Optimal control in a failure prone manufacturing system

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Contrôle Optimal d'un Système de Production perturbé par des Pannes

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Abstract:

This paper addresses the optimal control in a failure prone manufacturing system. A discrete-time model is used. A single commodity is produced and there is a constant demand. The optimal control policy minimizes the long run average cost incurred by holding inventory and by failing to meet the demand.

We show that the optimal control policy in any finite horizon problem is characterized by a critical number, which we call the *ideal inventory level* (or hedging point). The system should not produce at all if the resulting inventory level exceeds the ideal inventory level, and it should produce at full capacity if the resulting inventory level is less than the ideal inventory level, it should produce exactly enough to meet the ideal inventory level if the ideal inventory level is attainable.

We also show that the ideal inventory level increases and converges to a finite value as the horizon length increases. This implies that the optimal control policy in the infinite horizon problem is also characterized by a critical number.

Résumé:

Dans ce papier, nous étudions le contrôle optimal d'un système perturbé par des pannes. Le modèle utilisé est à temps discret. Le système fabrique un seul type de produit dont la demande est constant. Le contrôle consiste à minimiser le coût moyen engendré par les stocks et par les retards sur l'horizon infini.

Pour le cas de l'horizon fini, nous montrons que la politique optimale de contrôle pour chaque période est caractérisée par un nombre positif que nous appelons le niveau idéal du stock. La politique optimale de contrôle consiste à conduire l'état du stock vers ce niveau idéal. Plus précisément, on doit produire à la capacité maximale si celle-ci conduit à un stock inférieur ou égal au niveau idéal. On ne doit utiliser qu'une partie de la capacité (et même ne pas produire) dans le cas contraire de manière à approcher au mieux le niveau idéal.

Nous montrons ensuite que le niveau idéal du stock croît et converge vers une valeur finie lorsque la longueur de l'horizon tend vers l'infini. Cela implique que la politique optimale de contrôle dans le cas de l'horizon infini est également caratérisée par un niveau idéal du stock.

Keywords: Manufacturing Systems, Failures, Production Control

Mots-clés: Système de Production, Pannes, Contrôle Optimal

1. Introduction

The manufacturing system considered in this paper produces a single commodity. We use a **discrete-time model**. The manufacturing system is subject to breakdown, and it can be in two states in each period: the running state and the breakdown state. The transition between the two states is modelled as a discrete-time Markov chain. The time between failures is modelled as a geometrically distributed random variable with mean p^{-1}, while the repair time is modelled as a geometrically distributed random variable with mean r^{-1}. In other words, if the system is in running (resp. breakdown) state, it will be in breakdown (resp. running) state in the next period with probability p (resp. r). When the system is in running state, the production in each period can be up to a maximal quantity U; when it breaks down, it cannot produce at all. In the following, the quantity U is called production capacity.

The demand in each period is equal to a constant d. We assume that $U > d > 0$.

Let x_t be the inventory level at the end of period t. It may be negative, which corresponds to a backlog. Let u_t be the amount of production in period t. The inventory level can be determined by

$$x_t = x_t + u_t - d.$$

We suppose that the positive inventories incur a holding cost of c^+ per unit commodity per time period, while negative inventories incur a cost of c^-, with $c^+>0$ and $c^->0$.We seek an optimal control policy u_t such that the following performance index is minimized

$$\lim_{T\to\infty} \frac{1}{T} E\left[\sum_{t=1}^{T} (c^+ x^+_t + c^- x^-_t)\right] \qquad (1)$$

where $x^+_t := \max\{0, x_t\}$, and $x^-_t := \max\{0, - x_t\}$.

Let α_t denote the system state as follows

$$\alpha_t \begin{cases} =1 & \text{if the system is in the running state;} \\ =0 & \text{if the system is under repair.} \end{cases}$$

Clearly, $u_t = 0$ whenever $\alpha_t = 0$, and so we only need to determine the optimal production u_t when the system is in the running state, i.e. $\alpha_t = 1$.

We show in Section 4 that there exists an *ideal inventory level* z^* toward which the production should be aimed, that is

$$u_t \begin{array}{ll} = 0 & \text{if } x_{t-1} - d \geq z^* \\ = U & \text{if } x_{t-1} + U - d \leq z^* \\ = z^* + d - x_{t-1} & \text{otherwise.} \end{array} \tag{2}$$

When the system is in running state, it should produce nothing if the resulting inventory x_t exceeds the ideal inventory level, it should produce at full capacity if the resulting inventory level is less than the ideal inventory level, it should produce exactly enough to meet the ideal inventory level if the ideal inventory level is attainable. In the following, we call this policy a *critical number policy* with the critical number z^*.

Clearly, the production capacity of the system in the long term is equal to $Ur/(p+r)$. If $Ur/(p+r) \leq d$, the system does not have the capacity to meet the demand even if it produces at full capacity when it is up. Thus, $z^* = \infty$ under these conditions. We do not consider this trivial case in the following.

Literature survey

The production control in a failure prone manufacturing system is first addressed by Kimemia and Gershwin[1983], and pursued by Gershwin, Akella and Choong[1985], Akella, Choong, Gershwin[1984], Maimon and Gershwin[1988]. They use a continuous-time model in which machine failures are modelled as some Markov process. They show that the optimal control satisfies a Hamilton-Jacobi-Bellman (HJB) dynamic programming equation.

Akella and Kumar[1986], and Bielecki and Kumar[1988] address a simplified version of this problem with only one product and one machine. They show that the optimal control policy is in fact a critical number policy and give the close form of the optimal critical number (or ideal inventory level). Bielecki and Kumar claim that a zero-inventory policy may be optimal even in the presence of uncertainty.

Sharifnia[1988] addresses another simplified version of this problem with one product but multiple failure modes. He establishes equations for the ideal inventory levels.

Outline of paper

The paper is organized as follows. In Section 2, we present the main results of this paper and gives some informal arguments. In Section 3, we address the computation of the ideal inventory level. No analytical solution is obtained, but we propose a simulation-based technique. In Section 4, we show that the control policy in any finite horizon problem is a critical number policy (or hedging point strategy) and the critical number increases and converges to a finite value as the horizon length increases. This implies the optimality of the critical number policy in the infinite horizon problem. In Section 5, we show that the average inventory cost when using any critical number policy is bounded. Conclusions are given in Section 6.

2. Main Results

This paper is motivated not only by showing the optimality of critical number policy, but also by discovering why the optimal policy is a critical number policy. In this section, we present the main results of this paper and give some informal arguments.

We first examine a modified version of problem (1) with finite horizon T. We show that there exists an ideal inventory level z_t toward which the production u_t should be aimed, that is

$$
\begin{aligned}
u_t &= 0 && \text{if } x_{t-1} - d \geq z_t \\
&= U && \text{if } x_{t-1} + U - d \leq z_t \\
&= z_t + d - x_{t-1} && \text{otherwise.}
\end{aligned}
$$

The ideal inventory level z_t is time-dependent. Intuitively, the system breaks down more often over periods t-1, t, ..., T when given $\alpha_{t-1} = 1$ than it does over periods t, t+1, ..., T when given $\alpha_t = 1$. So one should try to keep higher ending inventory level in period t-1 than it should in period t. This implies that

$$z_1 \geq z_2 \geq \ldots \geq z_T = 0.$$

Figure 1 shows an example with T =10. z_t is the ideal inventory level and x_t is the optimal inventory level. During the first 5 periods, the system is up. First, the system produces at full capacity in order to meet the ideal inventory levels z_t. The ideal inventory level is met at the end of period 3. After that, the optimal policy consists in following the ideal invetory trajectory. The system fails to work in period 6 and the optimal inventory trajectory x_t leaves the ideal inventory trajectory.

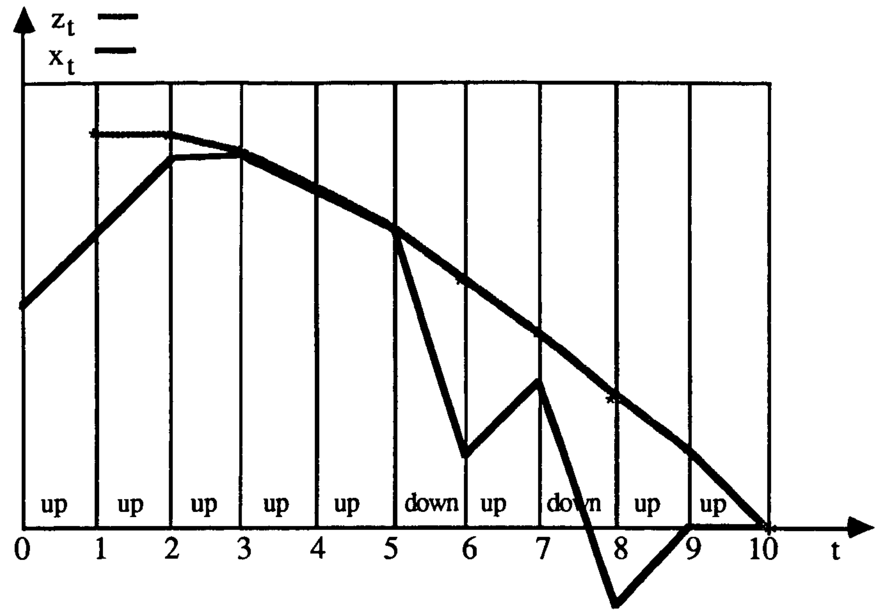

Figure 1: Ideal inventory and optimal inventory trajectories

Since the ideal inventory level z_t increases as t decreases for any finite horizon problem, the ideal inventory level z_1 increases as the horizon length T increases. Suppose that z_1 increases without limit as T increases. Then, the optimal control policy in the infinite horizon problem is to always produce at full capacity. Since the long term capacity of the system is greater than the demand, the inventory level will also increase without limit. Intuitively, this cannot be the optimal policy. We conclude that the ideal inventory level z_1 converges to a finite value as T increases. This yields that the optimal policy in the infinite horizon problem is also a critical number policy. To summarize, we have

$$z^* = \lim_{T\to\infty} z_1.$$

3. Ideal Inventory Level Computation

In this section, we examine the computation of the ideal inventory level z*. We show that the zero-inventory policy may be optimal even in the presence of machine failures.

Consider a policy $u_t = \pi^z(x_{t-1})$ given by a critical number z. It is

$$\begin{aligned} \pi^z(x_{t-1}) &= 0 && \text{if } \alpha_t = 0 \\ &= 0 && \text{if } x_{t-1} - d \geq z,\ \alpha_t = 1 \\ &= U && \text{if } x_{t-1} + U - d \leq z,\ \alpha_t = 1 \\ &= z^* + d - x_{t-1} && \text{otherwise.} \end{aligned} \tag{3}$$

The only quantity to choose is z.

We assume that when the policy (3) is used, the process (x_t, α_t) has a steady-state probability distribution

$$P^z(A, \alpha) := \lim_{t \to \infty} \text{Prob}\,(x_t \in A, \alpha_t = \alpha)$$

where A is an interval set. Define

$$Q^z(A) := P^z(A,0) + P^z(A,1).$$

The average inventory cost corresponding to the policy (3), denoted by J(z), can be computed by

$$J(z) = -\int_{-\infty}^{0} c^- x\, Q^z(dx) + \int_{0}^{+\infty} c^+ x\, Q^z(dx) \tag{4}$$

By similar procedure as that used by Bielecki and Kumar[1988], the following properties can be proved:

$$P^z((z, \infty), \alpha) = 0 \quad \text{for } \alpha = 0,1 \tag{5.1}$$
$$P^z(\{z\},1) =: \omega > 0 \tag{5.2}$$
$$P^z(A, \alpha) = P^{z+\theta}(A + \theta, \alpha) \qquad \text{for } \alpha = 0,1 \tag{5.3}$$

If the inventory level x_0 starts with a value greater than z, it will be depleted by the demand d until it hits z. After that, the policy (3) ensures that the inventory level never exceeds z. Thus, the property (5.1) follows. The property (5.2) claims that there is a strictly positive probability mass at z. Whenever x_t hits z with $\alpha_t = 1$, it stays at z until α_t switches to zero. So x_t spends a positive fraction of time at exactly the level z. This implies that the point z has a positive probability mass.

Property (5.3) means that the translation in z merely translates the probability distribution P^z. The properties (5.1), (5.2), and (5.3) can rewritten as follows

$$P^z((z,\infty),\alpha) = 0 \qquad \text{for } \alpha = 0,1 \tag{6.1}$$

$$P^z(\{z\},1) = P^0(\{0\},1) =: \omega > 0 \tag{6.2}$$

$$P^z(A,\alpha) = P^0(A - z,\alpha) \qquad \text{for } \alpha = 0,1 \tag{6.3}$$

We are now ready to optimize J(z). From the above properties, moving z from 0 to a negative value merely shifts the distribution P^0 by z units. From property (6.1), we have

$$J(z) - J(0) = c^- \,|z| \qquad \text{for } z \le 0.$$

Hence the ideal inventory level z* will never be negative.

Clearly, J(z) is a continous convex and piece-wise differentiable function.Its left-side derivative and right-side derivative always exist . For any $z \ge 0$, they can be computed as follows

$$\lim_{x \to z-} \frac{dJ(x)}{dx} = -c^- Q^0((-\infty, -z]) + c^+ Q^0((-z, 0]),$$

and

$$\lim_{x \to z+} \frac{dJ(x)}{dx} = -c^- Q^0((-\infty, -z)) + c^+ Q^0([-z, 0]).$$

A necessary and sufficient condition of the ideal inventory level z* is that the lef-side derivative is negative and the right-side derivative is positive, i.e.

$$-c^- Q^0((-\infty, -z^*]) + c^+ Q^0((-z^*, 0]) \le 0,$$

and (7)

$$-c^- Q^0((-\infty, -z^*)) + c^+ Q^0([-z^*, 0]) \ge 0.$$

The necessary and sufficient condition of z* = 0 is as follows:

$$-c^- Q^0((-\infty, 0)) + c^+ Q^0([0, 0]) \ge 0.$$

It implies that the ideal inventory level will be zero if the probability mass ω is great enough.

We have not obtained the analytic solution of the probability mass ω and the probability distribution function $Q^0(-z,0)$. They can be evaluated by simulation. Nevertherness, we give in the following example an analytical solution when U=2d.

Example: Let us consider the case in which the production capacity is equal to exactly twice of the demande, i.e. $U = 2d$. The assumption that the long term capacity is greater than the demand implies that

$$r > p$$

where r and p are the repair rate and failure rate respectively. We show in Appendix 1 that

$$z^*/d = \max\{0, \lceil v \rceil\}$$

where $\lceil v \rceil$ denotes the smallest integer greater or equal to v and

$$v = \frac{\ln\left(1 + \frac{r-p}{p(2-p-r)}\right) - \ln \frac{c^+ + c^-}{c^+}}{\ln(1-r) - \ln(1-p)}.$$

The necessary and sufficient condition that $z^* = 0$ is as follows:

$$\frac{r}{p} \geq 1 + \frac{2(1-p)}{p + c^+/c^-}.$$

Fix the problability p. The long term production capacity decreases toward the demand as the probability r decreases toward p. Then, the ideal inventory level increases without limit. The value of z^* given above verifies these issues, i.e.

$$\lim_{r \to p+} z^* \to \infty.$$

Consider now the continuous-time model as a limit case of the discrete-time model. Let Δ be the elementary interval length and let δ, λ and μ be the demande rate, the failure rate and the repair rate respectively, i.e.

$$d = \delta\, \Delta,\ p = \lambda\, \Delta, \text{ and } r = \mu\, \Delta.$$

It is quite easy to show that for $\mu > \lambda$,

$$\lim_{\Delta \to 0} z^* = \frac{\delta}{\mu - \lambda} \ln \frac{2\lambda(c^+ + c^-)}{c^+(\lambda + \mu)} \qquad \text{if } \frac{2\lambda(c^+ + c^-)}{c^+(\lambda + \mu)} > 1;$$

$$= 0 \qquad \text{otherwise}.$$

This limit is equal to the ideal inventory value given by Bielecki and Kumar[1988] for the continuous-time model. This leads us to believe that the other results of this paper may be true for the continuous-time model.

4. Optimality of Critical Number Policy

In this section, we show that the optimal control policy is a critical number policy. We first show that the optimal control policy is in fact a critical number policy in any finite time horizon problem and the critical number increases and converges to a finite value as the horizon length increases. This yields the optimality of critical number policy in the infinite horizon problem.

Let us consider a modified version of problem (1) with horizon T. Let $F_t(T,i,\xi)$ be minimal expected inventory cost over periods t, t+1, ..., T, when $x_t = \xi$ and $\alpha_t=i$, i.e.

$$F_t(T, i, \xi) = \min E\left\{ \sum_{s=t}^{T} g(x_s) \right\}$$

subject to

$$\begin{aligned} &x_s = x_{s-1} + u_s - d && \text{for } t < s \le T \\ &u_s \le \alpha_s U && \text{for } t < s \le T \\ &x_t = \xi \text{ and } \alpha_t = i \end{aligned}$$

where $g(x) = c^+x^+ + c^-x^-$.

The expected cost function $F_t(T,i,\xi)$ has the following properties:

1) $F_t(T,i,\xi)$ is a continuous, convex and piece-wise linear function in ξ;
2) For any $\forall\, \xi_1 < \xi_2 \le 0$, $F_t(T,i,\xi_1) > F_t(T,i,\xi_2)$;
3) There exists a finite $\xi^*_t(T,i) \ge 0$ that minimizes $F_t(T,i,\xi)$.

These properties are quite easy to show. For the clarity of the presentation, we do not give the proof.

It is clear that for any initial inventory level x_{t-1}, the optimal control u_t should minimize the expected cost $F_t(T,1,x_t)$ when the system is up. Intuitively, the convexity of the function $F_t(T,1,\xi)$ implies that the optimal control policy should drive the ending inventory x_t toward $\xi^*_t(T,1)$. The following theorem confirms these arguments.

Theorem 1: For the horizon T problem, the optimal control policy is a critical number policy. More precisely, there exist $z_t \geq 0$ for t =1, 2, ..., T such that when given x_{t-1}, the optimal control policy u_t is given by:

$$
\begin{aligned}
u_t &= 0 && \text{if } \alpha_t = 0 \\
&= 0 && \text{if } x_{t-1} - d \geq z_t,\ \alpha_t = 1 \\
&= U && \text{if } x_{t-1} + U - d \leq z_t,\ \alpha_t = 1 \\
&= z_t + d - x_{t-1} && \text{if } z_t - U \leq x_{t-1} - d \leq z_t,\ \alpha_t = 1.
\end{aligned}
$$

Proof: Notice that the ideal inventory level z_t is equal to $\xi^*_t(T,1)$. It is clear that u_t=0 when $\alpha_t = 0$. We examine separately the optimality in the three cases with $\alpha_t = 1$ where the optimal control is given.

Case 1: $x_{t-1} - d \geq \xi^*_t(T,1)$, $\alpha_t = 1$

Since $F_t(T,1,\xi)$ is convex in ξ and for any $u_t > 0$,

$$x_t = x_{t-1} + u_t - d > x_{t-1} - d \geq \xi^*_t(T,1),$$

then

$$F_t(T,1,x_t) \geq F_t(T,1,x_t - d).$$

This implies that $u_t = 0$ is the optimal control.

Case 2: $x_{t-1} + U - d \leq \xi^*_t(t,T,1)$, $\alpha_t = 1$

For any $u_t < U$, we have

$$x_t = x_{t-1} + u_t - d < x_{t-1} + U - d \leq \xi^*_t(T,1).$$

This yields that

$$F_t(T,1,x_t) \geq F_t(T,1,x_{t-1} + U - d).$$

Thus, $u_t = U$ is the opimal control.

Case 3: $\xi^*_t(T,1) - U < x_{t-1} - d < \xi^*_t(T,1)$, $\alpha_t = 1$

Let $u_t = \xi^*_t(T,1) + d - x_{t-1}$. The resulting x_t is equal to $\xi^*_t(T,1)$. Thus, $\xi^*_t(T,1) + d - x_{t-1}$ is the optimal control.

Q.E.D.

Since $F_T(T,1,\xi) = g(\xi)$, the function $F_T(T,1,\xi)$ reaches its minimum at $\xi = 0$. This yields that

$$z_T = \xi^*_T(T,1) = 0.$$

We examine now the dynamique property of the ideal inventory level. The following theorem claims that the system should try to keep higher ending inventory level in any period that it should do in the subsequent period.

Theorem 2: Let z_t be the optimal critical numbers for the horizon T problem. Then, we have

$$z_1 \geq z_2 \geq \ldots \geq z_T = 0,$$

and

$$z_t \leq z_{t+1} + d \qquad \forall 1 \leq t \leq T\text{-}1.$$

Proof: We will show the theorem by induction.

Since $z_{T-1} \geq 0$ and $z_T = 0$, it follows that $z_{T-1} \geq z_T$. $\forall\ \xi \geq d$, the expected cost can be $F_{T-1}(T,1,\xi)$ computed as

$$F_{T-1}(T,1,\xi) = c^+ \xi + c^+ (\xi - d).$$

This implies that

$$z_{T-1} \leq z_T + d.$$

Suppose by induction that the therorem is true for any $t \geq k$. That is

$$z_k \geq z_{k+1} \geq \ldots \geq z_T = 0$$

and

$$z_t \leq z_{t+1} + d \qquad \forall k \leq t \leq T\text{-}1.$$

By Lemma 1, we have

$$z_k \leq z_{k-1} \leq z_k + d.$$

Thus the theorem is proved.

Q.E.D.

Lemma 1: For any $k \leq T\text{-}1$, if the conditions

$$z_{t+1} \leq z_t \leq z_{t+1} + d \qquad \forall k \leq t \leq T\text{-}1$$

hold, then

$$z_k \leq z_{k-1} \leq z_k + d.$$

Proof: (see appendix 2).

Now we are ready to show the stability of the optimal control policy as the time horizon T increases. Intuitively, the average inventory cost when using the optimal control policy are most likely to be bounded. Thus, the ideal inventory level z_1 cannot increases without limit when the time horizon T increases. The following theorem addresses these issues.

Theorem 3: The ideal inventory level z_1 converges to a finite value when T increases. That is

$$\lim_{T \to \infty} z_1 = z^* < +\infty.$$

Proof: Let p^i_t be the probability that the system is in state i in period t given $\alpha_1 = 1$. p^0_t and p^1_t satisfy the following conditions:

$$p^1_t = \frac{r}{p+r} + \frac{p}{p+r}(1-p-r)^{t-1} \geq \min\left(p, \frac{r}{p+r}\right),$$

and

$$p^1_t + p^0_t = 1.$$

From the optimality of z_t and $\xi^*_t(T, 0)$, we have

$$\begin{aligned} &F_1(T,1, z_1) \\ &\quad = c^+ z_1 + E\{F_t(T, \alpha_2, x_2) / \alpha_1 = 1 \text{ and } x_1 = z_1\} \\ &\quad \geq c^+ z_1 + p^1_2 F_2(T,1,z_2) + p^0_2 F_2(T,0,\xi^*_2(T,0)). \end{aligned}$$

Similarly, it is true that $\forall\, t \leq T$,

$$\begin{aligned} &p^1_{t-1} F_{t-1}(T,1,z_{t-1}) + p^0_{t-1} F_{t-1}(T,0,\xi^*_{t-1}(T,0)) \\ &\quad \geq p_{1t-1}(t\text{-}1)\, c^+ z_{t-1} + p^1_t F_t(T,1,z_t) + p^0_t F_t(T,0,\xi^*_t(T,0)). \end{aligned}$$

Consequently, we have

$$F_1(T,1, z_1) \geq \sum_{t=1}^{T} p^1{}_t\, c^+ z_t.$$

By combining the above equations, we obtain

$$F_1(T,1, z_1) \geq B \sum_{t=1}^{T} z_t.$$

where $B = c^+ \min(p, r/(p+r)) > 0$.

In Section 5, we shall show that the average inventory cost J(0) exists and is bounded when the control policy (3) is applied with z = 0. This implies that the minimal average inventory cost is bounded, i.e.

$$\frac{F_1(T, 1, z_1)}{T} \leq J(0) + o(1) \qquad \forall\ T.$$

This implies that the mean of z_t is bounded. By theorem 2, z_1 converges to a finite number when T increases.

Q.E.D.

From theorem 3, we conclude that the optimal control policy for the infinite horizon problem is also a critical number policy. The critical number can be defined as

$$z^* = \lim_{T \to \infty} z_1.$$

5. Stability of Critical Number Policy

The purpose of this section is to show that the average inventory cost J(0) given by the policy (3) with z = 0 does exist and is bounded. Thus, J(z) exists and is bounded for any other z. The following theorem proves these arguments. Notice that the proof of theorem 4 is similar to the one given by Bielecki and Kumar [1988].

Theorem 4: Let x_t be the inventory level when using policy (3) with $z = 0$ and $x_0=0$. Then it follows that

$$-\lim_{t\to\infty} E[x_t] \leq A^* d$$

where A is a positive constant.

Proof: When applying the critical number policy (3) π^0 with $x_0 = 0$ and $\alpha_0 = 1$, x_t will never be positive. Figure 2 shows a typical inventory level trajectory. The inventory level starts at level 0, and it stays at level 0 while the system remains in the running state. It leaves the level 0 when the system breaks down and return to level 0 after a certain number of periods.

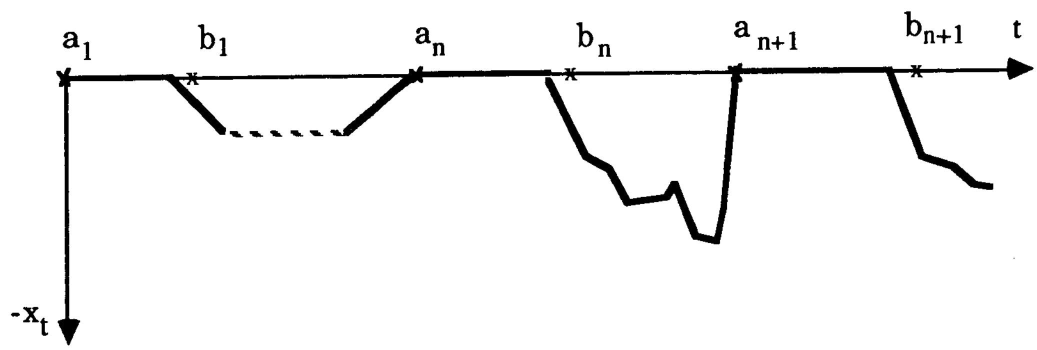

Figure 2: Inventory level trajectory of critical number policy

Let $\{a_n\}$ be the sequence of periods in which the inventory level come back to level 0. Let $\{b_n\}$ be the sequence of periods in which the inventory level leaves the level 0. Let T_n be the number of periods in which the inventory level stays at level 0 after a_n and let Γ_n be the number of periods that the inventory level stays negative after b_n.

Thus, we have

$$x_t = 0 \qquad \forall n > 0 \text{ and } t = a_n, a_n+1, \ldots, b_n - 1,$$
$$x_t < 0 \qquad \forall n > 0 \text{ and } t = b_n, b_n+1, \ldots, a_{n+1} - 1,$$

and

$$T_n = b_n - a_n \text{ and } \Gamma_n = a_{n+1} - b_n.$$

Clearly, the system should be in the running state when the ending inventory level is zero, i.e.

$$x_t = 0 \text{ and } \alpha_t = 1, \qquad \forall t = a_n, a_n+1, \ldots, b_n - 1.$$

Consider an elementary interval t for which $x_t = 0$ and $\alpha_t = 1$. The only way that the inventory level stays at level zero at the end of the next period is that the system stays up. As a consequence,

$$\begin{aligned}&\mathrm{Prob}\{\alpha_{t+1} = 1 \text{ and } x_{t+1} = 0 / \alpha_t = 1 \text{ and } x_t = 0\}\\&= \mathrm{Prob}\{\alpha_{t+1} = 1 / \alpha_t = 1\}\\&= 1 - p.\end{aligned}$$

This means that T_n is a geometrically distributed random variable with the mean p^{-1}.

We examine now the time periods b_n, b_n+1, ..., $a_{n+1}-1$ in which the ending inventory level is negatif. If the mean of Γ_n exists, it follows that

$$- E(x_t) \leq E(\Gamma_n)\, d.$$

We only need to show that $E[\Gamma_n]$ exist.

Since x_t leaves level 0 only when the system breaks down, we have

$$\alpha_{bn-1} = 1 \text{ and } x_{bn-1} = 0$$

and

$$\alpha_{bn} = 0 \text{ and } x_{bn} < 0.$$

Let

$k_0 := b_n$

$k_{m+1} := \min\{t > k_m : \alpha_t \neq \alpha_{km}\}$

be the successive periods in which the system state changes, and let

$Y_m := k_{2m-1} - k_{2m-2}$ and $Z_m := k_{2m} - k_{2m-1}$

be, respectively, the number of periods in which the system stays consecutively in states 0 and 1. Clearly, Y_m and Z_m are geometrically distributed random variables with mean r^{-1} and p^{-1} respectively.

Let y_t be the inventory level if the control policy is to produce always at full capacity for $t \geq b_n$. Then

$$y_{k_{2m}-1} = \sum_{i=1}^{m} (U-d)Z_i - \sum_{i=1}^{m} dY_i .$$

Let
$\tau := \min\{ k_{2m}: y_{k2m-1} \geq 0\}$.

By the definitions of T_n and Γ_n, we have

$$\tau - b_n = T_{n+1} + \Gamma_n.$$

By proving that the mean of τ exists we show the existence of the mean of Γ_n.

From the Principle of Large Deviation (Varadhan[1982]), the quantity

$$1 - \text{Prob}\left\{ \left| \frac{1}{m} \sum_{i=1}^{m} Z_i - p^{-1} \right| \leq \varepsilon \text{ and } \left| \frac{1}{m} \sum_{i=1}^{m} Y_i - r^{-1} \right| \leq \eta \right\}$$

decreases exponentially in m. This implies that

$$1 - \text{Prob}\left\{ \begin{array}{l} \sum_{i=1}^{m} (Z_i + Y_i) \leq m(p^{-1} + r^{-1} + \varepsilon + \eta) \\ \text{and } \sum_{i=1}^{m} ((U-d)Z_i - dY_i) \geq m\left[(U-d)(p^{-1} - \varepsilon) - d^*(r^{-1} + \eta)\right] \end{array} \right\}$$

also decreases exponentially in m. Since

$$Ur/(p+r) > d,$$

there exists ε, η so that

$$(U - d)(p^{-1} - \varepsilon) - d * (r^{-1} - \eta) > 0.$$

Thus, the quantity

$$1 - \text{Prob}\left\{ \sum_{i=1}^{m} (Z_i + Y_i) \leq m(p^{-1} + r^{-1} + \varepsilon + \eta) \text{ and } \sum_{i=1}^{m} ((U-d)Z_i - dY_i) \geq 0 \right\}$$

decreases exponentially in m. This implies that

$$1 - \text{Prob}[\, \tau - b_n \leq m(p^{-1} + r^{-1} + \varepsilon + \eta)]$$

also decreases exponentially in m and therefore also $1 - \text{Prob}[\, \tau - b_n \leq s]$. It follows that the mean of $\tau - b_n$ exists. *Q.E.D.*

6. Conclusion

We have shown in this paper that

1) The optimal control policy in any finite time horizon problem is a critical number policy;

2) The critical number increases and converges to a finite value as the remaining time horizon increases.

These two points yield the optimality of the critical number policy in the finite time horizon problem.

Futur research is necessary to examine the case of demand uncertainty and to extend the results in this paper to the continuous-time model.

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Appendix 1
Ideal Inventory Level Computation when U=2d

Since $Ur/(p+r) > d$ and $U= 2d$, we have

$$r > p.$$

When applying the control policy $\pi^0(x_{t-1})$ with $x_0 = 0$, the inventory level x_t takes value at some discrete points $-nd$ with integer $n \geq 0$. Let $q_i(n)$ be the probability that x_t takes value $-nd$ while the system is in state i in equilibrium. It yields that

$$\begin{aligned}
&q_1(n) = q_1(n)\,(1-p) + q_0(n)\,r && \text{for } n \geq 1;\\
&q_0(n) = q_1(n)\,p + q_0(n)\,(1-r) && \text{for } n > 1;\\
&q_0(1) = q_1(0)\,p;\\
&q_1(0) = q_1(0)\,(1-p) + q_0(1)\,r + q_1(1)\,(1-p);\\
&q_0(0) = 0;\\
&\textstyle\sum_{i,n} q_i(n) = 1.
\end{aligned}$$

Solving these equations, we obtain

$$\begin{aligned}
&q_1(n) = C\,a^{n+1} && \text{for } n \geq 1,\\
&q_0(n) = C\,a^{n} && \text{for } n \geq 1,\\
&q_1(0) = C\,a/p,\\
&q_0(0) = 0
\end{aligned}$$

where

$$a = (1-r)/(1-p) \text{ and } C = (pr - p^2)/[(1-r)(r+p)].$$

Let S(n) be the probability that the inventory level is equal to -nd, i.e.

$$S(n) = q_1(n) + q_0(n).$$

The average inventory cost J(z) can be computed as follows

$$J(z) = c^+ \sum_{nd < z} S(n)(z - nd) - c^- \sum_{nd \geq z} S(n)(z - nd) \qquad \forall\, z \geq 0.$$

From the necessary and sufficient condition of z^*, it is easy to show that $z^* = n^*d$ where n^* is the minimum of non negative integer n which satisfies

$$c^+ \sum_{i \leq n} S(i) - c^- \sum_{i > n} S(i) \geq 0.$$

After some manipulation, we obtain

$$n^* = \max\{0, \lceil v \rceil\}$$

where $\lceil v \rceil$ denotes the smallest integer greater or equal to v and

$$v = \frac{\ln\left(1 + \frac{1-a}{p+pa}\right) - \ln \frac{c^+ + c^-}{c^+}}{\ln a}.$$

The necessary and sufficient condition of $z^* = 0$ is

$$1 \geq \frac{c^+ + c^-}{c^+\left(1 + \frac{1-a}{p+pa}\right)}$$

which implies that

$$\frac{r}{p} \geq 1 + \frac{2(1-p)}{p + c^+ / c^-}.$$

Appendix 2
Proof of Lemma 1

Lemma 1: For any $k \leq T\text{-}1$, if the conditions

$$z_{t+1} \leq z_t \leq z_{t+1} + d \qquad \forall k \leq t \leq T\text{-}1 \tag{a1}$$

hold, then

$$z_k \leq z_{k\text{-}1} \leq z_k + d.$$

Proof: Let $\{\gamma_t(\varepsilon);\ t \geq k\text{-}1\}$ be a realization of the discrete-time Markov chain starting with $\gamma_{k\text{-}1}(\varepsilon) = 1$. Let $\{x^1{}_t(\xi, \varepsilon)\}$ be the optimal inventory trajectory starting with $x_{k\text{-}1}=\xi$ under the realization $\{\gamma_t(\varepsilon)\}$ and $\{u^1{}_t(\xi, \varepsilon)\}$ be the corresponding control. Let $\{x^2{}_t(\xi, \varepsilon)\}$ be the optimal inventory trajectory starting with $x_k = \xi$ under the realization $\{\gamma_{t\text{-}1}(\varepsilon)\}$ and $\{u^2{}_t(\xi, \varepsilon)\}$ be the corresponding control.

1) Proof of $z_k \leq z_{k\text{-}1}$

It is equivalent to show that $\forall \xi < z_k$, $\exists\ c > 0$ such that $\forall\ \Delta < c$

$$F_{k\text{-}1}(T, 1, \xi\ \text{-}\Delta) - F_{k\text{-}1}(T, 1, \xi) \geq F_k(T, 1, \xi\ \text{-}\Delta) - F_k(T, 1, \xi). \tag{a2}$$

By condition (a1), the inventory levels $x^1_t(\xi, \varepsilon)$ and $x^2_t(\xi, \varepsilon)$ can never exceed the ideal inventory level when $\xi < z_k$, i.e.

$$x^1_t(\xi, \varepsilon) \le z_t \text{ and } x^2_t(\xi, \varepsilon) \le z_t \qquad \forall k \le t \le T. \qquad \text{(a3)}$$

Since $z_{t-1} \ge z_t$ for t =k+1, k+2, ..., T, we have

$$x^1_{t-1}(\xi, \varepsilon) \ge x^2_t(\xi, \varepsilon) \qquad \forall k \le t \le T. \qquad \text{(a4)}$$

Let
$\lambda(\varepsilon) := \min\{t: T \ge t > k, \gamma_{t-1}(\varepsilon) = 1 \text{ and } u^2_t(\xi, \varepsilon) < U\}$

be the first period in which the production $u^2_t(\xi, \varepsilon)$ is less than the capacity. This implies that the ideal inventory level is attained at the end of period $\lambda(\varepsilon)$. By convention, $\lambda(\varepsilon) = T+1$ if $\lambda(\varepsilon)$ does not exist.

As a consequence, we have $\forall\ t < \lambda(\varepsilon)$

$$u^1_{t-1}(\xi, \varepsilon) = u^2_t(\xi, \varepsilon) = \gamma_{t-1}(\varepsilon) * U, \qquad \text{(a5)}$$

and

$$x^1_{t-1}(\xi, \varepsilon) = x^2_t(\xi, \varepsilon) \le z_t. \qquad \text{(a6)}$$

Let c be defined as follows:

$$c := \min_{\varepsilon / \lambda(\varepsilon) \le T} (U - u^2_{\lambda(\varepsilon)}(\xi, \varepsilon)).$$

From (a5) and (a6), it follows that $\forall \Delta < c$

$$x^1_{t-1}(\xi-\Delta, \varepsilon) = x^2_t(\xi-\Delta, \varepsilon) = x^2_t(\xi, \varepsilon) - \Delta \quad \forall\ \varepsilon \text{ and } t < \lambda(\varepsilon). \qquad \text{(a7)}$$

From (a3) and (a4), we have

$$x^2_{\lambda(\varepsilon)}(\xi-\Delta, \varepsilon) = x^2_{\lambda(\varepsilon)}(\xi, \varepsilon) = z_{\lambda(\varepsilon)} \qquad \forall\ \varepsilon, \qquad \text{(a8)}$$

and

$$x^1_{\lambda(\varepsilon)-1}(\xi-\Delta, \varepsilon) \le x^1_{\lambda(\varepsilon)-1}(\xi, \varepsilon) \le z_{\lambda(\varepsilon)-1} \qquad \forall\ \varepsilon. \qquad \text{(a9)}$$

From (a7), (a8) and (a9), we have

$$F_k(T, 1, \xi - \Delta)$$
$$= E\left\{ \sum_{t=k}^{\lambda(\varepsilon)-1} g[x_t^2(\xi, \varepsilon) - \Delta] + F_{\lambda(\varepsilon)}(T, 1, z_{\lambda(\varepsilon)}\right\}$$
$$= F_k(T, 1, \xi) + E\left\{ \sum_{t=k}^{\lambda(\varepsilon)-1} (g[x_t^2(\xi, \varepsilon) - \Delta] - g(x_t^2(\xi, \varepsilon)))\right\},$$

and

$$F_{k-1}(T, 1, \xi - \Delta)$$
$$= E\left\{ \sum_{t=k-1}^{\lambda(\varepsilon)-2} g[x_t^1(\xi, \varepsilon) - \Delta] + F_{\lambda(\varepsilon)-1}[T, 1, x^1_{\lambda(\varepsilon)-1}(\xi - \Delta, \varepsilon])\right\}$$
$$\geq F_{k-1}(T, 1, \xi) + E\left\{ \sum_{t=k}^{\lambda(\varepsilon)-1} (g[x_t^2(\xi, \varepsilon) - \Delta] - g[x_t^2(\xi, \varepsilon)])\right\}.$$

This yields that

$$F_{k-1}(T, 1, \xi - \Delta) - F_{k-1}(T, 1, \xi) \geq F_k(T, 1, \xi - \Delta) - F_k(T, 1, \xi).$$

2) Proof of $z(k-1) \leq z(k) + d$

It is equivalent to show that $\forall\ \xi > z_k$, $\exists\ c > 0$ such that $\forall\ \Delta < c$

$$F_{k-1}(T,1,\xi + d + \Delta) - F_{k-1}(T, 1, \xi + d) \geq F_k(T, 1, \xi + \Delta) - F_k(T, 1, \xi). \quad \text{(a10)}$$

We prove (a10) in two cases.

Case 1: $\xi \geq (k - T+1)\, d$

In this case, the initial inventory is greater than the total demand either with $x_{k-1} = \xi + d$ or with $x_k = \xi$. Thus we have

$$x^1{}_t(\xi+d, \varepsilon) \geq z_t \text{ and } u^1{}_t(\xi+d, \varepsilon) = 0,$$

and

$$x^2{}_t(\xi, \varepsilon) \geq z_t \text{ and } u^2{}_t(\xi, \varepsilon) = 0 \quad \forall t, \varepsilon.$$

Then, it is obvious that $\forall \Delta > 0$,

$$F_{k-1}(T,1,\xi + d + \Delta) - F_{k-1}(T, 1, \xi + d) = (k-T+2)\, \Delta,$$

and

$$F_k(T,1,\xi + \Delta) - F_k(T, 1, \xi) = (k-T+1)\, \Delta.$$

Thus the equation (a10) is true for any $\xi \geq (k - T+1)\, d$.

Case 2: $\xi < (k - T+1)\, d$

Let

$\mu(\varepsilon) := \min\{t: T \ge t > k$ such that $x^1_{t-1}(\xi+d, \varepsilon) = z_{t-1}$ and $u^1_{t-1}(\xi+d, \varepsilon) > 0$,
or $x^2_t(\xi, \varepsilon) = z_t$ and $u^2_t(\xi, \varepsilon) > 0\}$.

By convention, $\mu(\varepsilon) = T+1$ if $\mu(\varepsilon)$ does not exist.

As a consequence, $\forall t < \mu(\varepsilon)$

$$x^1_{t-1}(\xi+d, \varepsilon) = x^2_t(\xi, \varepsilon) + d \tag{a11}$$

and

$$u^1_{t-1}(\xi+d, \varepsilon) = u^2_t(\xi, \varepsilon). \tag{a12}$$

Let c be any positive value which satisfies the following conditions

$$\begin{array}{ll}
c \le z_t - x^1_t(\xi + d, \varepsilon) & \forall\ \varepsilon, t < \mu(\varepsilon) - 1 \text{ and } z_t - x^1_t(\xi + d, \varepsilon) > 0; \\
c \le z_t - x^2_t(\xi, \varepsilon) & \forall\ \varepsilon, t < \mu(\varepsilon) \text{ and } z_t - x^2_t(\xi, \varepsilon) > 0; \\
c \le u^1_{\mu(\varepsilon)-1}(\xi + d, \varepsilon) & \forall\ \varepsilon, \mu(\varepsilon) \le T \text{ and } u^1_{\mu(\varepsilon)-1}(\xi + d, \varepsilon) > 0; \\
c \le u^2_{\mu(\varepsilon)}(\xi, \varepsilon) & \forall\ \varepsilon, \mu(\varepsilon) \le T \text{ and } u^2_{\mu(\varepsilon)}(\xi, \varepsilon) > 0; \\
c \le - x^2_t(\xi, \varepsilon) & \forall\ \varepsilon, t, \mu(\varepsilon) = T + 1 \text{ and } x^2_t(\xi, \varepsilon) < 0.
\end{array}$$

Since $\xi < (k - T+1)\, d$ and the set of realization ε is finite, c exists.

For any $0 < \Delta < c$, the inventory levels $x^1_{t-1}(\xi+d, \varepsilon)$ and $x^2_t(\xi, \varepsilon)$ for $t < \mu(\varepsilon)$ increase Δ when ξ increases Δ, i.e.

$$x^1_{t-1}(\xi+d+\Delta, \varepsilon) = x^1_{t-1}(\xi+d, \varepsilon) + \Delta,$$

and (a13)

$$x^2_t(\xi+\Delta, \varepsilon) = x^2_t(\xi, \varepsilon) + \Delta.$$

Combining equations (a13) and (a11), the following equation is true for $t < \mu(\varepsilon)$

$$g(x^1_{t-1}(\xi+d+\Delta, \varepsilon)) - g(x^1_{t-1}(\xi+d, \varepsilon)) \ge g(x^2_t(\xi+\Delta, \varepsilon)) - g(x^2_t(\xi, \varepsilon)). \tag{a14}$$

Let us first consider the realizations in which $\mu(\varepsilon) \le T$. We have either

$$x^1_{\mu(\varepsilon)-1}(\xi+d, \varepsilon) = z_{\mu(\varepsilon)-1} \text{ and } u^1_{\mu(\varepsilon)-1}(\xi+d, \varepsilon) > 0$$

or

$$x^2_{\mu(\varepsilon)}(\xi, \varepsilon) = z_{\mu(\varepsilon)} \text{ and } u^2_{\mu(\varepsilon)}(\xi, \varepsilon) > 0.$$

In any case, it is easy to show that

$$x^1_{\mu(\varepsilon)-1}(\xi+d+\Delta, \varepsilon) \geq x^1_{\mu(\varepsilon)-1}(\xi+d, \varepsilon) \geq z_{\mu(\varepsilon)-1},$$

and

$$x^2_{\mu(\varepsilon)}(\xi, \varepsilon) \leq x^2_{\mu(\varepsilon)}(\xi+\Delta, \varepsilon) \leq z_{\mu(\varepsilon)}.$$

This means that

$$F_{\mu(\varepsilon)-1}(T,1,x^1_{\mu(\varepsilon)-1}(\xi+d+\Delta, \varepsilon)) - F_{\mu(\varepsilon)-1}(T,1,x^1_{\mu(\varepsilon)-1}(\xi+d, \varepsilon)) \geq 0, \tag{a15}$$

and

$$F_{\mu(\varepsilon)}(T,1,x^2_{\mu(\varepsilon)}(\xi+\Delta, \varepsilon)) - F_{\mu(\varepsilon)}(T,1,x^2_{\mu(\varepsilon)}(\xi, \varepsilon)) \leq 0. \tag{a16}$$

We consider now the realizations in which $\mu(\varepsilon) = T+1$. We have

$$x^1_{t-1}(\xi+d+\Delta, \varepsilon) = x^2_t(\xi+\Delta, \varepsilon) + d = x^2_t(\xi, \varepsilon) + d + \Delta \quad \forall t \leq T. \tag{a17}$$

Since $\xi < (k - T+1)\, d$ and $\mu(\varepsilon) = T+1$, then

$$x^2_T(\xi, \varepsilon) < 0 \text{ and } x^1_T(\xi+d, \varepsilon) \leq 0. \tag{a18}$$

Let l be the first period for which $x^2_t(\xi, \varepsilon)$ is negative. From (a11), we have

$$x^2_l(\xi, \varepsilon) < 0 \text{ and } x^1_{l-1}(\xi+d, \varepsilon) \geq 0. \tag{a19}$$

By the definition of Δ, we have

$$x^2_l(\xi+\Delta, \varepsilon) = x^2_l(\xi, \varepsilon) + \Delta < 0. \tag{a20}$$

From (a18),(a19) and (a20), we deduce that

$$g(x^1_{l-1}(\xi+d+\Delta,\varepsilon)) - g(x^1_{l-1}(\xi+d,\varepsilon)) + g(x^1_T(\xi+d+\Delta,\varepsilon)) - g(x^1_T(\xi+d,\varepsilon)) \geq c^+\Delta - c^-\Delta, \tag{a21}$$

and

$$g(x^2_l(\xi+\Delta, \varepsilon)) - g(x^2_l(\xi, \varepsilon)) = -c^-\Delta. \tag{a22}$$

By combining (a14), (a21) and (a22), it yields that $\forall \varepsilon$ such that $\mu(\varepsilon) = T+1$,

$$\sum_{t=k-1}^{T} (g(x^1_t(\xi+d+\Delta, \varepsilon)) - g(x^1_t(\xi+d, \varepsilon))) \geq \sum_{t=k}^{T} (g(x^2_t(\xi+\Delta, \varepsilon)) - g(x^2_t(\xi, \varepsilon))) \tag{a23}$$

By (a14), (a15), (a16) and (a23), we have

$$
\begin{aligned}
&F_k(T, 1, \xi + \Delta) \\
&\quad = E\left\{ \sum_{t=k}^{\mu(\varepsilon)-1} g(x_t^2(\xi, \varepsilon) + \Delta) + F_{\mu(\varepsilon)}(T, 1, x^2_{\mu(\varepsilon)}(\xi + \Delta, \varepsilon)) \right\} \\
&\quad \le F_k(T, 1, \xi) + E\left\{ \sum_{t=k}^{\mu(\varepsilon)-1} (g(x_t^2(\xi, \varepsilon) + \Delta) - g(x_t^2(\xi, \varepsilon))) \right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&F_{k-1}(T, 1, \xi + d + \Delta) \\
&\quad = E\left\{ \sum_{t=k-1}^{\mu(\varepsilon)-2} g(x_t^1(\xi + d, \varepsilon) + \Delta) + F_{\mu(\varepsilon)-1}(T, 1, x^1_{\mu(\varepsilon)-1}(\xi + d + \Delta, \varepsilon)) \right\} \\
&\quad \ge F_{k-1}(T, 1, \xi) + E\left\{ \sum_{t=k}^{\mu(\varepsilon)-1} (g(x_t^2(\xi, \varepsilon) + \Delta) - g(x_t^2(\xi, \varepsilon))) \right\}.
\end{aligned}
$$

It follows that

$$F_{k-1}(T,1,\xi + d + \Delta) - F_{k-1}(T, 1, \xi + d) \ge F_k(T, 1, \xi + \Delta) - F_k(T, 1, \xi).$$

Q.E.D.

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