Accumulated reward over the n first operational periods in fault-tolerant computing systems

Gerardo Rubino, Bruno Sericola

UNITÉ DE RECHERCHE
INRIA-RENNES

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

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Gerardo RUBINO
Bruno SERICOLA

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INSTITUT DE RECHERCHE EN INFORMATIQUE
ET SYSTÈMES ALÉATOIRES

Campus Universitaire de Beaulieu
35042 - RENNES CÉDEX
FRANCE
Téléphone : 99 36 20 00
Télex : UNIRISA 950 473 F
Télécopie : 99 38 38 32

Accumulated reward over the n first operational periods in fault-tolerant computing systems

Gerardo Rubino
Bruno Sericola

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Abstract :

We consider fault-tolerant computing systems with operational periods in which the system performs either correctly or in a degradated mode reached after a reconfiguration. In a degradated mode, possible repairs can put the system in a higher performance level. The system is modeled by a homogeneous Markov reward process with absorbing states. A subset of the state space is distinguished to represent the operational mode of the system. Moreover, a reward rate is associated to each one of its states to quantify the different performance levels. Some recent work has been done to evaluate the distribution of the total time spent in the n^{th} operational period. In this paper, we focus on the accumulated reward over the n first operational periods. The basic result is the distribution of this random variable and an algorithm to compute it. An example of a fault-tolerant system is given to illustrate the paper.

Récompense cumulée sur les n premières périodes opérationnelles dans les systèmes informatiques tolérant les pannes

Résumé :

On considère des systèmes informatiques tolérant les pannes avec des périodes opérationnelles durant lesquelles le système fonctionne soit correctement, soit dans un mode dégradé obtenu aprés une reconfiguration. Dans un mode dégradé, des réparations éventuelles peuvent placer le système à un niveau de performance plus élevé. Le système est modélisé par un processus markovien à taux de récompenses et avec des états absorbants. On différencie un sous-ensemble de l'espace d'état pour représenter le mode opérationnel du système. De plus, un taux de récompense est associé à chacun de ses états pour quantifier les différents niveaux de performance. Lors de travaux récents, on a calculé la distribution du temps total passé dans la $n^{\text{ème}}$ période opérationnelle. Dans ce rapport, nous étudions la récompense cumulée sur les n premières périodes opérationnelles. Le résultat central est une forme close de la distribution de cette variable aléatoire et un algorithme pour la calculer. Ces résultats sont illustrés par un exemple de système tolérant les pannes.

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE
(L. A. 227)
UNIVERSITÉ DE RENNES 1 I. N. S. A. DE RENNES

INSTITUT NATIONAL DE RECHERCHE
EN INFORMATIQUE ET EN AUTOMATIQUE
(LABORATOIRE DE RENNES)

1 Introduction

The increasing need to evaluate fault-tolerance in computer systems leads naturally to study the transient behaviour of models representing such systems. In particular, when fatal failures are taken into account, a steady state analysis is of no interest. Numerous works have been done concerning the transient analysis of fault-tolerant computing systems. In [1], Meyer introduces a useful measure called performability as a composite measure for performance and reliability. Informally, performability can be viewed as the probability that the system does up a certain amount of useful work over a finite mission time. An algorithm for the computation of accumulated reward in a finite utilization period can be found in [2] for degradable computer systems and in [3] in the more general case of repairable computer systems. Some recent work has been done to evaluate the distribution of the n^{th} operational period [4] (without rewards) and the distribution of accumulated reward until absorption [5]. In this paper, we focus on an intermediate measure which is the accumulated reward over the n first operational periods. The distribution of this random variable may be regarded as the probability that the system does up to a certain amount of useful work over the n first operational periods. The system is modeled by a homogeneous Markov reward process with absorbing states.

The remainder of the paper is organized as follows. In Section 2, we give explicitely the distribution of the accumulated reward over the n first operational periods and an algorithm to compute it. We give also a bound of the error made by replacing this distribution by the distribution of the accumulated reward until absorption, that is, by its limit when n grows to infinity. In Section 3, an example of a fault-tolerant computing system is given to illustrate these results. Section 4 is devoted to some conclusions.

2 Model description and solution

The evolution in time of the system is represented by a finite state homogeneous Markov process. Since we are interested in the transient behaviour of the process, all the recurrent states can be collapsed into only one absorbing state. So, let $X = \{X_t, t \geq 0\}$ be a homogeneous Markov process with finite state space denoted by $E = \{1, ..., N, a\}$. States $1, ..., N$ are transient and a is absorbing. The process X is given by its transition rate matrix $A = (A(i,j),\ i,j \in E)$ (infinitesimal generator) where

$$A(i,i) = -\sum_{j \neq i} A(i,j).$$

The initial probability distribution of X is denoted by the row vector α.

In the sequel, we will denote by 1^T the column vector with all its coordinates equal to 1 (we will always use row vectors and $(.)^T$ denotes the transpose operator) and by I the identity matrix, their dimensions being defined by the context.

Let r_i be the reward rate (or the performance level) associated with the state $i \in E$. We denote by B the subset of E containing the operational states and by B' the subset of E containing the other transient states, so we have: $E = B \cup B' \cup \{a\}$. With the operational states we associate strictly positive reward rates and we define the $(|B| \times |B|)$ diagonal matrix R_B whose i^{th} entry is equal to r_i. We will see later that the reward rates associated to the others states have no influence on the results.

The distribution of the n^{th} sojourn time in a subset of states for an irreducible and homogeneous Markov process (without rewards rates) can be found in [4]. For $n \geq 1$, let $S_{i,B,n}$ denote the total time that X spends in state $i \in B$ during its n^{th} visit to B. By definition, over a path in which X visits B only $n' < n$ times, $S_{i,B,n}$ is equal to 0. For every $n \geq 1$, the random variable $S_{B,n}$ representing the accumulated reward over the n^{th} operational period is

$$S_{B,n} \stackrel{\text{def}}{=} \sum_{i \in B} r_i S_{i,B,n}.$$

In this work, we focus on the random variable $TS_{B,n}$ representing the accumulated reward over the n first operational periods. That is, for every $n \geq 1$

$$TS_{B,n} \stackrel{\text{def}}{=} \sum_{k=1}^{n} S_{B,k}.$$

The distribution of $TS_{B,\infty}$ which represents the accumulated reward over all the operational periods (that is, the accumulated reward in the subset B until absorption), can be found in [5].

Let P be the transition probability matrix of the uniformized chain of X. The matrices A and P are related by

$$P = I + A/\lambda$$

where $\lambda \geq max(-A(i,i), i \in E)$ is the rate of a Poisson process independent of the uniformized chain. The main relation between X and its uniformized chain is [6]

$$\mathbb{P}(X_t = j / X_0 = i) = \sum_{k=0}^{+\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} P^k(i,j).$$

We decompose the matrices A, P and the vector α into submatrices and subvectors with respect to the partition $\{B,\ B',\ \{a\}\}$ of the state space E as follows.

$$A = \begin{pmatrix} A_B & A_{BB'} & A_{Ba} \\ A_{B'B} & A_{B'} & A_{B'a} \\ 0 & 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P_B & P_{BB'} & P_{Ba} \\ P_{B'B} & P_{B'} & P_{B'a} \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha = (\alpha_B \ , \ \alpha_{B'} \ , \ \alpha_a)$$

To obtain the distribution of $S_{B,n}$, the reader can find in [4, Theorem 3.1] the analogous result in the particular case of Markov processes without rewards. In our case, we have

$$\mathbb{P}(S_{B,n} \leq t) = 1 - v_1 G^{n-1} e^{R_B^{-1} A_B t} 1^T \tag{1}$$

$$\text{where } v_1 = \alpha_B + \alpha_{B'}(I - P_{B'})^{-1} P_{B'B} = \alpha_B - \alpha_{B'} A_{B'}^{-1} A_{B'B}$$

$$\text{and } G = (I - P_B)^{-1} P_{BB'} (I - P_{B'})^{-1} P_{B'B} = A_B^{-1} A_{BB'} A_{B'}^{-1} A_{B'B}.$$

The proof is basically the same as in [4, Theorem 3.1].

Let us analyze the random variable $TS_{B,n}$. Define the vectors

$$\begin{aligned} u_B(n,t) &\stackrel{\text{def}}{=} (\mathbb{P}(S_{B,n} \leq t / X_0 = i), i \in B), \\ u_{B'}(n,t) &\stackrel{\text{def}}{=} (\mathbb{P}(S_{B,n} \leq t / X_0 = i), i \in B') \end{aligned}$$

and the $| B' | \times | B |$ matrix $H \stackrel{\text{def}}{=} (I - P_{B'})^{-1} P_{B'B}$. Observe that

$$\mathbb{P}(TS_{B,n} \leq t) = \alpha_B u_B^T(n,t) + \alpha_{B'} u_{B'}^T(n,t) + \alpha_a.$$

To derive the distribution of $TS_{B,n}$, we will use the following lemma which gives an expression of the vector $u_{B'}^T(n,t)$ as a function of the vector $u_B^T(n,t)$.

Lemma 2.1 $\forall n \geq 1$, $u_{B'}^T(n,t) = 1^T - H\left(1^T - u_B^T(n,t)\right)$

Proof. For every $i \in B'$,

$$\begin{aligned} \mathbb{P}(TS_{B,n} \leq t / X_0 = i) &= \sum_{j \in B} P(i,j) \mathbb{P}(TS_{B,n} \leq t / X_0 = j) \\ &\quad + \sum_{j \in B'} P(i,j) \mathbb{P}(TS_{B,n} \leq t / X_0 = j) + P(i,a). \end{aligned}$$

This gives in matrix notation

$$\begin{aligned} u_{B'}^T(n,t) &= P_{B'B} u_B^T(n,t) + P_{B'} u_{B'}^T(n,t) + P_{B'a} \\ &= 1^T - H\left(1^T - u_B^T(n,t)\right) \end{aligned}$$

since $P_{B'B} 1^T + P_{B'} 1^T + P_{B'a} = 1^T$. □

The main result of this paper is given by the next theorem.

Theorem 2.2 $\forall n \geq 1$, $\mathbb{P}(TS_{B,n} \leq t) = 1 - w_n e^{M_n t} 1^T$

where $w_n = (v_1\ 0\ \cdots\ 0)$ is the vector with length $n \times |B|$ (each 0 represents here the vector of length $|B|$ with all its elements equal to 0) and M_n is the $(n \times |B|) \times (n \times |B|)$ following matrix.

$$M_n = \begin{pmatrix} Q_1 & Q_2 & 0 & 0 & & \cdots & & & 0 & 0 \\ 0 & Q_1 & Q_2 & 0 & 0 & & \cdots & & 0 & 0 \\ 0 & 0 & Q_1 & Q_2 & 0 & 0 & & \cdots & 0 & 0 \\ \vdots & & & & \ddots & \ddots & & & \vdots & \vdots \\ \vdots & & & & & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & & & \cdots & & & 0 & Q_1 & Q_2 \\ 0 & 0 & & & \cdots & & & & 0 & Q_1 \end{pmatrix}$$

with $Q_1 \stackrel{\text{def}}{=} R_B^{-1} A_B = -\lambda R_B^{-1}(I - P_B)$ and $Q_2 \stackrel{\text{def}}{=} -R_B^{-1} A_{BB'} A_{B'}^{-1} A_{B'B} = \lambda R_B^{-1} P_{BB'} H$. Each 0 represents here the $|B| \times |B|$ matrix with all its elements equal to the real number 0.

Proof. For $n = 1$, the result is immediat since $TS_{B,1} = S_{B,1}$ (see equation (1)).

Let $n \geq 2$ and $i \in B$. Writing the renewal equations for X we have

$$\begin{aligned} \mathbb{P}(TS_{B,n} \leq t / X_0 = i) &= \sum_{j \in B} P(i,j) \int_0^{t/r_i} \mathbb{P}(TS_{B,n} \leq t - r_i s / X_0 = j) \lambda e^{-\lambda s} ds \\ &\quad + \sum_{j \in B'} P(i,j) \int_0^{t/r_i} \mathbb{P}(TS_{B,n-1} \leq t - r_i s / X_0 = j) \lambda e^{-\lambda s} ds \\ &\quad + P(i,a) \int_0^{t/r_i} \lambda e^{-\lambda s} ds \end{aligned}$$

This gives, after a variable change,

$$\begin{aligned} \mathbb{P}(TS_{B,n} \leq t / X_0 = i) &= \sum_{j \in B} \frac{\lambda}{r_i} e^{-\lambda t / r_i} \int_0^t e^{\lambda s / r_i} P(i,j) \mathbb{P}(TS_{B,n} \leq s / X_0 = j) ds \\ &\quad + \sum_{j \in B'} \frac{\lambda}{r_i} e^{-\lambda t / r_i} \int_0^t e^{\lambda s / r_i} P(i,j) \mathbb{P}(TS_{B,n-1} \leq s / X_0 = j) ds \\ &\quad + (1 - e^{-\lambda t / r_i}) P(i,a) \end{aligned}$$

In matrix notation we then obtain

$$u_B^T(n,t) = \lambda R_B^{-1} e^{-\lambda R_B^{-1} t} \int_0^t e^{\lambda R_B^{-1} s} P_B u_B^T(n,s) ds$$

$$+\lambda R_B^{-1}e^{-\lambda R_B^{-1}t}\int_0^t e^{\lambda R_B^{-1}s}P_{BB'}u_{B'}^T(n-1,s)ds$$
$$+(I-e^{-\lambda R_B^{-1}t})P_{Ba}$$

and using Lemma 2.1

$$\begin{aligned} u_B^T(n,t) &= \lambda R_B^{-1}e^{-\lambda R_B^{-1}t}\int_0^t e^{\lambda R_B^{-1}s}P_B u_B^T(n,s)ds \\ &\quad +\lambda R_B^{-1}e^{-\lambda R_B^{-1}t}\int_0^t e^{\lambda R_B^{-1}s}P_{BB'}\left[1^T H\left(1^T-u_B^T(n-1,s)\right)\right]ds \\ &\quad +(I-e^{-\lambda R_B^{-1}t})P_{Ba} \end{aligned}$$

After some algebraic manipulations and using the relation $P_{Ba}=1^T-P_B 1^T-P_{BB'}1^T$ we have

$$\begin{aligned} u_B^T(n,t) &= \lambda R_B^{-1}e^{-\lambda R_B^{-1}t}\left[\int_0^t e^{\lambda R_B^{-1}s}P_B u_B^T(n,s)ds+\int_0^t e^{\lambda R_B^{-1}s}P_{BB'}H u_B^T(n-1,s)ds\right] \\ &\quad +(I-e^{-\lambda R_B^{-1}t})(1^T-P_B 1^T-P_{BB'}H 1^T) \end{aligned}$$

Taking now the derivative with respect to the variable t, we obtain

$$\frac{d}{dt}u_B^T(n,t) = \lambda R_B^{-1}(I-P_B)(1^T-u_B^T(n,t))-\lambda R_B^{-1}P_{BB'}H(1^T-u_B^T(n-1,t))$$

that is

$$\frac{d}{dt}u_B^T(n,t) = -Q_1(1^T-u_B^T(n,t))-Q_2(1^T-u_B^T(n-1,t)) \tag{2}$$

Let us denote by $l_B(n,t)$ the vector

$$l_B(n,t) \stackrel{\text{def}}{=} \left(1^T-u_B^T(n,t);\ \ 1^T-u_B^T(n-1,t);\ \ \cdots;\ \ 1^T-u_B^T(1,t)\right)$$

whose length is $n\times|B|$. For every $n\geq 1$, equation (2) leads to

$$\frac{d}{dt}l_B^T(n,t)=M_n l_B^T(n,t)$$

that is, since $l_B^T(n,0)=1^T$,

$$l_B^T(n,t)=e^{M_n t}1^T$$

Finally,

$$\begin{aligned} \mathbb{P}(TS_{B,n}\leq t) &= \alpha_B u_B^T(n,t)+\alpha_{B'}u_{B'}^T(n,t)+\alpha_a \\ &= 1-\alpha_B(1^T-u_B^T(n,t))-\alpha_{B'}(1^T-u_{B'}^T(n,t)) \\ &= 1-v_1(1^T-u_B^T(n,t)) \quad \text{using Lemma 2.1} \\ &= 1-w_n l_B^T(n,t) \end{aligned}$$

which is the proposed form of the distribution. $\square$

Remark that when n grows to infinity, we obtain the distribution of $TS_{B,\infty}$ using equation (2), that is

$$\mathbb{P}(TS_{B,\infty} \leq t) = 1 - v_1 e^{(Q_1+Q_2)t} 1^T.$$

In order to compute the distribution of $TS_{B,n}$ for a fixed n, one can proceed as follows. Let β be a positive real number such that $\beta \geq max(Q_1(i,i), i \in B)$ and denote by P_n the $(n\times|B|) \times (n\times|B|)$ matrix

$$P_n \stackrel{\text{def}}{=} I + M_n/\beta.$$

The matrix P_n is a function of the $|B|\times|B|$ matrices $P_1 \stackrel{\text{def}}{=} I + Q_1/\beta$ and $P_2 \stackrel{\text{def}}{=} Q_2/\beta$. With this notation, we obtain for every $n \geq 1$

$$\mathbb{P}(TS_{B,n} \leq t) = \sum_{k=0}^{+\infty} e^{-\beta t} \frac{(\beta t)^k}{k!} (1 - w_n P_n^k 1^T)$$

Denoting now by $x_B^T(n,k)$ the vector composed by the $|B|$ first elements of $P_n^k 1^T$, it follows that

$$\mathbb{P}(TS_{B,n} \leq t) = \sum_{k=0}^{K} e^{-\beta t} \frac{(\beta t)^k}{k!} (1 - v_1 x_B^T(n,k)) + e(K)$$

where $e(K)$ is the error obtained by truncating the previous series to K steps. It is easy to verify that

$$e(K) \leq 1 - \sum_{k=0}^{K} e^{-\beta t} \frac{(\beta t)^k}{k!}$$

and so K can be evaluated beforehand for a given error tolerance. Since P_n (as M_n) is a block upper triangular matrix, we can decompose the matrix P_n^k into four submatrices in the following way.

$$P_n^k = \begin{pmatrix} P_1^k & W_{n-1,k} \\ 0 & P_{n-1}^k \end{pmatrix}.$$

For instance, $W_{n-1,1}$ is the $|B| \times ((n-1)\times|B|)$ matrix $(P_2\ 0\ \cdots\ 0)$. These considerations lead to recursive relations.

$$W_{n-1,k} = P_1 W_{n-1,k-1} + W_{n-1,1} P_{n-1}^{k-1}$$

and

$$\begin{aligned}
x_B^T(n,k) &= P_1^k 1^T + W_{n-1,k} 1^T \\
&= P_1^k 1^T + P_1 W_{n-1,k-1} 1^T + W_{n-1,1} P_{n-1}^{k-1} 1^T \\
&= P_1^k 1^T + P_1 W_{n-1,k-1} 1^T + P_2 x_B^T(n-1,k-1) \\
&= P_1 (P_1^{k-1} 1^T + W_{n-1,k-1} 1^T) + P_2 x_B^T(n-1,k-1) \\
&= P_1 x_B^T(n,k-1) + P_2 x_B^T(n-1,k-1)
\end{aligned}$$

This last relation induces a simple algorithm to evaluate $x_B^T(n,k)$ with initial values $x_B^T(0,j)=0$ for every $j\geq 0$ and $x_B^T(1,0)=1^T$.

Another quantity of interest is the error committed by replacing the distribution of $TS_{B,n}$ by the distribution of $TS_{B,\infty}$.

Remarking that

$$x_B^T(n,k)=(P_1+P_2)^k 1^T \quad \text{for every } 0\leq k\leq n-1$$

we obtain

$$\mathbb{P}(TS_{B,n}\leq t)=\sum_{k=0}^{n-1} e^{-\beta t}\frac{(\beta t)^k}{k!}(1-v_1(P_1+P_2)^k 1^T)+e(n-1)$$

On the other hand, we have

$$\mathbb{P}(TS_{B,\infty}\leq t)=\sum_{k=0}^{n-1} e^{-\beta t}\frac{(\beta t)^k}{k!}(1-v_1(P_1+P_2)^k 1^T)+e'(n-1)$$

This leads to

$$\mathbb{P}(TS_{B,n}\leq t)-\mathbb{P}(TS_{B,\infty}\leq t)=e(n-1)-e'(n-1)\leq e(n-1)=1-\sum_{k=0}^{n-1} e^{-\beta t}\frac{(\beta t)^k}{k!}$$

Finally, let us consider the moments of the accumulated reward $TS_{B,n}$. The first moment can be obtained directly from the distribution of the random variable $S_{B,n}$.

$$\mathbb{E}(TS_{B,n})=\sum_{k=1}^{n}\mathbb{E}(S_{B,k})=-v_1\,(I-G^n)\,(I-G)^{-1}Q_1^{-1}1^T$$

which can also be written

$$\mathbb{E}(TS_{B,n})=-v_1\,(I-G^n)\,(Q_1+Q_2)^{-1}1^T.$$

To get higher order moments we need the distribution of $TS_{B,n}$ since the variables $S_{B,n}$ are not independent. As a direct corollary of Theorem 2.2, we have

$$\mathbb{E}(TS_{B,n}^k)=(-1)^k k!\,w_n M_n^{-k}1^T.$$

It is immediat to verify that the $[i,j]$ block of M_n^{-1} is given by

$$M_n^{-1}[i,j]=\begin{cases} 0 & \text{if } i>j \\ (-1)^{j-i}Q_1^{-1}\left(Q_2Q_1^{-1}\right)^{j-i}=G^{j-i}Q_1^{-1} & \text{if } i\leq j \end{cases}.$$

When $n=+\infty$, the moments of the total reward $TS_{B,\infty}$ are

$$\mathbb{E}(TS_{B,\infty})=(-1)^k k!\,v_1(Q_1+Q_2)^{-k}1^T.$$

To finish this section, we will develop the computation for the important particular case of second order moments.

$$\begin{aligned}
\mathbb{E}(TS^2_{B,n}) &= 2w_n M_n^{-2} 1^T \\
&= 2v_1 \sum_{j=1}^{n} M_n^{-2}[1,j] 1^T \\
&= 2v_1 \sum_{j=1}^{n} \sum_{h=1}^{j} G^{h-1} Q_1^{-1} G^{j-h} Q_1^{-1} 1^T \\
&= 2v_1 \sum_{h=1}^{n} \sum_{j=h}^{n} G^{h-1} Q_1^{-1} G^{j-h} Q_1^{-1} 1^T \\
&= 2v_1 \sum_{h=1}^{n} G^{h-1} Q_1^{-1} \sum_{i=0}^{n-h} G^{i} Q_1^{-1} 1^T \\
&= 2v_1 \sum_{h=1}^{n} G^{h-1} Q_1^{-1} \left(I - G^{n-h+1}\right) (I-G)^{-1} Q_1^{-1} 1^T \\
&= 2v_1 (I - G^n)(Q_1+Q_2)^{-2} 1^T - 2v_1 \left(\sum_{h=1}^{n} G^{h-1} Q_1^{-1} G^{n-h+1} \right) (Q_1+Q_2)^{-1} 1^T.
\end{aligned}$$

3 Application

In this section we will illustrate the proposed techniques with an example of a fault-tolerant system composed by two elements. Let us consider a system with two identical processors working independently and simultaneously. Each processor has an exponentially distributed life-time with rate ϕ. When there is a fault, we assume that, with a constant probability c, it is not fatal and the system starts a recovery procedure that tries to put it back in an operational state. After an exponentially distributed delay with rate ρ (in which there is no useful work), this reconfigurating procedure succeeds with constant probability d and puts the system again in an operational state but with only one unit performing user tasks. The time needed to repair the faulted unit is exponentially distributed with rate μ. The corresponding continuous time homogeneous Markov process is represented in Figure 1 where states U_i, $i = 1, 2$ correspond to the system with i processors operational, that is, performing useful user tasks. State F_1 represents the system trying to reconfigure after the fault of a processor; from the user point of view, the system is not operational in that state. The absorbing state 0 corresponds to the system completly down.

The subset of the state space composed by the operational states is $B = \{U_2, U_1\}$. To distinguish between the states in B, we consider rewards associated to each one. For instance, the reader may think of r_i as a quantity proportional to the thruput of the system when there are i processors performing user

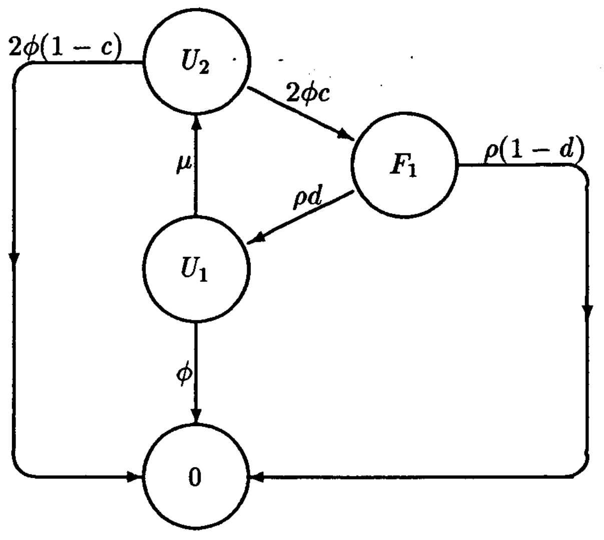

Figure 1: 2 components with failure and repair

tasks, $i = 1, 2$. The random variable $TS_{B,\infty}$ represents then "the total amount of useful work done by the system until its crash" and this paper focus on the analysis of $TS_{B,n}$, "the total amount of useful work done by the system during the n first operational periods".

The basic data for the analysis of the distribution of $TS_{B,n}$ are

$$Q_1 = \begin{pmatrix} \frac{-2\phi}{r_2} & 0 \\ \frac{\mu}{r_1} & -\frac{\phi+\mu}{r_1} \end{pmatrix}$$
$$Q_2 = \begin{pmatrix} 0 & \frac{2\phi cd}{r_2} \\ 0 & 0 \end{pmatrix}$$

Let us consider a numerical example with the following parameter values.

$$\phi = 0.005$$
$$\mu = 1$$
$$c = 0.99$$
$$d = 0.9$$

Assume a perfect parallelism in the system and take $r_1 = 1$ and $r_2 = 2$. We computed the distribution function of some of the random variables $TS_{B,n}$. The resulting curves are plotted in Figure 2 together with the distribution of $TS_{B,\infty}$.

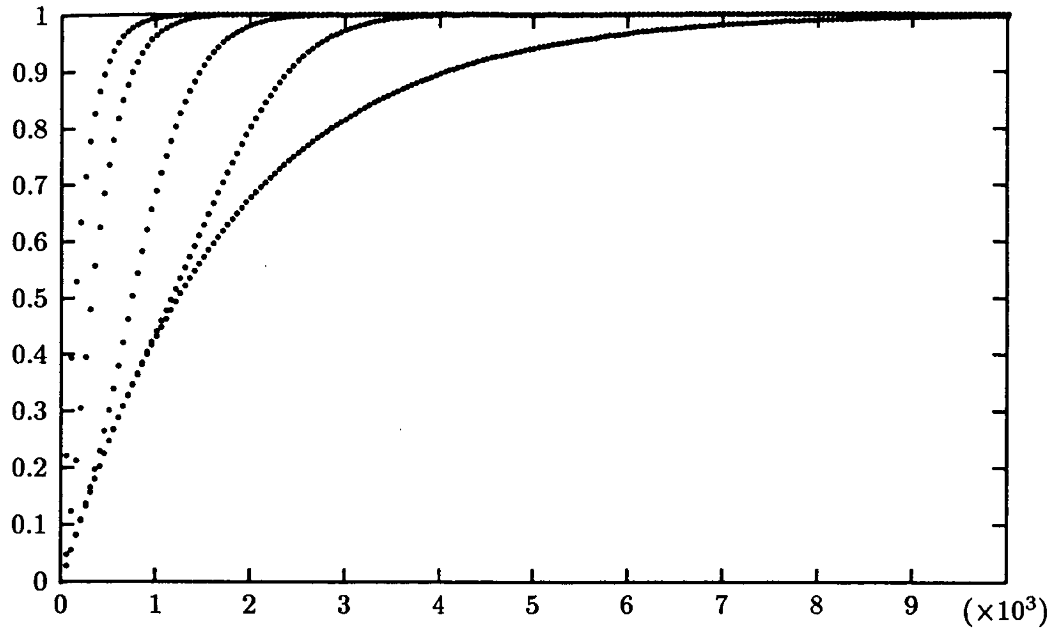

Figure 2: From top to bottom, $\mathbb{P}(TS_{B,n} \leq t)$, $n = 1, 2, 5, 10, \infty$.

In Table 3, we give the values of the corresponding first and second order moments (V denotes the variance and CV the coefficient of variation, that is, the square root of variance divided by the absolute value of the mean).

n	1	2	5	10	∞
$\mathbb{E}(TS_{B,n})$	200	378.20	800.43	1239.39	1770.97
$V(TS_{B,n})$	4×10^4	7.9525×10^4	2.4553×10^5	7.1782×10^5	3.1362×10^6
$CV(TS_{B,n})$	1.0	0.74564	0.61905	0.68360	0.99998

Table 1: Mean and variance in the example.

Note the coefficient of variaton is strictly less than one for $n > 1$. It is minimal for $n = 5$.

Last, observe that in this particular example, all the analyzed distributions do not depend on the parameter p since B' is reduced to only one state. Such a depenency appears if we consider "efficiency" measures as, for instance, the ratio between $\mathbb{E}(TS_{B,n})$ and the mean time elapsed until the n^{th} operational period.

4 Conclusions

The main contribution of this work is the closed form of the accumulated reward distribution over the n first operational periods in fault-tolerant computing systems. A simple algorithm to compute it is given and for large values of n, an error bound can be used to approach this distribution by the accumulated reward until absortion. Moments are also considered and special attention is devoted to the two first ones. Formally, given a homogeneous Markov process, the distribution of linear combinations of the n first sojourn times in proper subsets of the state space has been explicitely calculated.

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