gaussian law for the weak topology on probability laws.*

Remark. If P_t is bounded and $v_\varepsilon + v'_\varepsilon = O(\varepsilon)$, so that the initial law is already nearly Gaussian with small enough variance, then no particular assumption is made on τ; on the other hand, if v_ε and v'_ε are of order $\varepsilon^{-\alpha}$ for some α, and if P_t is of order $\varepsilon^{-\alpha}e^{-ct} + 1$, then the conditional law is nearly Gaussian for times $\tau \gg \log(1/\varepsilon)$.

Proof. First note that since $v'_\varepsilon e^{-c\tau} = O(\varepsilon)$ for any c, it follows from (3.1.3) that $X_\tau - M_\tau$ is of order $\sqrt{\varepsilon}$ so ξ_τ is of order 1. On the other hand, consider the Hermite polynomials defined by induction by $H_0 = 1$ and for $k \geq 1$,

$$H_k(x) = xH_{k-1}(x) - H'_{k-1}(x). \tag{3.5.7}$$

Consider a n-uple of non negative integers $(k_1, \ldots, k_n)$ and suppose that at least one element, say for instance k_i, is non zero; by putting

$$f(x) = \prod_{j \neq i} H_{k_j}(x^j)\, H_{k_i - 1}(x^i), \tag{3.5.8}$$

the variable $f(\xi_\tau)$ is of order 1 so we can apply theorem 3.5.1; by considering the ith component of (3.5.2), we deduce from (3.5.7) that

$$\mathbb{E}\Big[H_{k_1}(\xi^1_\tau) \ldots H_{k_n}(\xi^n_\tau) \;\Big|\; \mathcal{Y}_t\Big] = O(\sqrt{\varepsilon}). \tag{3.5.9}$$

If the above expression were zero, it would caracterize the standard Gaussian law. Thus all the polynomial conditional statistics of ξ_τ converge in probability to those of the standard Gaussian law. $\square$

4. Asymptotic smoothing

In this section, we consider a family of deterministic times T and we look for the conditional law of the whole path $(X_t, t \leq T)$ given $\mathcal{Y}_T$. We will use the framework of §3 with some restrictions ($\gamma = 0$, h linear). We first state the theorem in §4.1; the other subsections are devoted to the proof.

§4.1 Statement of the result

Our aim is to study the signal process X_t, $t \leq T$, under the filtration $\mathcal{F}_t \vee \mathcal{Y}_T$; more precisely, we want to know whether this is a semimartingale and if it is, find its decomposition, or at least an approximation; this will enable us to compute approximate conditional statistics of the process. With reference to §3, we are going to assume two additional conditions; firstly, we will suppose $\gamma = 0$, so that the signal and observation noises are independent; if indeed $\gamma \neq 0$, since Y_t is clearly not a $\mathcal{F}_t \vee \mathcal{Y}_T$ semimartingale, the process X_t will not be a semimartingale; secondly, we will suppose that h is linear; this will avoid us difficulties due to some anticipating processes. We will also restrict ourselves to the case where $X_t - M_t$ is of order $\sqrt{\varepsilon}$ on the whole time interval.

We first define a suboptimal Gaussian smoother $(\overline{M}_t, \overline{P}_t)$ which is supposed to approximate the conditional mean and variance of X_t given $\mathcal{Y}_T$; in order to construct it, we linearize (0.3) around the extended Kalman filter and we consider the fixed-interval linear smoother for the linearized system (see [18] for the formulas of linear smoothing). This can be formulated as the

Definition 4.1.1. *Consider the system (0.3) with $\gamma = 0$ and let (M_t, P_t) be an extended Kalman filter. The associated Gaussian smoother is then the solution of the backward equations*

$$\dot{\overline{M}}_t = \beta(t, M_t) + \left(\beta'(t, M_t) + a(t, M_t)P_t^{-1}\right)(\overline{M}_t - M_t) \tag{4.1.1}$$

and

$$\dot{\overline{P}}_t = \left(\beta'(t, M_t) + a(t, M_t)P_t^{-1}\right)\overline{P}_t + \overline{P}_t\left(\beta'(t, M_t) + a(t, M_t)P_t^{-1}\right)^* - a(t, M_t) \tag{4.1.2}$$

with final condition $(\overline{M}_T, \overline{P}_T) = (M_T, P_T)$.

We now want to prove that the conditional law of X_t given $\mathcal{Y}_T$ is approximately the Gaussian law with mean $\overline{M}_t$ and covariance $\varepsilon\overline{P}_t$. In a part of the subsequent statements, we do not require v_ε (see (3.1.1)) to be finite; when $v_\varepsilon = +\infty$, this means by convention that the law of X_0 has not necessarily a density. The basic result is the

Theorem 4.1.2. *Assume the conditions of theorem 3.1.1 with $v'_\varepsilon = 0$ and P_t uniformly bounded; suppose moreover that a is uniformly elliptic, $\gamma = 0$, h is linear and that the exponential moments of $X_t - M_t$ are uniformly bounded. Let Π be the observable variable* *with values in the space of probability measures on C_T^n defined as follows: for each observed path, it is the law of the Gaussian process x_t with initial law $\mathcal{N}(\overline{M}_0, \varepsilon\overline{P}_0)$ and solution of*

$$\begin{aligned} dx_t =& a(t, M_t)\Big[P_t^{-1}(x_t - M_t) - \overline{P}_t^{-1}(x_t - \overline{M}_t)\Big]dt + \beta(t, M_t)dt \\ &+ \beta'(t, M_t)(x_t - M_t)dt + \sqrt{\varepsilon}\sigma(t, M_t)dw_t \end{aligned} \tag{4.1.3}$$

for a standard Wiener process w_t. We also let $\Pi(x_0, dx)$ be the law defined in (4.1.3) with initial value x_0 and we define

$$\overline{\xi}_t = (\varepsilon\overline{P}_t)^{-1/2}(X_t - \overline{M}_t). \tag{4.1.4}$$

Then

(a) if $v_\varepsilon = O(\varepsilon)$, for any family of $\mathcal{Y}_T \otimes \mathcal{B}(\mathbb{R}^n)$ measurable C^1 functions $f(x)$ such that f and f' have uniformly polynomial growth,

$$\mathbb{E}\Big[f(\overline{\xi}_0)\overline{\xi}_0^\star - f'(\overline{\xi}_0) \,\Big|\, \mathcal{Y}_T\Big] = O(\sqrt{\varepsilon}); \tag{4.1.5}$$

(b) there exists a measurable process $\check{X}_t$ such that $\check{X}_0 = X_0$, $\check{X}_t - X_t$ is of order ε, and the conditional law of $(\check{X}_t, t \leq T)$ given $\mathcal{F}_0 \vee \mathcal{Y}_T$ is equal to $\Pi(X_0, .)$.

Note that for each t, the law of x_t under Π is $\mathcal{N}(\overline{M}_t, \varepsilon\overline{P}_t)$. The result (a) says that the conditional law of X_0 is approximately Gaussian (as in §3.5) and (b) implies that this approximate Gaussian structure is propagated. In particular, as it will be proved subsequently, the theorem implies that the conditional law of $(X_t, t \leq T)$ given $\mathcal{Y}_T$ is close to Π and this can be stated as the

Corollary 4.1.3. *Under the assumptions of theorem 4.1.2, let k be a fixed integer, let $0 \leq t_1, \ldots, t_k \leq T$ be families of numbers and let f be a family of real-valued observable uniformly bounded functions defined on $\mathbb{R}^{kn}$ which are uniformly (in x and ε) continuous. If x_t is a continuous path, define*

$$\check{\xi}_t(x) = (\varepsilon\overline{P}_t)^{-1/2}(x_t - \overline{M}_t). \tag{4.1.6}$$

Then

(a) if $v_\varepsilon = O(\varepsilon)$, one has

$$\mathbb{E}\big[f(\overline{\xi}_{t_1}, \ldots, \overline{\xi}_{t_k}) \mid \mathcal{Y}_T\big] - \int f(\check{\xi}_{t_1}(x), \ldots, \check{\xi}_{t_k}(x))\Pi(dx) \longrightarrow 0 \tag{4.1.7}$$

in probability as $\varepsilon \to 0$;
(b) if X_0 is bounded in $L^{\infty-}$ and $t_i \gg \log(1/\varepsilon)$ for $1 \le i \le k$, the same convergence holds.

For each deterministic time t, the law of $\check{\xi}_t(x)$ under Π is the standard Gaussian law, so the corollary implies in particular the convergence of the conditional law of $\overline{\xi}_t$ to the standard Gaussian law. Note that the results (b) of theorem 4.1.2 and corollary 4.1.3 hold even if the law of X_0 is not absolutely continuous with respect to the Lebesgue measure; in particular for $t = T$, we obtain a result for the filtering problem.

§4.2 Estimation of the initial state

In this subsection, we see how the technique of §3 can be used in order to approximate the conditional law of X_0 given $\mathcal{Y}_T$ and obtain the first part of the theorem. We first prove the

Lemma 4.2.1. *Under the assumptions of theorem 4.1.2, the process $\overline{P}_t$ is uniformly bounded and elliptic, the process $X_t - \overline{M}_t$ is of order $\sqrt{\varepsilon}$ and for any $K > 0$,*

$$\mathbb{E} \exp K \int_0^t |X_s - \overline{M}_s| ds \le C_K e^{C_K \sqrt{\varepsilon} t}. \tag{4.2.1}$$

Proof. Recall that A_t is the process defined in (3.1.2) and let $\zeta_{s,t}$ be the associated fundamental solution as considered in definition 1.1.4; since A_t is exponentially stable, $\zeta_{s,t}$ is of order $e^{-c(t-s)}$ for $s \le t$. Then the solution of (4.1.2) can be written as

$$\overline{P}_t = \zeta^\star_{t,T} P_T \zeta_{t,T} + \int_t^T \zeta^\star_{t,s} a(s, M_s) \zeta_{t,s} ds. \tag{4.2.2}$$

Since a and P_T are bounded, we deduce that $\overline{P}_t$ is also bounded. By writing the equation satisfied by $\overline{P}_t^{-1}$, since a is uniformly elliptic, we prove that it is also bounded so $\overline{P}_t$ is elliptic. However, for the study of $X_t - \overline{M}_t$, we cannot directly apply the technique which was used for $X_t - M_t$ because $\overline{M}_t$ is anticipating for $\mathcal{F}_t$. Thus we use an auxiliary process; consider the solution of

$$\dot{\rho}_t = \beta(t, \rho_t) + K_0(X_t - \rho_t) \tag{4.2.3}$$

with $\rho_0 = X_0$, for some positive K_0 which is large enough, so that $X_t - \rho_t$ is easily estimated from the results of §1.3: it is of order $\sqrt{\varepsilon}$, its exponential moments are bounded and it satisfies an estimate of type (4.2.1). Thus we only have to prove that $\overline{M}_t - \rho_t$ is of order $\sqrt{\varepsilon}$ and satisfies also an estimate of type (4.2.1). But

$$\frac{d}{dt}(\overline{M}_t - \rho_t) = \beta(t, M_t) - \beta(t, \rho_t) - A_t^\star(\overline{M}_t - \rho_t) - A_t^\star(\rho_t - M_t) + K_0(\rho_t - X_t) \quad (4.2.4)$$

so that

$$\overline{M}_t - \rho_t = \zeta_{t,T}^\star(M_T - \rho_T) - \int_t^T \zeta_{t,s}^\star\big[\beta(s, M_s) - \beta(s, \rho_s) + A_s^\star(M_s - \rho_s) + K_0(\rho_s - X_s)\big]ds. \quad (4.2.5)$$

Since $M_t - \rho_t$ and $\rho_t - X_t$ are of order $\sqrt{\varepsilon}$, the process $\overline{M}_t - \rho_t$ is also of order $\sqrt{\varepsilon}$ and

$$|\overline{M}_u - \rho_u| \leq C \int_u^T e^{-c(v-u)}\Big(|\rho_v - X_v| + |X_v - M_v|\Big)dv + C|M_T - \rho_T|e^{-c(T-u)} \quad (4.2.6)$$

so

$$\begin{aligned}\int_s^t |\overline{M}_u - \rho_u|du \leq C|M_T - \rho_T| &+ C\int_s^t \Big(|\rho_u - X_u| + |X_u - M_u|\Big)du \\ &+ C\int_t^T e^{-c(u-t)}\Big(|\rho_u - X_u| + |X_u - M_u|\Big)du. \end{aligned} \quad (4.2.7)$$

The exponential moments of $M_T - \rho_T$ are bounded; the exponential moments of the second term are estimated from (3.1.4) and a similar property for $\rho_t - X_t$ (see the construction of ρ_t); finally, from the Jensen inequality

$$\begin{aligned}&\mathbb{E}\exp K\int_t^T e^{-c(u-t)}\Big(|\rho_u - X_u| + |X_u - M_u|\Big)du \\ &\leq c\int_t^T e^{-c(u-t)}\mathbb{E}\exp\frac{K}{c}\Big(|\rho_u - X_u| + |X_u - M_u|\Big)du\end{aligned} \quad (4.2.8)$$

which is uniformly bounded. The lemma then follows. □

Sketch of the proof of theorem 4.1.2, part (a). Recall that the process Λ_t was defined in (3.2.1) and consider the process L_t, $0 \leq t \leq T$, defined by

$$\begin{aligned}L_t^{-1} = \Lambda_t^{-1}\exp\Big\{ &- \frac{1}{\sqrt{\varepsilon}}\int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1}\sigma(s, X_s)d\widetilde{W}_s \\ &- \frac{1}{2\varepsilon}\int_0^t \big|\sigma^\star(s, X_s)\overline{P}_s^{-1}(X_s - \overline{M}_s)\big|^2 ds\Big\}.\end{aligned} \quad (4.2.9)$$

One can check that for ε fixed, some exponential moment of $|\overline{M}_t|^2$ is bounded, so $\Lambda_t L_t^{-1}$ is a $(\mathcal{F}_t \vee \mathcal{Y}_T, \widetilde{\mathbb{P}})$ martingale and if $\overline{\mathbb{P}}$ is the probability measure on $\mathcal{F}_T$ absolutely continuous with respect to $\widetilde{\mathbb{P}}$ with density $\Lambda_T L_T^{-1}$, the process

$$\overline{W}_t = \widetilde{W}_t + \frac{1}{\sqrt{\varepsilon}}\int_0^t \sigma^\star(s, X_s)\overline{P}_s^{-1}(X_s - \overline{M}_s)ds \quad (4.2.10)$$

is a $(\mathcal{F}_t \vee \mathcal{Y}_T, \mathbb{P})$ Brownian motion. Note that Y_t is measurable with respect to the initial σ-field of this filtration, so it is not a semimartingale; however, since we have assumed $\gamma = 0$, the process X_t is a semimartingale with decomposition

$$dX_t = a(t,X_t)\big[P_t^{-1}(X_t - M_t) - \overline{P}_t^{-1}(X_t - \overline{M}_t)\big]\,dt + \beta(t,X_t)dt + \sqrt{\varepsilon}\sigma(t,X_t)d\overline{W}_t. \quad (4.2.11)$$

We want to obtain a formula for L_t similar to (3.2.7). Note first that (4.2.9) can be written as

$$\begin{aligned} L_t = \Lambda_t \exp \frac{1}{\varepsilon}\Big\{ &\int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1}\big(dX_s - \beta(s,X_s)ds\big) \qquad (4.2.12)\\ &+ \int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1} a(s,X_s)\Big[\frac{1}{2}\overline{P}_s^{-1}(X_s - \overline{M}_s) - P_s^{-1}(X_s - M_s)\Big]ds\Big\}. \end{aligned}$$

On the other hand, since $\overline{M}_t$ and $\overline{P}_t$ are absolutely continuous and $\mathcal{Y}_T$ measurable, they are $\mathcal{F}_t \vee \mathcal{Y}_T$ semimartingales and we deduce from Itô's formula that

$$\begin{aligned} (X_t - \overline{M}_t)^\star \overline{P}_t^{-1}(X_t - \overline{M}_t) =& (X_0 - \overline{M}_0)^\star \overline{P}_0^{-1}(X_0 - \overline{M}_0) + 2\int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1} dX_s\\ &- 2\int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1} \dot{\overline{M}}_s ds\\ &- \int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1} \dot{\overline{P}}_s \overline{P}_s^{-1}(X_s - \overline{M}_s)ds \qquad (4.2.13) \end{aligned}$$

so that

$$\begin{aligned} 2\varepsilon \log\big(L_t \Lambda_t^{-1}\big) =& (X_t - \overline{M}_t)^\star \overline{P}_t^{-1}(X_t - \overline{M}_t) - (X_0 - \overline{M}_0)^\star \overline{P}_0^{-1}(X_0 - \overline{M}_0)\\ &+ 2\int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1}\Big[\dot{\overline{M}}_s - \beta(s,X_s) - a(s,X_s)P_s^{-1}(X_s - M_s)\Big]ds\\ &+ \int_0^t (X_s - \overline{M}_s)^\star \overline{P}_s^{-1}\Big[\dot{\overline{P}}_s + a(s,X_s)\Big]\overline{P}_s^{-1}(X_s - \overline{M}_s)ds. \end{aligned} \qquad (4.2.14)$$

By writing this equation at time $t = T$, and using (4.1.1), (4.1.2) and (3.2.7) (since $\gamma = 0$ and h is linear, one has $\psi_1' = 0$), we obtain

$$2\varepsilon \log L_T = (X_0 - M_0)^\star P_0^{-1}(X_0 - M_0) - (X_0 - \overline{M}_0)^\star \overline{P}_0^{-1}(X_0 - \overline{M}_0) + \int_0^T \psi_4(s,X_s)ds + \psi_5(T) \qquad (4.2.15)$$

where

$$\psi_4(t,x) = \psi_2(t,x) - 2(x-\overline{M}_t)^\star \overline{P}_t^{-1}\Big(\beta(t,x)-\beta(t,M_t)-\beta'(t,M_t)(x-M_t)\Big)$$
$$+(x-\overline{M}_t)^\star \overline{P}_t^{-1}\big(a(t,x)-a(t,M_t)\big)\Big[\overline{P}_t^{-1}(x-\overline{M}_t)-2P_t^{-1}(x-M_t)\Big] \quad (4.2.16)$$

and $\psi_5(T)$ is an observable variable. The variable L_T can be viewed as a functional of $(X_0,\overline{W},Y)$ and we want to differentiate it with respect to X_0; we can indeed define a differentiation operator $\overline{\nabla}_0$ as in definition 3.3.1 and check that X_t is differentiable in $L^{\infty-}$ and that its derivative $\overline{Z}_t$ is solution of

$$d\overline{Z}_t = a(t,X_t)(P_t^{-1}-\overline{P}_t^{-1})\overline{Z}_t dt + \beta'(t,X_t)\overline{Z}_t dt + \sqrt{\varepsilon}\sigma_j'(t,X_t)\overline{Z}_t dW_t^j$$
$$+\sigma(t,X_t)\big((\sigma^\star)'(t,X_t),P_t^{-1}(X_t-M_t)-\overline{P}_t^{-1}(X_t-\overline{M}_t)\big)\overline{Z}_t dt \quad (4.2.17)$$

with the notation defined in (3.3.3). We deduce that L_T is also differentiable and that

$$2\varepsilon\overline{\nabla}_0 \log L_T = 2(X_0-M_0)^\star P_0^{-1} + 2(X_0-\overline{M}_0)^\star \overline{P}_0^{-1} + \int_0^T \psi_4'(t,X_t)\overline{Z}_t dt. \quad (4.2.18)$$

By proceeding as in §3.4, we can also prove the integration by parts formula

$$\mathbb{E}\Big[f(\overline{\xi}_0)\overline{\nabla}_0 \log L_T \;\Big|\; \mathcal{Y}_T\Big] = -\mathbb{E}\Big[\frac{p_0'}{p_0}(X_0)f(\overline{\xi}_0) + f'(\overline{\xi}_0)(\varepsilon\overline{P}_0)^{-1/2} \;\Big|\; \mathcal{Y}_T\Big]. \quad (4.2.19)$$

From (4.2.18), (4.2.19) and (3.1.1), the proof of (4.1.5) is easily reduced to the proof of

$$\int_0^T \psi_4'(t,X_t)\overline{Z}_t dt = O(\varepsilon). \quad (4.2.20)$$

The process $\psi_4'(t,X_t)$ is of order ε so it is sufficient to prove that $\overline{Z}_t$ is of order e^{-ct}. First note that the process W_t is a $(\mathcal{F}_t\vee\mathcal{Y}_T,\tilde{\mathbb{P}})$ semimartingale, so under $\mathbb{P}$, it is also a $\mathcal{F}_t\vee\mathcal{Y}_T$ semimartingale and it is a $\mathcal{F}_t$ Brownian motion; moreover, the process

$$\overline{A}_t = a(t,M_t)(P_t^{-1}-\overline{P}_t^{-1}) + \beta'(t,M_t) \quad (4.2.21)$$

can be shown to be exponentially stable (it is $(\overline{P}_t,k)$ stable for some $k>0$) and equation (4.2.17) can be written as

$$d\overline{Z}_t = \overline{A}_t\overline{Z}_t dt + \big(a(t,X_t)-a(t,M_t)\big)(P_t^{-1}-\overline{P}_t^{-1})\overline{Z}_t dt$$
$$+\big(\beta'(t,X_t)-\beta'(t,M_t)\big)\overline{Z}_t dt + \sqrt{\varepsilon}\sigma_j'(t,X_t)\overline{Z}_t dW_t^j$$
$$+\sigma(t,X_t)\big((\sigma^\star)'(t,X_t),P_t^{-1}(X_t-M_t)-\overline{P}_t^{-1}(X_t-\overline{M}_t)\big)\overline{Z}_t dt. \quad (4.2.22)$$

From lemma 4.2.1 and (3.1.4), the assumptions of lemma 1.4.1 are satisfied so we can deduce that $\overline{Z}_t$ is of order e^{-ct}. $\square$

The process W_t is a $(\mathcal{F}_t \vee \mathcal{Y}_T, \overline{\mathbb{P}})$ semimartingale, so since $\mathbb{P}$ and $\overline{\mathbb{P}}$ are mutually absolutely continuous, it is also a $(\mathcal{F}_t \vee \mathcal{Y}_T, \mathbb{P})$ semimartingale; its martingale part is a Brownian motion and in this subsection, we want to compute its finite variation part. The main tool will be the differentiation of variables $\Phi(X_0, \overline{W}, Y)$ with respect to $\overline{W}$, more precisely with respect to absolutely continuous perturbations of $\overline{W}$. The theory of these perturbations is generally called the stochastic calculus of variations, or the Malliavin calculus, and has been widely studied (see [28]); the fundamental result of this calculus is an integration by parts formula and actually, the result that we will prove (lemma 4.3.2 below) is a result of the same type. We will adopt the

Definition 4.3.1. *Let Φ and Ψ_t be functionals of $(X_0, \overline{W}, Y)$. The variable Φ will be said to be differentiable with respect to $\overline{W}$ with derivative Ψ_t if for any bounded $\mathcal{F}_t \vee \mathcal{Y}_T$ adapted process u_t, one has*

$$\lim_{\delta \to 0} \frac{1}{\delta}\Big(\Phi(X_0, \overline{W} + \delta \int_0^{\cdot} u_s ds, Y) - \Phi(X_0, \overline{W}, Y)\Big) = \int_0^T \Psi_t u_t dt \tag{4.3.1}$$

where the limit holds in probability. In this case, we will note $\overline{D}_t \Phi = \Psi_t$.

Using the calculations of §4.2, one can deduce from (4.2.11) and (4.2.15) that X_s and L_T are differentiable with respect to $\overline{W}$ and that

$$\overline{D}_t X_s = \sqrt{\varepsilon}\, \overline{Z}_s \overline{Z}_t^{-1} \sigma(t, X_t) 1_{\{t \le s\}}, \tag{4.3.2}$$

$$2\varepsilon \overline{D}_t \log L_T = \int_t^T \psi_4'(s, X_s) \overline{D}_t X_s ds. \tag{4.3.3}$$

On the other hand, one can check the existence, for each fixed ε, of a $\alpha > 0$ such that Λ_T and $L_T \Lambda_T^{-1}$ are respectively in $L^{1+\alpha}(\widetilde{\mathbb{P}})$ and $L^{1+\alpha}(\overline{\mathbb{P}})$; by putting $q = (1+\alpha)^2/(1+2\alpha)$ and by means of the Hölder inequality, this implies that

$$\begin{aligned}
\overline{\mathbb{E}} L_T^q &= \overline{\mathbb{E}}\Big[\big(L_T \Lambda_T^{-1}\big)^{(\alpha^2+\alpha)/(1+2\alpha)} \big(L_T \Lambda_T^{-1}\big)^{(1+\alpha)/(1+2\alpha)} \Lambda_T^q\Big] \\
&\le \Big[\overline{\mathbb{E}}\big(L_T \Lambda_T^{-1}\big)^{(\alpha^2+\alpha)/\alpha}\Big]^{\alpha/(1+2\alpha)} \Big[\overline{\mathbb{E}}\big(L_T \Lambda_T^{-1} \Lambda_T^{1+\alpha}\big)\Big]^{(1+\alpha)/(1+2\alpha)} \\
&\le \Big[\overline{\mathbb{E}}\big(L_T \Lambda_T^{-1}\big)^{1+\alpha}\Big]^{\alpha/(1+2\alpha)} \Big[\widetilde{\mathbb{E}} \Lambda_T^{1+\alpha}\Big]^{(1+\alpha)/(1+2\alpha)} \\
&< \infty.
\end{aligned} \tag{4.3.4}$$

One checks similarly that

$$\limsup_{\delta\to 0}\frac{1}{\delta}\Big\|L_T(\overline{W}+\delta\int_0^{\cdot} u_s ds)-L_T(\overline{W})\Big\|_q^{\overline{\mathbb{P}}}<\infty \tag{4.3.5}$$

for some $q>1$. We are going to verify that these two properties imply the

Lemma 4.3.2. *The process*

$$\check{W}_t=\overline{W}_t-\int_0^t \mathbb{E}\big[\overline{D}_s\log L_T \mid \mathcal{F}_s\vee\mathcal{Y}_T\big]^\star ds \tag{4.3.6}$$

is a $(\mathcal{F}_t\vee\mathcal{Y}_T,\mathbb{P})$ *Brownian motion.*

Proof. For any bounded $\mathcal{F}_t\vee\mathcal{Y}_T$ adapted process u_t,

$$\int_0^T u_s^\star d\overline{W}_s=\lim_{\delta\to 0}\frac{1}{\delta}\Big(\exp\Big\{\delta\int_0^T u_s^\star d\overline{W}_s-\frac{\delta^2}{2}\int_0^T |u_s|^2 ds\Big\}-1\Big) \tag{4.3.7}$$

where the limit holds in $L^{\infty-}(\overline{\mathbb{P}})$. Thus, since L_T is in a $L^q(\overline{\mathbb{P}})$, $q>1$,

$$\begin{aligned}\mathbb{E}\int_0^T u_s^\star d\overline{W}_s&=\overline{\mathbb{E}}\Big[L_T\int_0^T u_s^\star d\overline{W}_s\Big]\\&=\lim_{\delta\to 0}\frac{1}{\delta}\Big(\overline{\mathbb{E}}\Big[L_T\exp\Big\{\delta\int_0^T u_s^\star d\overline{W}_s-\frac{\delta^2}{2}\int_0^T|u_s|^2ds\Big\}\Big]-1\Big).\end{aligned} \tag{4.3.8}$$

But from the Girsanov theorem, the multiplication by the exponential in the expectation is equivalent to a perturbation on $\overline{W}$ so that

$$\mathbb{E}\int_0^T u_s^\star d\overline{W}_s=\lim_{\delta\to 0}\frac{1}{\delta}\overline{\mathbb{E}}\Big[L_T(\overline{W}+\delta\int_0^{\cdot} u_s ds)-L_T(\overline{W})\Big]. \tag{4.3.9}$$

The integrability condition (4.3.5) shows that we can exchange the limit and the expectation and therefore

$$\begin{aligned}\mathbb{E}\int_0^T u_s^\star d\overline{W}_s&=\overline{\mathbb{E}}\int_0^T \overline{D}_s L_T\, u_s ds\\&=\overline{\mathbb{E}}\Big[L_T\int_0^T \overline{D}_s\log L_T\, u_s ds\Big]\\&=\mathbb{E}\int_0^T \overline{D}_s\log L_T\, u_s ds.\end{aligned} \tag{4.3.10}$$

Thus $\check{W}_t$ is a martingale; from its quadratic variation, it is necessarily a Brownian motion. □

Proof of theorem 4.1.2, part (b). When considered as a $(\mathcal{F}_t \vee \mathcal{Y}_T, \mathbb{P})$ semimartingale, the process X_t has the decomposition

$$
\begin{aligned}
dX_t =& a(t,X_t)\big[P_t^{-1}(X_t - M_t) - \overline{P}_t^{-1}(X_t - \overline{M}_t)\big]\,dt + \beta(t,X_t)dt \\
&+ \sqrt{\varepsilon}\sigma(t,X_t)\mathbb{E}\big[\overline{D}_t \log L_T \mid \mathcal{F}_t \vee \mathcal{Y}_T\big]^\star dt + \sqrt{\varepsilon}\sigma(t,X_t)d\check{W}_t.
\end{aligned}
\tag{4.3.11}
$$

On the other hand, consider the solution of

$$
\begin{aligned}
d\check{X}_t =& a(t,M_t)\big[P_t^{-1}(\check{X}_t - M_t) - \overline{P}_t^{-1}(\check{X}_t - \overline{M}_t)\big]\,dt + \beta(t,M_t)dt \\
&+ \beta'(t,M_t)(\check{X}_t - M_t) + \sqrt{\varepsilon}\sigma(t,M_t)d\check{W}_t
\end{aligned}
\tag{4.3.12}
$$

with initial condition $\check{X}_0 = X_0$. Then the law of $\check{X}$ conditioned by X_0 is $\Pi(X_0, .)$, so in order to prove the theorem, it is sufficient to prove that $X_t - \check{X}_t$ is of order ε. Using the process $\overline{A}_t$ defined in (4.2.21), the equation for $X_t - \check{X}_t$ can be written as

$$
\begin{aligned}
d(X_t - \check{X}_t) =& \overline{A}_t(X_t - \check{X}_t)dt + \big(a(t,X_t) - a(t,M_t)\big)\Big(P_t^{-1}(X_t - M_t) - \overline{P}_t^{-1}(X_t - \overline{M}_t)\Big)dt \\
&+ \Big(\beta(t,X_t) - \beta(t,M_t) - \beta'(t,M_t)(X_t - M_t)\Big)dt \\
&+ \sqrt{\varepsilon}\big(\sigma(t,X_t) - \sigma(t,M_t)\big)d\check{W}_t \\
&+ \sqrt{\varepsilon}\sigma(t,X_t)\mathbb{E}\big[\overline{D}_t \log L_T \mid \mathcal{F}_t \vee \mathcal{Y}_T\big]^\star dt.
\end{aligned}
\tag{4.3.13}
$$

The process $\overline{D}_t \log L_T$ is given by (4.3.3) and by proceeding as in the proof of (4.2.20), we can deduce that it is of order $\sqrt{\varepsilon}$; moreover the process $\overline{A}_t$ is exponentially stable so we can apply lemma 1.3.2 and deduce from (4.3.13) that $X_t - \check{X}_t$ is of order ε. □

Proof of corollary 4.1.3. We first deduce from theorem 4.1.2, part (b), that in the left-hand side of (4.1.7) one can replace the process X_t involved in $\overline{\xi}_t$ by $\check{X}_t$; thus, defining

$$
\phi(\eta) = \int f(\check{\xi}_{t_1}(x), \ldots, \check{\xi}_{t_k}(x))\Pi(\overline{M}_0 + (\varepsilon\overline{P}_0)^{1/2}\eta, dx),
\tag{4.3.14}
$$

it is sufficient to prove that

$$
\mathbb{E}\big[\phi(\overline{\xi}_0) \mid \mathcal{Y}_T\big] - \int \phi(\eta)\pi_0(d\eta) \longrightarrow 0
\tag{4.3.15}
$$

in probability, where π_0 is the standard Gaussian measure on $\mathbb{R}^n$. By using the exponential stability of the process $\overline{A}_t$ defined in (4.2.21), we prove that if η and η' are two vectors of $\mathbb{R}^n$ and x_t and x'_t are the solutions of (4.1.3) with initial values $\overline{M}_0 + (\varepsilon\overline{P}_0)^{1/2}\eta$ and $\overline{M}_0 + (\varepsilon\overline{P}_0)^{1/2}\eta'$, then

$$
\left|x_t - x'_t\right| \le Ce^{-ct}\sqrt{\varepsilon}|\eta - \eta'|
\tag{4.3.16}
$$

so

$$\left|\check{\xi}_t(x) - \check{\xi}_t(x')\right| \leq Ce^{-ct}|\eta - \eta'|. \tag{4.3.17}$$

Thus, if τ is the infimum of t_i, $1 \leq i \leq k$, we have

$$\left|\phi(\eta) - \phi(\eta')\right| \leq \rho\left(e^{-c\tau}|\eta - \eta'|\right) \tag{4.3.18}$$

where ρ is a bounded function converging to 0 at 0 linked with the uniform modulus of continuity of f. In case (a) we can deduce from theorem 4.1.2, case (a), that the conditional law of $\overline{\xi}_0$ converges to π_0 (see the proof of corollary 3.5.2), so since the functions ϕ are uniformly continuous, we obtain (4.3.15); in case (b), we can define on a enlargement of the probability space an independent standard Gaussian variable η_0; the distance between $\overline{\xi}_0$ and η_0 is of order $1/\sqrt{\varepsilon}$ and

$$\begin{aligned}\left|\mathbb{E}[\phi(\overline{\xi}_0) \mid \mathcal{Y}_T] - \int \phi(\eta)\pi_0(d\eta)\right| &\leq \mathbb{E}\Big[\left|\phi(\overline{\xi}_0) - \phi(\eta_0)\right| \Bigm| \mathcal{Y}_T\Big] \\ &\leq \mathbb{E}\Big[\rho\big(e^{-c\tau}|\overline{\xi}_0 - \eta_0|\big) \Bigm| \mathcal{Y}_T\Big]\end{aligned} \tag{4.3.19}$$

which converges to 0 because ρ is bounded and $e^{-c\tau}|\overline{\xi}_0 - \eta_0|$ converges in probability to 0. □

5. Estimation of some non continuous functionals

Results of §2 imply that continuous functionals of X can be approximated by means of some Kalman-like suboptimal filters. Now if the functional is not continuous, other tools can sometimes be used; for instance if the functional is given by a stochastic differential equation driven by X_t, we can use the results of [24] in order to estimate the difference between this functional and the solution of the same equation driven by some observable approximation M_t of X_t; the following result says that under the assumptions of §3, the convergence holds when M_t is the extended Kalman filter if the time interval is not too large.

Theorem 5.1. *Consider the time interval $[0,T]$ with $T = K/\varepsilon$, $K > 0$. Suppose that X_t is solution of (0.3) with $\beta = \varepsilon b$; assume that X_0 is bounded in probability, that $b(t,x)$ has uniformly linear growth, that σ and γ are bounded, that h' is bounded and Lipschitz; let M_t be a Kalman-like filter with a bounded gain G_t and suppose that $\sup_{t\leq T}|X_t - M_t|$ converges in probability to 0, that $\widehat{X}_t - M_t = O(\varepsilon)$ in L^1 and that $X_t - \widehat{X}_t = O(\sqrt{\varepsilon})$ in*

L^2. Let $f(t,x)$ and $g(x)$ be families of respectively observable and deterministic functions which have uniformly linear growth and are uniformly Lipschitz; let U_t be the solution of

$$dU_t = \varepsilon f(t,U_t)dt + g(U_t)dX_t \tag{5.1}$$

with some deterministic initial value $U_0 = u_0$. If V_t is the solution of

$$dV_t = \varepsilon f(t,V_t)dt + g(V_t)dM_t \tag{5.2}$$

with $V_0 = u_0$, then $\sup_{t\leq T}|U_t - V_t|$ converges in probability to 0.

Sketch of the proof. It is well-known that the process

$$I_t = \frac{1}{\sqrt{\varepsilon}}\Big(Y_t - \int_0^t \mathbb{E}\big[h(s,X_s) \mid \mathcal{Y}_s\big]ds\Big) \tag{5.3}$$

is a standard $\mathcal{Y}_t$ Brownian motion: it is the innovation process. Now write (1.1.1) in the form

$$dM_t = \varepsilon b(t,M_t)dt + G_t\mathbb{E}\big[h(t,X_t) - h(t,M_t) \mid \mathcal{Y}_t\big]dt + \sqrt{\varepsilon}G_t dI_t \tag{5.4}$$

which is the decomposition of M_t as a $\mathcal{Y}_t$ semimartingale and note that since $\widehat{X}_t - M_t$ and $|X_t - \widehat{X}_t|^2$ are of order ε, the second term is of order ε in L^1 (apply Taylor's formula to the function h at point $\widehat{X}_t$); thus, by integrating on $[0,T]$, it appears that the variation on $[0,T]$ of the finite variation part of M_t is bounded in probability; its quadratic variation is also bounded. The same property holds for X_t considered with the filtration $\mathcal{F}_t \supset \mathcal{Y}_t$. Moreover we have assumed that $\sup_t |X_t - M_t|$ converges in probability to 0. Thus we can apply theorem 2.4.3 of [24] and deduce the theorem. □

Some equations which are more general than (5.1) can also be dealt with; for instance the function g may be allowed to depend on t and to be random; we refer to [24] for the set of assumptions which has to be used in this case. In theorem 5.1, the time is at most of order $1/\varepsilon$ because each error on the estimation of U_s is propagated on the whole time interval $[s,T]$ without damping, so that for large times $|U_t - V_t|$ may be large; however, if (5.1) is a stable equation, so that a perturbation of order 1 on the state at time s leads to a perturbation of order $e^{-c\varepsilon(t-s)}$ at time $t \geq s$, then the difference $U_t - V_t$ remains small as $T \to \infty$; in this case, if we consider (M_t, V_t) as a suboptimal filter for the signal (X_t, U_t), this filter has two time scales: a fast component M_t (memory length of order 1) and a slow component V_t (memory length of order $1/\varepsilon$).

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Imprimé en France

par

l'Institut National de Recherche en Informatique et en Automatique