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SUR LES SOUS-SOLUTIONS ET LES SUPER-SOLUTIONS DES EQUATIONS D'HAMILTON-JACOBI-BELLMAN

UN ALGORITHME ACCELERE POUR RESOUDRE DES PROBLEMES DE CONTROLE OPTIMAL

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RESUMÉ

On considère des schêmas discretisés de problèmes de contrôle optimal en temps continu. On cherche le coût optimal definie par les équations de HJB associées. On présente un algorithme qui construit une suite de supersolutiones à partir d'un ensemble de subsolutiones de telles équationes et qui converge en un nombre fini d'itérations. On donne aussi une estimation de l'erreur d'approximation commis.

ABSTRACT

We consider a discrete scheme for optimal control problems in continuous time. We look for the optimal cost given by the corresponding HJB equations. We present an algorithm which generates a sequence of supersolutions from a set of subsolutions of such equations, convergent in a finite number of iterations. We also give a bound for the approximation error.

1. INTRODUCTION

We present in this paper an accelerated algorithm to solve a nonlinear fixed-point problem originated by discretization of optimal control problems in continuous time.

Generally speaking, it is possible to solve a great number of optimization problems by finding solutions to the associated variational/quasi-variational inequalities (VI), or equalities of Hamilton-Jacobi-Bellman (HJB) type. When we apply finite difference methods or finite element methods (see [1], [2] and [3]), these VI become *discrete* variational or quasi variational inequalities (DVI) which can be considered as optimality conditions for a Markov chain optimal control problem. Such DVI are solved by finding a fixed-point of a nonlinear contractive operator (it can be shown that this is equivalent to getting the solution of these DVI). When doing suitable iterations and with a proper choice of the starting point, the later problem can be considered as the equivalent one of generating a sequence of subsolutions which grow up to the desired solution. An important feature of our algorithm consists on defining another sequence of **supersolutions** whose limit is the fixed point. In this way, we can also give a bound for the approximation error of the method. The solution of the fixed-point problem is frequently found by an iterative procedure which can become very slow when the contraction factor of the operator is very close to unity (see [4]). The proposed accelerated algorithm is based on the resolution of linear systems which appear implicitly in the contractive operator definition. In our computational implementation we have applied a conjugate gradient algorithm for the linear system resolution (see [5]).

2. DESCRIPTION OF THE DISCRETIZED PROBLEM

We employ here a notation akin to that introduced in [1], that is, let $\mathcal{B} = \left\{ \beta \right\}$ be a finite set of indexes such that for each $\beta \in \mathcal{B}$ there exists a matrix A^{β} with the following properties:

(1) A^{β} is a square matrix of order N.

(2) $A^{\beta}_{ij} \geq 0 \qquad \forall i, \forall j.$

(3) $\sum_{j} A^{\beta}_{ij} < 1 \qquad \forall i.$

For $w \in \mathbb{R}^N$, we define $M^{\beta}w = A^{\beta}w + f^{\beta}$, where $f^{\beta} \in \mathbb{R}^N$.

An operator $M : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is introduced by the definition:

$$(Mw)_i = \min_{\beta \in \mathcal{B}} (M^{\beta}w)_i = \min_{\beta \in \mathcal{B}} \left\{ \sum_{j} A^{\beta}_{ij} w_j + f^{\beta}_i \right\}$$

Due to properties (1) - (3), M is a contractive operator, in the following sense:

If we introduce in $\mathbb{R}^N$ the norm $\| w \| = \max_{1 \leq i \leq N} |w_i|$, then there exists $\eta \in [0,1)$ such that

$$\| Mw - M\tilde{w} \| \leq \eta \, \| w - \tilde{w} \| \qquad \forall\, w, \tilde{w} \in \mathbb{R}^N \tag{1}$$

The accelerated algorithm proposed in this paper computes the solution of the following problem:

$$\mathcal{F}ind\ \bar{w} \in \mathbb{R}^N \ / \ M\bar{w} = \bar{w} \tag{2}$$

Because of (1), there exists a unique solution of equation (2) which can be computed by the following algorithm:

ALGORITHM 0

Step 0: Take $w^0 \in R^N$, $\nu = 0$, and start the procedure.

Step 1: Define $w^{\nu+1} = \min_{\beta \in \mathcal{B}} (M^\beta w^\nu)$ (i.e. $w^{\nu+1} = Mw^\nu$).

Step 2: If $w_i^\nu = w_i^{\nu+1}$ then stop; else set $\nu = \nu + 1$ and go to Step 1.

This algorithm produces either a finite sequence w^ν whose last element is $w^{\bar{\nu}} = \bar{w}$ (solution of fixed-point problem (2)) or an infinite sequence w^ν converging to $\bar{w}$. Also, the following bound for the approximation error holds:

$$\| w^\nu - \bar{w} \| \leq \eta^\nu \| w^0 - \bar{w} \|$$

3. ACCELERATED ALGORITHM

In the computation of Algorithm 0 it is common to observe that the index $\bar{\beta}(i)$ which realizes the minimum remains constant for each component i, after a finite number of iterations. So Algorithm 0 is essentially used to solve the linear system

$$w_i = (M^{\bar{\beta}(i)} w)_i$$

through iteration

$$w_i^{\nu+1} = (M^{\bar{\beta}(i)} w^\nu)_i \qquad (3)$$

Therefore we have defined an accelerated algorithm that uses this experimentally observed property by solving (when suitable), the linear system implicitly defined in (3).

This new algorithm converges in a finite number of steps.

Description of the accelerated algorithm (Algorithm 1):

Step 0: Take $w^0 \in \mathbb{R}^N$, $p_{max} \geq 1$. Set $\nu = 0$, $\mu = 0$, and start the procedure.

Step 1: Define $w_i^{\nu+1}(\mu) = \min_{\beta \in \mathcal{B}} (M^\beta w^\nu(\mu))_i$.

Determine $\bar{\beta}(i,\nu)$ such that $w_i^{\nu+1}(\mu) = (M^{\bar{\beta}(i,\nu)} w^\nu(\mu))_i$.

Step 2: If $w_i^\nu(\mu) = w_i^{\nu+1}(\mu)$, then stop; else go to Step 3.

Step 3: For $\nu \geq 1$, compute $q = \text{card}\left\{ i \ / \ \bar{\beta}(i,\nu) \neq \bar{\beta}(i,\nu-1) \right\}$.

Step 4: If $q = 0$ then set $p = p + 1$, else set $p = 0$.

Step 5: If $p \leq p_{max}$ then set $\nu = \nu + 1$ and go to Step 1; else go to Step 6.

Step 6: Define $\hat{\beta}(i,\mu) = \bar{\beta}(i,\nu)$ and solve the system

$$y_i(\mu) = (M^{\hat{\beta}(i,\mu)} y(\mu))_i$$

Step 7: Set $\nu = 0$, $w_i^0(\mu+1) = y_i(\mu)$, $\mu = \mu + 1$ and go to Step 1.

4. CONVERGENCE OF ALGORITHM 1:

For the sake of simplicity, convergence of Algorithm 1 will be proved for those cases where the following condition (C1) is satisfied:

For any vector of indexes $\gamma = (\gamma_1, \ldots, \gamma_N)$, $\gamma_i \in \mathcal{B}$, *define* $z(\gamma)$ *as the solution of the system:*

$$z(\gamma)_i = (M^{\gamma(i)} z(\gamma))_i$$

Then we say (C1) is verified if $\gamma \neq \tilde{\gamma} \Rightarrow z(\gamma) \neq \tilde{z}(\tilde{\gamma})$.

THEOREM:

If (C1) holds, then Algorithm 1 produces a decreasing sequence of elements $y(\mu)$ that converges in a finite number of steps to the fixed-point $\bar{w}$ of operator M.

Proof:

First we prove the following assertion:

Proposition 1:

$\forall\ \mu$, Step 6 is reached after a finite number of cycles through loop (1,2,3,4,5), allowing the generation of a new element $y(\mu+1)$.

Let us assume by contradiction that there is an infinite number of cycles (1-...-5), then the corresponding sequence $w^{\nu}(\mu)$ will converge to the unique solution $\bar{w}$ of the fixed-point problem (2).

If we suppose there are two different vectors γ and $\tilde{\gamma}$ in the calculation of $w^{\nu}(\mu)$ which appear an infinite number of times

Finally

$$y_i(\mu) = \lim_{k\to\infty} \Big[(M^{\hat{\beta}})^k w^{\bar{\nu}} \Big]_i \le \Big[(M^{\hat{\beta}})^k w^{\bar{\nu}} \Big]_i \le w_i^{\bar{\nu}} \le$$

$$\le w_i^{\bar{\nu}-1} \le \ldots \le w_i^1(\mu) \le w_i^0(\mu) = y_i(\mu-1) \qquad (9)$$

Thus

$$y_i(\mu) \le y_i(\mu-1) \qquad \forall\ \mu \ge 1,\ \forall\ i$$

■

Considering there is only a finite number of possible variations $\hat{\beta}(.,\mu)$, we can only generate a finite number of different $y(\mu)$; that is why after a finite number of cycles, the test in Step 2 is satisfied and Algorithm 1 finishes, having found the fixed-point of operator M.

■

Remarks:

(1) Property (9): $y_i(\mu) \le y_i(\mu-1) \quad \forall\ \mu \ge 1,\ \forall\ i$ holds independently of the initial point w^0 chosen.

(2) When that initial point w^0 is a subsolution (i.e. $w^0 \le \min_\beta (M^\beta w^0)_i$) we can write

$$w_i^0(0) \le w_i^1(0) \le \ldots \le w_i^{\bar{\nu}(0)}(0) \le y_i(0)$$

From the monotony of M^β and definition of $\bar{w}$ ($\bar{w} = M\bar{w}$ is the fixed point) we will have

$$w_i^{\nu}(0) \le \bar{w}_i \qquad \forall\ i,\quad \forall\ \nu \le \bar{\nu}(0)$$

and

$$\bar{w} = \lim_{\mu \to \infty} y(\mu) \le y(\mu) \qquad \forall\ \mu \ge 0.$$

Therefore we can obtain the following bound for the convergence of Algorithm 1:

$$w_i^o(0) \le w_i^{\nu}(0) \le w_i^{\bar{\nu}}(0) \le \bar{w}_i \le y_i(\mu) \le w_i^{\tilde{\nu}}(\mu) \le y_i(\mu-1)$$

$$\forall\ i,\quad \forall\ \nu \le \bar{\nu}(0),\quad \forall\ \mu \ge 1,\ \forall\ \tilde{\nu} \le \bar{\nu}(\mu).$$

Consequently

$$0 \le y_i(\mu) - \bar{w}_i \le w_i^{\tilde{\nu}}(\mu) - \bar{w}_i \ ; \qquad \forall\ \mu,\ \forall\ \tilde{\nu} \le \bar{\nu}(\mu).$$

(3) Vectors $y(\mu)$ are supersolutions since they verify

i) $y_i(\mu) \ge \bar{w}_i \quad \forall\ i$, due to Remark (2).

ii) $y_i(\mu) \ \ge\ (My(\mu))_i \qquad \forall\ i,\ \ \forall\ \mu \ge 0$, because

$$y_i(\mu) = \left[M^{\hat{\beta}(i,\mu)} y(\mu)\right]_i \ge \min_{\beta} \left[M^{\beta} y(\mu)\right]_i = \left[My(\mu)\right]_i .$$

(4) $\bar{w}$ is the maximum subsolution of (2) and the minimum supersolution of (2).

(5) When condition (C1) is not verified, Algorithm 1 can be modified, in order to keep convergence, using similar techniques to those described in [8] (developed there for the special case of differential games with stopping times). In our case, these modifications take the following form:

Step 0': Take $w^0 \in \mathbb{R}^N$, $p_{max} \geq 1$. Set $\nu = 0$, $\mu = 0$, $\varepsilon_0 > 0$ and start the procedure.

Step 1': Define $w_i^{\nu+1}(\mu) = \min_{\beta \in \mathcal{B}} (M^\beta w^\nu(\mu))_i$.

Determine $\bar{\beta}_{\varepsilon_\nu}(i,\nu) = \left\{ \beta \in \mathcal{B} \;/\; |w_i^{\nu+1}(\mu) - (M^\beta w^\nu(\mu))_i| \leq \varepsilon_\nu \right\}$

Step 5': If $p \leq p_{max}$ then set $\nu = \nu+1$, $\varepsilon_\nu = \sqrt{\|w^{\nu+1}(\mu) - w^\nu(\mu)\|}$ and go to Step 1; else go to Step 6'.

Step 6': Let it be $\hat{\beta}(i,\mu)$ any $\beta \in \mathcal{B}$ such that $w_i^{\nu+1}(\mu) = (M^\beta w^\nu(\mu))_i$

Solve the system

$$y_i(\mu) = (M^{\hat{\beta}(i,\mu)} y(\mu))_i$$

[6] Algorithm 1 can also be applied in cases where M is not strictly contractive (i.e. it is only verified inequality $\|Mw - M\tilde{w}\| \leq \|w - \tilde{w}\|$) but a fixed power of M have this property, i.e. there is an integer s such that $\|M^s w - M^s \tilde{w}\| \leq \eta \|w - \tilde{w}\|$, with $\eta < 1$. Property of convergence is conserved. An application of this type can be seen in [7].

[7] Algorithm 1 can be extended using others stopping rules in Step 5. It is also transformed in a convergent algorithm which accelerates Algorithm 0. These extensions, its properties and the corresponding proofs of convergence are considered in [9].

5. PRACTICAL CONSIDERATIONS AND NUMERICAL EXAMPLES

We show in this section some comparisons between computing times of Algorithm 0 and Algorithm 1.

We can observe the strong dependence of the acceleration phenomenum on the contraction factor η. The proposed Algorithm shows its efficiency specially when η is close to unity.

The results shown have been produced in a VAX 780, for the solution of an optimal control problem posed for a divergent multilevel system with stochastic demand (see [6]).

η	CPU time Alg. 0 (in sec.)	CPU time Alg. 1 (in sec.)	% Reduction
0.50	22.33	17.76	23.13
0.86	70.58	20.00	71.66
0.91	108.71	21.01	81.68
0.96	300.73	22.09	92.65
0.99	991.02	22.43	97.74

6. CONCLUSIONS

We present Algorithm 1 and we show it will always converge after a finite number of steps. If condition (C1) holds, the proof of the convergence can be simplified, but in absence of (C1), the algorithm can be modified (as shown in Remark 5) so as to assure its convergence.

Algorithm 1 is specially suitable for solving fixed-point problems whose contraction factor is very close to unity.

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