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MATHEMATICAL ANALYSIS OF THE PROPAGATION OF ELASTIC GUIDED WAVES IN HETEROGENEOUS MEDIA

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862

Mathematical Analysis of the Propagation of Elastic Guided Waves in Heterogeneous Media.

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Résumé

Dans cet article, nous nous intéressons à la propagation d'ondes élastiques dans des milieux isotropes hétérogènes, invariants par translation dans une direction. Nous faisons l'analyse théorique de l'existence d'ondes guidées et de leurs propriétés. En particulier, les seuils, ou fréquences de coupures, sont étudiés en détail. Le principal outil mathématique est la théorie spectrale des opérateurs autoadjoints, et plus spécialement le principe du Max-Min.

Abstract

In this article, we are concerned by the propagation of elastic waves in isotropic heterogeneous media, invariant under translation in one direction. We give a theorical analysis of the existence of guided waves and of their properties. In particular the thresholds, or cut-off frequencies, are studied in detail. The main mathematical tool is the spectral theory of selfadjoint operators, and more specifically the Max-Min principle.

Mots-clés

Ondes guidées, Elastodynamique, Milieux hétérogènes, Fréquences de coupure, Théorie spectrale, Principe du Min-Max.

Keywords

Guided waves, Elastodynamics, Heterogeneous media, Cut off frequencies, Spectral theory, Min-Max principle.

MATHEMATICAL ANALYSIS OF THE PROPAGATION OF ELASTIC GUIDED WAVES IN HETEROGENEOUS MEDIA

ANALYSE MATHEMATIQUE DE LA PROPAGATION D'ONDES ELASTIQUES GUIDEES EN MILIEU HETEROGENE

Alain BAMBERGER, Yves DERMENJIAN, Patrick JOLY

0. INTRODUCTION

The question of the existence of guided waves when the domain of propagation is infinite and invariant under translation in one space dimension is not a trivial problem since it generally leads to an eigenvalue problem for an unbounded selfadjoint operator with non compact resolvent in an appropriate Hilbert space. The cases corresponding to scalar propagation equations, namely the Schrödinger equation, the acoustic wave equation, the water wave equation, are now well-known and have been investigated from various points of view by several authors, i.e. [D.G.1], [We.1 & 2], [Wi.1 & 2]. The case of wave phenomena governed by hyperbolic systems is much more complicated and has retained much less attention in the mathematical literature. Recently, the case of Maxwell's equations was studied by [Ba.Bo.] (See also [G.1], [We.3]).

In this paper we are interested in the elastodynamic equations ([Ac.],[E.S.],[Mi.]) which govern the propagation of elastic waves in solids and more specifically by the guided waves propagating in media which are invariant under translation in the direction x_3. In our case, the phenomenon of waveguide is provoked by a local variation in the plane (x_1, x_2) of the coefficients characterizing the elastic behaviour of the material, that is to say, in the case of linear isotropic media, the density ρ and the Lamé's parameters λ and μ. The situation is slightly more complicated than in the case of Maxwell's equations since there exist two types of waves (instead of one for Maxwell's equations) propagating in a homogeneous medium (i.e. when ρ, λ, μ are constant) : the P-waves propagating with the velocity $V_P = ((\lambda+2\mu)/\rho)^{1/2}$ and the S-waves propagating with the velocity $V_S = (\mu/\rho)^{1/2}$. In [B.J.K.], the authors studied another type of elastic guided waves : the surface waves . In their model problem, the medium is homogeneous but the propagation domain is the exterior of a infinitely long borehole. In that case, the boundary condition, namely the free surface condition plays a very important role in the mechanism of surface wave (see, also, [G.2]). In our case, the only phenomenon which will generate the guided waves will be, as we said before, the local variations of ρ, λ, μ as functions of the two space variables x_1 and x_2 and the main objective of this article is to find some conditions on these functions to guarantee the existence of guided waves. In a second step, we shall study some properties of these waves.

Before giving the outline of the paper, let us give our notation. The unknown function is the displacement field $U(x,t) = (U_1(x, t) , U_2(x, t) , U_3(x, t))$ for x given in $\mathbb{R}^3$ and t in $\mathbb{R}^+$. $U(x, t)$ obeys the linear elastodynamic equations [Mi.] . In the case of a homogeneous isotropic medium, (ρ, λ, μ) are constant, these equations can be written :

$$(0.1) \qquad \frac{\partial^2 U}{\partial t^2} = \left(\frac{\lambda + \mu}{\rho}\right) \nabla(\nabla . U) + \frac{\mu}{\rho} \Delta U$$

When $\rho(x)$, $\lambda(x)$, $\mu(x)$ are functions of x_1 and x_2 , (0.1) is no longer valid and the mathematical model takes a more complicated form that we shall give in section 1.

By definition, a guided wave (or guided mode) is a solution of the elastodynamic equations in the form :

$$(0.2) \qquad U_j(x,t) = \tilde{u}_j(x_1, x_2) \exp i(\omega t - \beta x_3) \qquad j = 1, 2, 3$$

where :

- $\omega > 0$ is the pulsation of the mode
- $\beta > 0$ is the wave number
- $(\tilde{u}_j(x_1, x_2) , j = 1, 2, 3)$ is a complex valued vector field which must satisfy :

$$(0.3) \qquad 0 < \sum_{j=1}^{3} \int_{\mathrm{R}^2} |\tilde{u}_j(x_1, x_2)|^2 \, dx_1 \, dx_2 < +\infty$$

This last condition means physically that the displacement field remains concentrated in a bounded region of the plane (x_1, x_2). Such solutions do not appear in a homogeneous medium but can appear, as we shall see, if the coefficients vary with (x_1, x_2). But in any case , the guided mode exists if and only if ω and β satisfy a relation which is by definition the dispersion relation of the modes. Plugging (0.2) into the equations reduces the problem to the research of the eigenvalues and the eigenfunctions of a selfadjoint operator $\boldsymbol{A}(\beta)$ in the Hilbert space $L^2(\mathrm{R}^2, \rho \, dx_1 dx_2)$. In such an approach, β appears as a parameter, ω^2 is the eigenvalue and $\tilde{u}$ is the corresponding eigenfunction. Thus, all the results we obtain stem from the spectral analysis of $\boldsymbol{A}(\beta)$ and the main tool of the analysis is the Max-Min principle ([R.S.2]).

This article is organized as follows. In section 1, we present the mathematical framework, give the mathematical formulation of our problem and obtain some important preliminary results about the bilinear form $a(\beta;.,.)$ associated with the operator $\boldsymbol{A}(\beta)$. In section 2, we determine the essential spectrum of $\boldsymbol{A}(\beta)$ and study the properties of possible eigenvalues embedded in the essential spectrum. The main results of the paper can be found in section 3 in which we study in detail the discrete spectrum of $\boldsymbol{A}(\beta)$. Our two main existence results (theorem 3.5 and theorems 3.6 & 3.7) are given in section 3.1. In section 3.2, we introduce the very important notion of thresholds and study in detail the properties of these thresholds (theorems 3.8-3.14).

1. MATHEMATICAL FRAMEWORK

1.1 The equations

Looking for guided modes with pulsation ω and wavenumber β one reduces to solve the system :

$$(1.1) \qquad A(\beta)u = \omega^2 u , \qquad u \in L^2(\mathrm{R}^2)^3 ,$$

where the differential operator $A(\beta)$ is defined by :

$$(1.2)\quad \begin{cases} (A(\beta)u)_k = -\dfrac{1}{\rho}\displaystyle\sum_{j=1}^{2}\dfrac{\partial\sigma^{\beta}_{kj}(u)}{\partial x_j} - \dfrac{\beta}{\rho}\,\sigma^{\beta}_{k3}(u)\,, \qquad k = 1, 2 \\[2ex] (A(\beta)u)_3 = -\dfrac{1}{\rho}\displaystyle\sum_{j=1}^{2}\dfrac{\partial\sigma^{\beta}_{3j}(u)}{\partial x_j} + \dfrac{\beta}{\rho}\,\sigma^{\beta}_{33}(u)\,, \end{cases}$$

where the symmetric matrix $\sigma^{\beta}(u)$ is derived from :

$$(1.3)\quad \begin{cases} \sigma^{\beta}(u) = \lambda\,(\mathrm{div}^{\beta}(u))\,\mathrm{I} + 2\mu\,\varepsilon^{\beta}(u)\,, \\[1ex] \mathrm{div}^{\beta}(u) = \dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} - \beta\,u_3\,, \end{cases} \qquad \begin{cases} \varepsilon^{\beta}_{kj}(u) = \dfrac{1}{2}\left(\dfrac{\partial u_k}{\partial x_j} + \dfrac{\partial u_j}{\partial x_k}\right) \qquad k, j = 1, 2\,, \\[1ex] \varepsilon^{\beta}_{k3}(u) = \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_k} + \beta\,u_k\right) \qquad k = 1, 2\,, \\[1ex] \varepsilon^{\beta}_{33}(u) = -\,\beta\,u_3\,. \end{cases}$$

To obtain these formulations, it suffices to start from the linear elastodynamic equations [Mi.] :

$$(1.4)\quad \begin{cases} \dfrac{\partial^2 U_i}{\partial t^2} = \dfrac{1}{\rho}\displaystyle\sum_{j=1}^{2}\dfrac{\partial}{\partial x_j}\left(\sigma_{ij}(U)\right) \qquad i = 1, 2, 3\,, \\[2ex] \sigma_{ij}(U) = \lambda\,(\mathrm{div}\;U)\,\delta_{ij} + 2\mu\,\varepsilon_{ij}(U) \qquad i,j = 1, 2, 3\,, \\[2ex] \varepsilon_{ij}(U) = \dfrac{1}{2}\left(\dfrac{\partial U_i}{\partial x_j} + \dfrac{\partial U_j}{\partial x_i}\right) \qquad i,j = 1, 2, 3\,, \end{cases}$$

and to consider displacement fields in the form (0.2) .The following change of unknown functions

$$u_1 = \tilde{u}_1 , \quad u_2 = \tilde{u}_2 , \quad u_3 = i\,\tilde{u}_3 ,$$

leads then to equations (1.1),(1.2) which permits us to work only with real coefficients and to consider only real valued functions .

From now on, we shall use only two space coordinates, namely x_1 and x_2, instead of three. That is why we shall set $x = (x_1, x_2)$.

1.2 The assumptions

We shall suppose that the functions $\rho(x)$, $\lambda(x)$ and $\mu(x)$ are measurable, positive and bounded and that they satisfy the following conditions :

(i) There exists a strictly positive real number R and three positive constant ρ_∞, λ_∞, μ_∞ such that :

$$\rho(x) = \rho_\infty , \quad \lambda(x) = \lambda_\infty , \quad \mu(x) = \mu_\infty \qquad \text{for } |x| \geq R \tag{1.5}$$

(ii)

$$\left| \begin{aligned} & 0 < \rho_- = \underset{x \in \mathbb{R}^2}{\text{ess. inf.}}\, \rho(x) \leq \rho_+ = \underset{x \in \mathbb{R}^2}{\text{ess. sup.}}\, \rho(x) < +\infty \\ & 0 < \lambda_- = \underset{x \in \mathbb{R}^2}{\text{ess. inf.}}\, \lambda(x) \leq \lambda_+ = \underset{x \in \mathbb{R}^2}{\text{ess. sup.}}\, \lambda(x) < +\infty \\ & 0 < \mu_- = \underset{x \in \mathbb{R}^2}{\text{ess. inf.}}\, \mu(x) \leq \mu_+ = \underset{x \in \mathbb{R}^2}{\text{ess. sup.}}\, \mu(x) < +\infty \end{aligned} \right. \tag{1.6}$$

A particular example of functions $\rho(x)$, $\lambda(x)$, $\mu(x)$ satisfying such assumptions is the following:

- Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^2$ and let $((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty))$ be six strictly positive constants, we define:

$$(\rho(x), \lambda(x), \mu(x)) = \begin{cases} (\rho_0, \lambda_0, \mu_0) & \text{if } x \in \mathcal{O} \\ (\rho_\infty, \lambda_\infty, \mu_\infty) & \text{if } x \notin \mathcal{O} \end{cases}$$

Such an example defines what we shall call a "jump coefficient" medium (we also suppose that $\mathcal{O}$ is homotopic to a point, i.e. $\mathcal{O}$ has no hole).

1.3 Mathematical formulation

In the sequel we consider only real valued functions and real Hilbert spaces. In particular we shall set :

$$H = L^2(R^2, R^3) = L^2(R^2)^3$$

equipped with the following inner product :

$$(u, v) = \sum_{j=1}^{3} \int_{R^2} u_j(x)\, u_j(x)\, \rho(x)\, dx$$

We shall denote by $\|u\| = (u, u)^{1/2}$ the corresponding Hilbert space norm. Let us introduce:

$$V = H^1(R^2, R^3) = H^1(R^2)^3$$

We can define on V the following symmetric bilinear form :

$$a(\beta; u, v) = \sum_{i,j=1}^{3} \int_{R^2} \sigma_{ij}^{\beta}(u)\, \varepsilon_{ij}^{\beta}(v)\ dx. \qquad (u, v) \in V \times V$$

that we can also write :

$$a(\beta; u, v) = \int_{R^2} \lambda(x)\, \mathrm{div}^{\beta} u\ \mathrm{div}^{\beta} v\ dx + 2 \sum_{i,j=1}^{3} \int_{R^2} \mu(x)\, \varepsilon_{ij}^{\beta}(u)\, \varepsilon_{ij}^{\beta}(v)\ dx. \tag{1.7}$$

Formally we have

$$a(\beta; u, v) = (A(\beta)u, v)$$

We can give now the two equivalent formulations of our problem .

<u>Variational formulation</u>

> Find $u \in V$, $u \neq 0$, such that
>
> $a(\beta; u, v) = \omega^2\, (u, v)$, $\forall\, v \in V$.

<u>Spectral formulation</u>

Let $A(\beta)$ be the positive and selfadjoint operator in H, with domain $D(A(\beta))$ dense in H, defined by :

$$\left| \begin{array}{ll} D(\boldsymbol{A}(\beta)) = \left\{ u \in V \,;\, \frac{\partial}{\partial x_j} \sigma^{\beta}_{kj}(u) \in L^2(\mathbb{R}^2),\ k=1,2,3,\ j=1,2 \right\} \\ \boldsymbol{A}(\beta)u = A(\beta)u \qquad \text{if } u \in D(\boldsymbol{A}(\beta)). \end{array} \right.$$

Our problem is equivalent to the following one:

$$\left| \begin{array}{l} \text{Find } u \in D(\boldsymbol{A}(\beta)),\ u \neq 0, \text{ such that} \\ \boldsymbol{A}(\beta)u = \omega^2 u. \end{array} \right.$$

The properties of the operator $\boldsymbol{A}(\beta)$ (selfadjointness, ...) as well as the equivalence of the two formulations are a consequence of the Lax-Milgram theorem , of the coerciveness result (1.8) which we shall establish in the next section, and of the identity :

$$\forall (u, v) \in D(\boldsymbol{A}(\beta)) \times V \qquad a(\beta; u, v) = (\boldsymbol{A}(\beta)u, v)$$

We are thus led to study an eigenvalue problem for an unbounded selfadjoint operator in which the wavenumber β appears simply as a parameter. Let us note that, as the resolvent of $\boldsymbol{A}(\beta)$ is not compact, even the question of the existence of eigenvalues for $\boldsymbol{A}(\beta)$ is not obvious and this is why we need a rather complete study of the bilinear form $a(\beta; u, v)$ to be able to state precise results.

1.4 Properties of the bilinear form $a(\beta; u, v)$

Lemma 1.1

With $|\nabla u|^2 = \sum_{j=1}^{3} |\nabla u_j|^2$, $\nabla u_j = (\frac{\partial u_j}{\partial x_1}, \frac{\partial u_j}{\partial x_2})$, *one has*

$$(1.8) \qquad a(\beta; u, u) \geq \mu_- \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx + \beta^2 \int_{\mathbb{R}^2} |u|^2 dx \right) , \forall\, u \in V.$$

Proof of lemma 1.1 :

From (1.7), we deduce the inequality :

$$(1.9) \qquad a(\beta; u, u) \geq 2\mu_- \int_{\mathbb{R}^2} \left(\sum_{i,j=1}^{3} |\varepsilon^{\beta}_{ij}(u)|^2 \right) dx$$

Then it is sufficient to prove that :

$$2\int_{\mathbb{R}^2} \Big(\sum_{i,j=1}^{3} |\varepsilon_{ij}^{\beta}(u)|^2 \Big)\, dx = \int_{\mathbb{R}^2} \big(|\nabla u|^2 + \beta^2 |u|^2 \big)\, dx + \int_{\mathbb{R}^2} |\mathrm{div}^{\beta} u|^2\, dx$$

To obtain this identity, we first note that :

$$2\sum_{i,j=1}^{3} |\varepsilon_{ij}^{\beta}(u)|^2 = 2\Big(\Big|\frac{\partial u_1}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_2}\Big|^2 + \beta^2 |u_3|^2 \Big) + \Big|\frac{\partial u_1}{\partial x_2}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_1}\Big|^2 + 2\,\frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}$$

$$+ \Big|\frac{\partial u_3}{\partial x_1}\Big|^2 + \beta^2 |u_1|^2 + 2\beta u_1 \frac{\partial u_3}{\partial x_1} + \Big|\frac{\partial u_3}{\partial x_2}\Big|^2 + \beta^2 |u_2|^2 + 2\beta u_2 \frac{\partial u_3}{\partial x_2}$$

Then, thanks to integration by parts, we get out of the terms $\int_{\mathbb{R}^2} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\, dx$, $i \neq j$, and $\int_{\mathbb{R}^2} u_j \frac{\partial u_3}{\partial x_j}\, dx$ and get (1.8).■

We give now a decomposition of the bilinear form $a(\beta; u, v)$ that we shall use in the next sections.

Proposition 1.2

(1.10) $a(\beta; u, u) = \beta^2 \frac{\mu_\infty}{\rho_\infty} \| u \|^2 + b(\beta; u, u) + p(\beta; u, u)$

where we have set:

$$b(\beta; u, u) = b_0(u, u) + b_1(\beta; u, u) + b_2(u, u)$$

(1.11)

$$b_0(u, u) = \int_{\mathbb{R}^2} \left[\lambda \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \Big)^2 + 2\mu \Big(\Big|\frac{\partial u_1}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_2}\Big|^2 \Big) + \mu \Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \Big)^2 \right] dx$$

$$b_1(\beta, u, u) = \int_{\mathbb{R}^2} (\lambda + \mu) \left[\beta^2 |u_3|^2 - 2\beta u_3 \Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \Big) \right] dx$$

$$b_2(u, u) = \int_{\mathbb{R}^2} \mu |\nabla u_3|^2\, dx$$

(1.12) $$p(\beta, u, u) = \beta^2 \int_{\mathbb{R}^2} \Big(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty} \Big) \rho |u|^2\, dx + 2\beta \int_{\mathbb{R}^2} (\mu - \mu_\infty) \left[\frac{\partial}{\partial x_1}(u_1 u_3) + \frac{\partial}{\partial x_2}(u_2 u_3) \right] dx$$

The bilinear form $b(\beta; u, u)$ is positive since one has the identity

(1.13) $$b_0(u, u) + b_1(\beta; u, u) = \int_{\mathbb{R}^2} \mu \left[\Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \Big)^2 + 2\sum_{j=1}^{2} \Big| \frac{\partial u_j}{\partial x_j} - \frac{\beta}{2} u_3 \Big|^2 \right] dx + \int_{\mathbb{R}^2} \lambda\, |\mathrm{div}^{\beta} u|^2\, dx$$

Proof :

By definition of $a(\beta; u, v)$ we have immediately, after having developed some terms:

$$a(\beta; u, u) = \int_{\mathbb{R}^2} \lambda\, |\mathrm{div}^{\beta} u|^2\, dx + \int_{\mathbb{R}^2} \mu \left[2\Big(\Big|\frac{\partial u_1}{\partial x_1}\Big|^2 + \Big|\frac{\partial u_2}{\partial x_2}\Big|^2 \Big) + \Big(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \Big)^2 \right] dx$$

$$+\int_{\mathbb{R}^2} \mu|\nabla u_3|^2 \, dx + \sum_{j=1}^{2} 2\beta \int_{\mathbb{R}^2} \mu \frac{\partial u_3}{\partial x_j} u_j \, dx + \beta^2 \int_{\mathbb{R}^2} \mu \left(|u_1|^2 + |u_2|^2 + 2|u_3|^2\right) dx$$

Then it suffices to remark that

$$\int_{\mathbb{R}^2} \mu \frac{\partial u_3}{\partial x_j} u_j \, dx = \int_{\mathbb{R}^2} (\mu - \mu_\infty)\left(\frac{\partial u_3}{\partial x_j} u_j + \frac{\partial u_j}{\partial x_j} u_3\right) dx - \int_{\mathbb{R}^2} \mu \frac{\partial u_j}{\partial x_j} u_3 \, dx$$

$$\int_{\mathbb{R}^2} \lambda \left|\mathrm{div}^\beta u\right|^2 dx = \int_{\mathbb{R}^2} \lambda \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)^2 dx + \int_{\mathbb{R}^2} \lambda \left[\beta^2 |u_3|^2 - 2\beta u_3 \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right)\right] dx$$

$$\int_{\mathbb{R}^2} \mu \left(|u_1|^2 + |u_2|^2 + |u_3|^2\right) dx = \int_{\mathbb{R}^2} \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho |u|^2 \, dx + \frac{\mu_\infty}{\rho_\infty} \|u\|^2$$

to obtain the identity (1.10). To get (1.13), we simply use the equality :

$$2\left(\frac{\partial u_j}{\partial x_j}\right)^2 + \frac{\beta^2}{2}|u_3|^2 - 2\beta \frac{\partial u_j}{\partial x_j} u_3 = 2\left(\frac{\partial u_j}{\partial x_j} - \frac{\beta}{2} u_3\right)^2 \blacksquare$$

Remark 1.3

In the decomposition (1.10), the main property of the bilinear form $b(\beta; u, v)$ is its positivity. The interest of the bilinear form $p(\beta; u, v)$ comes from the following compactness property :

(1.14) *For any sequence (u^n) converging weakly in V, we have, if u is the limit of (u^n):*

$$\lim_{n \to +\infty} p(\beta; u^n, u^n) = p(\beta; u, u)$$

To prove (1.14), we simply note that, as (u^n) is bounded in $H^1(\mathbb{R}^2)^3$, (u^n) is compact in $L^2(|x| \leq R)^3$. The result follows immediately, as $\mu - \mu_\infty = 0$ for $|x| \geq R$.

2. SPECTRAL STUDY OF THE OPERATOR

In this section we give the main spectral properties of $\boldsymbol{A}(\beta)$ with the exception of the discrete spectrum. We shall study the discrete spectrum of $\boldsymbol{A}(\beta)$ in section 3. This section is divided in two parts : in the first one we determine the essential spectrum of $\boldsymbol{A}(\beta)$ and in the second one we study the properties related to eigenvalues embedded in the essential spectrum.

2.1 The essential spectrum of $\boldsymbol{A}(\beta)$

We recall that the discrete spectrum $\sigma_d(\boldsymbol{A}(\beta))$ of $\boldsymbol{A}(\beta)$ is the set of finite multiplicity eigenvalues which are isolated in the spectrum, $\sigma(\boldsymbol{A}(\beta))$, of $\boldsymbol{A}(\beta)$. By definition, the essential spectrum $\sigma_{ess}(\boldsymbol{A}(\beta))$ is the complement of $\sigma_d(\boldsymbol{A}(\beta))$ in $\sigma(\boldsymbol{A}(\beta))$. We have the following characterisation of the essential spectrum:

Characterisation 2.1 (cf [Sch.])

A number σ belongs to $\sigma_{ess}(\boldsymbol{A}(\beta))$ if and only if :

$$(2.1)\quad \left| \begin{array}{l} \text{There exists a sequence } (u^n) \text{ in } \mathbf{D}(\boldsymbol{A}(\beta)) \text{ such that} \\ \|u^n\|^2 = 1 \\ u^n \rightarrow 0 \quad \text{in } \boldsymbol{H} \ \ (\text{weakly}) \\ \boldsymbol{A}(\beta)\, u^n \rightarrow 0 \quad \text{in } \boldsymbol{H} \ \ (\text{strongly}) \end{array} \right.$$

Theorem 2.2

$$\sigma_{ess}(\boldsymbol{A}(\beta)) = [\ \beta^2 \frac{\mu_\infty}{\rho_\infty}\ , +\infty\ [$$

Proof :

(i) $\sigma_{ess}(\boldsymbol{A}(\beta)) \subset [\ \beta^2 \frac{\mu_\infty}{\rho_\infty}\ , +\infty\ [$

The inequality (1.8) shows that any sequence (u^n) satisfying (2.1) is bounded in $\boldsymbol{V}$. From property (1.14), we know that, as (u^n) converges weakly in $\boldsymbol{V}$ to 0,

$$\lim_{n\rightarrow+\infty} p(\beta;\, u^n,\, u^n) = 0.$$

We shall now use the decomposition (1.10).

$$0 = \lim_{n\rightarrow+\infty} ((\boldsymbol{A}(\beta) - \sigma)\, u^n,\, u^n)$$

$$= \lim_{n\rightarrow+\infty} a(\beta;\, u^n,\, u^n)\ -\ \sigma$$

$$= (\beta^2 \frac{\mu_\infty}{\rho_\infty} - \sigma)\ +\ \lim_{n\rightarrow+\infty} b(\beta;\, u^n, u^n)$$

Then, as $b(\beta; u^n, u^n) \geq 0$, we deduce immediately

$$\sigma \geq \beta^2 \frac{\mu_\infty}{\rho_\infty}$$

(ii) $\sigma_{ess}(\boldsymbol{A}(\beta)) \supset [\, \beta^2 \frac{\mu_\infty}{\rho_\infty}, +\infty \,[$

We construct, for each σ in the interval $[\beta^2\mu_\infty/\rho_\infty, +\infty[$, a sequence (u^n) verifying (2.1). For this, let us denote by $A_\infty(\beta)$ the operator corresponding to constant coefficients $\rho=\rho_\infty$, $\lambda=\lambda_\infty$, $\mu=\mu_\infty$. We have, in the sense of distributions :

$$A_\infty(\beta)\, u_S = \frac{\mu_\infty}{\rho_\infty}(k_1^2 + k_2^2 + \beta^2)\, u_S$$

where

$$u_S\,(x_1, x_2) = (k_1^2 + k_2^2)^{-1/2} \cos(k_1x_1 + k_2x_2) \begin{bmatrix} k_2 \\ -k_1 \\ 0 \end{bmatrix}, \quad (k_1,k_2) \in \mathbb{R}^2$$

(Physically, the function u_S corresponds to a shear wave or S-wave, see [Mi.]). Unfortunately, u_S does not belong to the domain of $\boldsymbol{A}(\beta)$. That is why we introduce a cut-off function ϕ_n to define

$$\left| \begin{array}{l} u^n = \phi_n\, u_S \\ \phi_n(x) = \alpha_n\, \phi(\frac{x_1 - 3n}{n}, \frac{x_2}{n}) \\ \phi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| \geq 2 \end{cases}, \quad \phi \in C_0^\infty(\mathbb{R}^2) \\ \rho_\infty \displaystyle\int_{\mathbb{R}^2} |\phi_n(x)|^2\, |u_S(x)|^2\, dx = 1 \qquad \text{(determines } \alpha_n) \end{array} \right.$$

Then, if one chooses (k_1, k_2) such that $\sigma = \frac{\mu_\infty}{\rho_\infty}(k_1^2 + k_2^2 + \beta^2)$, it is rather easy to check that the sequence (u^n) satisfies (2.1). The theorem follows immediately since $\sigma_{ess}(\boldsymbol{A}(\beta))$ is closed.

2.2 About the eigenvalues embedded in the essential spectrum

2.2.1 Properties of the eigenfunctions

Lemma 2.3

Any eigenfunction u associated with an eigenvalue ω^2 *such that* $\omega^2 > \frac{\mu_\infty}{\rho_\infty}\beta^2$ *satisfies :*

(2.2) $$\frac{\partial u_3}{\partial x_i} - \beta\, u_i = 0 \quad \text{if } i = 1, 2 \text{ and } |x| \geq R.$$

Proof: When $|x| \geq R$, the three Lamé's coefficients ρ, λ and μ are constant and the equations satisfied by u are written :

(2.3) $$\begin{cases} \mu_\infty(\Delta u_1 - \beta^2 u_1) + (\lambda_\infty + \mu_\infty)\dfrac{\partial}{\partial x_1}(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} - \beta\, u_3) = -\rho_\infty\, \omega^2\, u_1 \\ \mu_\infty(\Delta u_2 - \beta^2 u_2) + (\lambda_\infty + \mu_\infty)\dfrac{\partial}{\partial x_2}(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} - \beta\, u_3) = -\rho_\infty\, \omega^2\, u_2 \\ \mu_\infty(\Delta u_3 - \beta^2 u_3) + (\lambda_\infty + \mu_\infty)\,\beta\,(\dfrac{\partial u_1}{\partial x_1} + \dfrac{\partial u_2}{\partial x_2} - \beta\, u_3) = -\rho_\infty\, \omega^2\, u_3 \end{cases}$$

Let us introduce, for i =1, 2 :

$$r_{i3} = \frac{\partial u_3}{\partial x_i} - \beta\, u_i \qquad \in \boldsymbol{H}.$$

Combining equations (2.3), it is easy to see that

$$\Delta r_{i3} + \frac{\rho_\infty}{\mu_\infty}(\omega^2 - \beta^2\frac{\mu_\infty}{\rho_\infty})r_{i3} = 0 \qquad i = 1, 2\,.$$

To complete the proof, it suffices to apply Rellich's theorem ([Wi.2], p. 56).∎

Lemma 2.4

Any eigenfunction u associated with an eigenvalue ω^2 *such that* $\omega^2 > \frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty}\beta^2$ *satisfies :*

$$u(x) = 0 \quad \textit{for } |x| \geq R$$

Proof: From (2.2), we deduce that :

$$\Delta u_3 = \beta\,(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}) \qquad \text{for } |x| \geq R$$

Plugging this equality into the last equation of (2.3) leads to :

$$\Delta u_3 + \frac{\rho_\infty}{\lambda_\infty + 2\mu_\infty}(\omega^2 - \beta^2 \frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty})u_3 = 0$$

By Rellich's theorem, we know that $u_3(x) = 0$ for $|x| \geq R$. Then, using (2.2), we have $u_1(x) = u_2(x) = 0$ for $|x| \geq R$.

2.2.2 Multiple jump coefficient guide : a non existence result

We first state a result concerning a jump coefficient guide.

Corollary 2.5

For a jump coefficient guide, any eigenvalue ω^2 of $\boldsymbol{A}(\beta)$ satisfies

$$\omega^2 \leq \beta^2 \frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty}$$

Proof : Let us recall that (see section 1.2) :

$$\begin{cases} \rho(x) = \rho_0, \quad \mu(x) = \mu_0, \quad \lambda(x) = \lambda_0 & \text{if } x \in \mathcal{O} \\ \rho(x) = \rho_\infty, \quad \mu(x) = \mu_\infty, \quad \lambda(x) = \lambda_\infty & \text{if } x \notin \mathcal{O} \end{cases}$$

Reasoning as for Lemma 2.3, we show that the functions $r_{i3} = \frac{\partial u_3}{\partial x_i} - \beta u_i$ satisfy :

$$\Delta r_{i3} + \frac{\rho_\infty}{\mu_\infty}(\omega^2 - \beta^2 \frac{\mu_\infty}{\rho_\infty})r_{i3} = 0 \qquad \text{in } R^2 \setminus \mathcal{O}$$

This proves that r_{i3} is analytic in $R^2 \setminus \mathcal{O}$. As $R^2 \setminus \mathcal{O}$ is connected and $r_{i3} = 0$ for $|x| \geq R$, we deduce that $r_{i3} = o$ in $R^2 \setminus \mathcal{O}$. By the same manipulations as in the proof of lemma 2.4, we now see that

$$\Delta u_3 + \frac{\rho_\infty}{\lambda_\infty + 2\mu_\infty}(\omega^2 - \beta^2 \frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty})u_3 = 0$$

which proves that u_3 is analytic in $R^2 \setminus \mathcal{O}$ and equal to 0 for $|x| \geq R$. As $R^2 \setminus \mathcal{O}$ is connected, we can state that :

$$u_3 = 0 \qquad \text{in } \mathrm{R}^2 \setminus \boldsymbol{O}$$

and therefore, as $r_{i3} = 0$ in $\mathrm{R}^2 \setminus \boldsymbol{O}$, that :

$$u = 0 \qquad \text{in } \mathrm{R}^2 \setminus \boldsymbol{O}.$$

Now, let $A_0(\beta)$ be the differential operator defined by ((1.2),(1.3)) when $\rho(x) = \rho_0$, $\mu(x) = \mu_0$ and $\lambda(x) = \lambda_0$. In the sense of distributions, we have

$$\begin{cases} A_0(\beta)u = \omega^2 u & \text{in } \boldsymbol{O} \\ A_0(\beta)u = \omega^2 u & \text{in } \mathrm{R}^2 \setminus \boldsymbol{O} \qquad (\text{ as } u = 0 \text{ }) \end{cases}$$

Let $a_0(\beta;.,.)$ be the bilinear form given by (1.7) when $\rho(x) = \rho_0$, $\mu(x) = \mu_0$ and $\lambda(x) = \lambda_0$. As the support of u is included in $\overline{\boldsymbol{C}}$, it is easy to see that, for any v in $\mathrm{D}(\mathrm{R}^2)$ ($\equiv C_0^\infty(\mathrm{R}^2)$)

$$a_0(\beta;\, u,\, v) = a(\beta;\, u,\, v)$$

from which we deduce

$$< \rho_0 A_0(\beta)u\, ,\, v > = \omega^2\, (u,\, v)$$

This proves that :

$$A_0(\beta)u = \omega^2 u \qquad \textit{in } \mathrm{D}'(\mathrm{R}^2).$$

We can mimick the first part of the proof, replacing ρ_∞, μ_∞ and λ_∞ by ρ_0, μ_0 and λ_0 and show that u is identically 0 which contradicts the fact that u is an eigenfunction .■

The proof of corollary 2.5 leads to a natural generalization of the previous result. First of all, let us make a definition.

Definition 2.6

Assume that there exists N bounded connected open sets $\boldsymbol{O}_j$ (*in* R^2), $j = 1,2,\ldots,N$ *and strictly positive constants* $\{(\rho_j, \mu_j, \lambda_j), j = 1, 2,\ldots,N,$ *such that :*

$$\begin{cases} O_j \cap O_k = \emptyset & \text{if } j \neq k \\ R^2 \setminus \overline{\bigcup_{j=1}^{N} O_j} \text{ is connected} \\ \rho(x) = \rho_j,\ \lambda(x) = \lambda_j,\ \mu(x) = \mu_j & \text{if } x \in O_j \\ \rho(x) = \rho_\infty,\ \lambda(x) = \lambda_\infty,\ \mu(x) = \mu_\infty & \text{if } x \in R^2 \setminus \overline{\bigcup_{j=1}^{N} O_j} \end{cases}$$

We shall say that such assumptions define a *<u>multiple jump coefficient guide</u>*.

We can now enounce our result :

> **<u>Theorem 2.7</u>**
>
> *For a multiple jump coefficient guide, the operator* $\boldsymbol{A}(\beta)$ *has no eigenvalue in the interval* $]\ \beta^2 \dfrac{\lambda_\infty + 2\mu_\infty}{\rho_\infty},\ +\infty\ [$

<u>Proof</u>: We can suppose without loss of generality that the sets O_j are numbered in such a way that, for any k, the set $(R^2 \setminus \bigcup_{j=1}^{k} \overline{O_j})$ is connected. It is then easy to see that the result can be obtained by multiple iterations of the proof of corollary 2.5. One first shows that u is equal to zero in $(R^2 \setminus \bigcup_{j=1}^{N} \overline{O_j})$, then in $(R^2 \setminus \bigcup_{j=1}^{N-1} \overline{O_j})$, and so on ...

<u>Remark</u>

The proof shows that the number N can be no finite in theorem 2.7, and that the open set O can have a hole in corollary 2.5.

<u>2.2.3 Multiplicity and accumulation points of eigenvalues of $\boldsymbol{A}(\beta)$</u>

These two questions can be studied in a similar way since in both cases we have to work with a sequence (u^n) in $\mathbf{D}(\boldsymbol{A}(\beta))$ which converges weakly in $\boldsymbol{H}$ to 0, with $\|u^n\| = 1$. To express that σ is an eigenvalue of $\boldsymbol{A}(\beta)$ with infinite multiplicity, we simply write :

(2.4) $\qquad \boldsymbol{A}(\beta)\, u^n = \sigma\, u^n \quad , n = 1, 2, \ldots,$

while σ is an accumulation point of the eigenvalues of $\boldsymbol{A}(\beta)$ if and only if there exists a sequence σ_n of real numbers such that

$$(2.5) \qquad \begin{cases} \boldsymbol{A}(\beta)\, u^n = \sigma_n\, u^n \\ \sigma_n \to \sigma \quad (n \to +\infty) \end{cases}$$

Proposition 2.8

Any eigenvalue σ of $\boldsymbol{A}(\beta)$, distinct from $\beta^2 \dfrac{\mu_\infty}{\rho_\infty}$ and $\beta^2 \dfrac{\lambda_\infty + 2\mu_\infty}{\rho_\infty}$, has finite multiplicity.

Proof: Suppose that u^n satisfies (2.4) and converges weakly in $\boldsymbol{H}$ to 0. As in theorem 2.2 we can conclude that $\sigma \geq \beta^2 \dfrac{\mu_\infty}{\rho_\infty}$. Suppose now that $\sigma > \beta^2 \dfrac{\mu_\infty}{\rho_\infty}$ and let (u^n) be a sequence of eigenfunctions associated with σ such that:

$$\begin{cases} \|u^n\| = 1 \\ u^n \to 0 \qquad \text{(weakly) in } \boldsymbol{H} \end{cases}$$

To study $a(\beta; u^n, u^n)$, we use the decomposition (1.10). After having remarked that :

$$(2.6) \qquad \begin{aligned} b_1(\beta; u^n, u^n) = &\int_{R^2} [\, (\lambda+\mu) - (\lambda_\infty+\mu_\infty)\,]\,[\, \beta^2 |u_3^n|^2 - 2\beta\, (\frac{\partial u_1^n}{\partial x_1} + \frac{\partial u_2^n}{\partial x_2})\, u_3^n \,]\, dx \\ &- (\lambda_\infty+\mu_\infty) \int_{R^2} [\, \beta^2 |u_3^n|^2 - 2\beta\, (\frac{\partial u_1^n}{\partial x_1} + \frac{\partial u_2^n}{\partial x_2})\, u_3^n \,]\, dx \end{aligned}$$

we integrate by parts the second term of the right hand side of (2.6) and use lemma 2.3 to replace $\partial u_3^n / \partial x_j$ by βu_j when $|x| \geq R$. Then we obtain:

$$a(\beta; u^n, u^n) - \beta^2 \frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty} \| u^n \|^2 = b_0(u^n, u^n) + b_2(u^n, u^n) + \beta^2 (\lambda_\infty+\mu_\infty) \int_{|x|>R} (|u_1^n|^2 + |u_2^n|^2)\, dx + \tilde{p}(\beta; u^n, u^n)$$

where $\tilde{p}(\beta; ., .)$ as the same compactness property (1.14) as $p(\beta; ., .)$. As b_0 and b_2 are positive, $\|u^n\| = 1$, and $a(\beta; u^n, u^n) = \sigma$, we get:

$$\sigma - \beta^2 \frac{\lambda_\infty + 2\mu_\infty}{\rho_\infty} \geq \tilde{p}(\beta; u^n, u^n)$$

Taking the limit when n goes to infinity shows that $\sigma \geq \beta^2 (\lambda_\infty+2\mu_\infty)/\rho_\infty$. Now suppose that $\sigma > \beta^2 (\lambda_\infty+2\mu_\infty)/\rho_\infty$. Applying lemma 2.4, we can state that the sequence (u^n) converges strongly in $\boldsymbol{H}$ to 0 (and not only weakly) : indeed by compactness, (u^n) converges strongly in $L^2(|x| < R)^3$ and, by lemma 2.4, the support of u^n is included in the ball $|x| \leq R$. This contradicts the equality $\|u^n\| = 1$ and completes the proof of the theorem.■

It is natural to think that the absence of eigenvalues in the interval $]\beta^2 (\lambda_\infty + 2\mu_\infty) /\rho_\infty, +\infty[$ is a general result. But this remains a conjecture which could be proven if one could use for systems a unique continuation theorem as it exists for the operator $-\Delta$ for instance (see theorem XIII.63in [R.S.2]).
The situation concerning the eigenvalues in the interval $]\beta^2 \mu_\infty /\rho_\infty, \beta^2 (\lambda_\infty + 2\mu_\infty) /\rho_\infty[$ is much less clear. In fact there exist some examples of functions $(\rho(x), \lambda(x), \mu(x))$ for which such eigenvalues do exist. We refer the reader to a forthcoming paper in preparation for a more detailed analysis of this particular point.

Proposition 2.9

The eigenvalues of $\boldsymbol{A}(\beta)$ can accumulate only at infinity or at $\beta^2 \dfrac{\lambda_\infty+2\mu_\infty}{\rho_\infty}$.

Proof:

It is similar to the one of proposition 2.8. A priori, a sequence of finite multiplicity eigenvalues of $\boldsymbol{A}(\beta)$ could converge to $\beta^2 \mu_\infty /\rho_\infty$ but the result of theorem 3.8 will exclude this possibility.■

3. THE DISCRETE SPECTRUM OF $\boldsymbol{A}(\beta)$, EXISTENCE AND PROPERTIES OF THE GUIDED WAVES

This section, in which we study the eigenvalues of $\boldsymbol{A}(\beta)$ which are not embedded in the essential spectrum, contains the main results of this article. We have divided this section in two parts. The sub-section 3.1 is devoted to our two main existence results. These results lead us to introduce the important notion of thresholds, that we study in details in the subsection 3.2.
The main tool of the analysis is the well-known max-min principle [R.S.2]. We are going a statement of this principle for the particular case of our operator $\boldsymbol{A}(\beta)$. Let us first introduce some notation.

Definition 3.1

For any integer $m \geq 1$, we shall denote by $s_m(\beta)$ the real number defined by one of the two equivalent formulas (cf.[D.S.], p. 1544)

$$(3.1) \quad s_m(\beta) = \sup_{(v_1,v_2,\ldots,v_{m-1}) \in \boldsymbol{H}} \left(\inf_{\substack{v \in [v_1,v_2,\ldots,v_{m-1}]^\perp \\ v \neq 0}} \frac{a(\beta; v, v)}{\|v\|^2} \right)$$

$$(3.2) \quad s_m(\beta) = \inf_{(v_1,\ldots,v_m) \in V} \left(\sup_{\substack{v \in [v_1,\ldots,v_m] \\ v \neq 0}} \frac{a(\beta;\, v,\, v)}{\|v\|^2} \right)$$

where we have set

$$[v_1,\ldots v_m] = \{ v = \sum_{j=1}^{m} \alpha_j v_j \,;\, (\alpha_1,\ldots,\alpha_m) \in \mathrm{R}^m \}$$

$$[v_1,\ldots,v_m]^{\perp} = \{ v \in V \,;\, (v,v_j) = 0,\ 1 \le j \le m \}$$

Joining the result of theorem 2.2 to the Max-Min principle permits us to enonce the following theorem:

<u>**Theorem 3.2**</u>
The sequence $(s_m(\beta))$ is an increasing sequence of real numbers converging to $\beta^2 \mu_\infty / \rho_\infty$. For each m, the following alternative holds:
(i) $s_m(\beta) < \beta^2 \mu_\infty / \rho_\infty$: the operator $\boldsymbol{A}(\beta)$ has at least m eigenvalues, counted with their multiplicity, strictly smaller than $\beta^2 \mu_\infty / \rho_\infty$, which are:

$$s_1(\beta) \le s_2(\beta) \le \ldots \le s_m(\beta)$$

(ii) $s_m(\beta) = \beta^2 \mu_\infty / \rho_\infty$: the operator $\boldsymbol{A}(\beta)$ has at most m eigenvalues strictly smaller than $\beta^2 \mu_\infty / \rho_\infty$.

<u>3.1 Existence of guided modes</u>

In fact in this section we discuss the existence of discrete spectrum. Let us first remark that, from the coerciveness inequality (1.8), we immediately deduce the:

<u>**Lemma 3.3**</u>

$$\sigma_d(\boldsymbol{A}(\beta)) \subseteq [\ \beta^2 \frac{\mu_-}{\rho_+},\ \beta^2 \frac{\mu_\infty}{\rho_\infty})$$

Therefore, the discrete spectrum of $\boldsymbol{A}(\beta)$ will be empty as soon as $\mu_- / \rho_+ = \mu_\infty / \rho_\infty$. Let us state a precise result in the following corollary.

<u>**Corollary 3.4**</u>
Assume that:

$\rho(x) \le \rho_\infty$ *a.e.* $x \in \mathrm{R}^2$

$\mu(x) \ge \mu_\infty$ *a.e.* $x \in \mathrm{R}^2$

then $\boldsymbol{A}(\beta)$ has no discrete spectrum.

This simple result shows that the question of the existence of discrete spectrum is not trivial. Besides, theorem 3.2 gives us a method to prove the existence of eigenvalues in the interval $[\beta^2 \mu_- / \rho_+, \beta^2 \mu_\infty / \rho_\infty[$: if we are able to construct m appropriate test functions $(v_1,\ldots, v_1)$ in V such that:

(3.3) $\quad \forall\, v \in [v_1, \ldots, v_2]$, $\quad a(\beta; v, v) - \beta^2 \dfrac{\mu_\infty}{\rho_\infty} < 0$

then point (i) of theorem 3.2 holds and we know that $\boldsymbol{A}(\beta)$ has at least m eigenvalues, namely $s_1(\beta)$, $s_2(\beta), \ldots, s_m(\beta)$.

3.1.1 A first existence result

We set:

$$\left(\frac{\mu}{\rho}\right)_- = \underset{x \in \mathbb{R}^2}{\text{ess. inf.}} \frac{\mu(x)}{\rho(x)}$$

Theorem 3.5

If the inequality $\left(\dfrac{\mu}{\rho}\right)_- < \dfrac{\mu_\infty}{\rho_\infty}$ *holds, then, for each integer* $m \geq 1$, *there exists* $\beta^*_m \geq 0$, *such that*

$$s_m(\beta) < \beta^2 \frac{\mu_\infty}{\rho_\infty} \quad \text{for} \quad \beta > \beta^*_m,$$

which means that $\boldsymbol{A}(\beta)$ *has at least m eigenvalues in the interval* $[\beta^2 \dfrac{\mu}{\rho_+}, \beta^2 \dfrac{\mu_\infty}{\rho_\infty})$, *namely* $(s_1(\beta), s_2(\beta), \ldots, s_m(\beta))$.

Proof:

From the inequality $(\mu/\rho)_- < \mu_\infty/\rho_\infty$ we deduce the existence of a non negligible measurable set C such that $\mu(x)/\rho(x) < \mu_\infty/\rho_\infty$,a.e. $x \in C$. There exists an open set U such that $C \cap U$ and $C \cap (\mathbb{R}^2 \setminus \overline{U})$ have a strictly positive measure. Repeating this result, we prove the existence, for any $m \geq 1$, of m open sets $(U_1, U_2, \ldots, U_m)$, $U_j \cap U_k = \emptyset$ if $k \neq j$, and m compact subsets $(C_1, \ldots, C_m)$ of C, with non zero measure such that:

$U_k \supset C_k \qquad k = 1, 2, \ldots, m$

$\dfrac{\mu(x)}{\rho(x)} < \dfrac{\mu_\infty}{\rho_\infty}$,a.e. $x \in C_k$, $k = 1, 2, \ldots, m$.

For each k, we can find u_1^k in $H_0^1(U_k)$ such that

$$\int_{U_k} \left(\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho}\right) \rho)\, |u_1^k|^2\, dx > 0 \quad \text{and} \quad \int_{U_k} |u_1^k|^2 \rho) dx = 1$$

Let $\tilde{u}_1^k$ the function of $\boldsymbol{V}$ equal to u_1^k in U_k and equal to zero everywhere else. The functions $u^k = (\tilde{u}_1^k, 0, 0)$ generate a m-dimensional subspace $\boldsymbol{V}_m$ of $\boldsymbol{V}$. Moreover, as the U_k are disjoint, the vectors u^k constitute an orthonormal basis of V_m and give also the principal directions of the quadratic form defined on V_m by $a(\beta;\cdot,\cdot)$. Therefore, there exists an integer k, $1 \le k \le m$, such that:

$$\sup_{\substack{v \in V_m \\ v \neq 0}} \frac{a(\beta;\, v, v)}{\|v\|^2} = a(\beta;\, u^k, u^k).$$

But, thanks to equality (1.10):

$$(3.4)\quad a(\beta;\, u^k, u^k) = \beta^2 \frac{\mu_\infty}{\rho_\infty} + \int_{U_k} \left[(\lambda + 2\mu)\left(\frac{\partial u_1^k}{\partial x_1}\right)^2 + \mu \left(\frac{\partial u_1^k}{\partial x_2}\right)^2 \right] dx \;-\; \beta^2 \int_{U_k} \left(\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho}\right) \rho\, |u_1^k|^2\, dx$$

which proves that, for β large enough:

$$a(\beta;\, u^k, u^k) < \beta^2 \frac{\mu_\infty}{\rho_\infty}$$

__Remark__: In the case where the inequality $\mu(x)/\rho(x) < \mu_\infty/\rho_\infty$ holds in some open ball, the proof of theorem 3.5 is simpler and leads to a more precise result that we shall give in section 3.2 (theorem 3.10).In fact, the idea of the proof is to compare the number $s_m(\beta)$ with the eigenvalue of an "interior" Dirichlet problem for a classical scalar elliptic operator in this ball.

3.1.2. A second existence result: generalized Stoneley waves.

One can naturally wonder whether the condition $(\mu/\rho)_- < \mu_\infty/\rho_\infty$ is necessary for the existence of a discrete spectrum or not. Our next result will prove that it is not. In fact, we have the general inequality:

$$\frac{\mu_-}{\rho_+} \le \left(\frac{\mu}{\rho}\right)_- \le \frac{\mu_\infty}{\rho_\infty}$$

It can occur that $(\mu/\rho)_- = \mu_\infty/\rho_\infty$ and that $\mu_-/\rho_+ < (\mu/\rho)_-$ which preserves the possibility of the presence of eigenvalues of $\boldsymbol{A}(\beta)$ in the interval $]\, \beta^2 \mu_-/\rho_+, \beta^2 (\mu/\rho)_- [$. The existence of the generalized Stoneley waves will give an example of such eigenvalues.

We shall obtain our main result for a jump coefficient guide associated with $(\rho_0, \lambda_0, \mu_0)$ and $(\rho_\infty, \lambda_\infty, \mu_\infty)$ (cf. section 1.2). We shall define in appendix what is the Stoneley's equation associated with $((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty))$. To state our result we only have to define the set $\mathrm{E_S}$ as the set of coefficients $((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty))$ for which the corresponding Stoneley's equation admits at least one real solution in the interval $]\, 0, (\mu/\rho)_-^{1/2} [$ where $(\mu/\rho)_-$ is equal to Min $(\mu_0/\rho_0, \mu_\infty/\rho_\infty)$. $\mathrm{E_S}$ is a non empty subset of $(\mathrm{R}^+)^3 \times (\mathrm{R}^+)^3$ and has been studied by several authors ([Ca.], [Mi.] p.165, [E.S.] p.539). When $((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty))$ belongs to $\mathrm{E_S}$, we denote by V_{st} the smallest solution in the interval $]\, 0, (\mu/\rho)_-^{1/2} [$ of the Stoneley's equation. We can state the following theorem.

THEOREM 3.6

Suppose that $((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty))$ *belongs to* $\mathbf{E}_S$. *Then for the corresponding jump coefficient guide whose interface* Γ *is of class* C^1, *one has, for any* $m \in \mathbf{N}^*$:

$$(3.5) \qquad \limsup_{\beta \to +\infty} \frac{s_m(\beta)}{\beta^2} \leq V_{st}^2 \qquad \left(< \frac{\mu_\infty}{\rho_\infty}\right),$$

In particular, there exists an increasing sequence $(\beta_m^*)_{m\geq 1}$ *of positive real numbers such that, for* $\beta > \beta_m^*$, *the operator* $A(\beta)$ *has at least* m *eigenvalues in the interval* $]0, \beta^2 \frac{\mu_\infty}{\rho_\infty}[$ *which are* $s_1(\beta) \leq \ldots \leq s_m(\beta)$.

We shall give the proof of this result later in this section. In fact, an attentive reading of this proof will show that the previous theorem can be generalized as follows

Theorem 3.7

The conclusions of theorem 3.6 still hold as soon as the three following conditions are satisfied :

(i) There exist three open sets O, O_1, O_2 *such that*

. $\bar{O} = \bar{O}_1 \cup \bar{O}_2$, $\quad O_1 \cap O_2 = \emptyset$

. $(\rho(x), \lambda(x), \mu(x)) = (\rho_i, \lambda_i, \mu_i)$ if $x \in O_i$, i = 1, 2

. *the interface* $\Gamma = \partial O_1 \cap \partial O_2$ *is of class* C^1

(ii) $((\rho_1, \lambda_1, \mu_1), (\rho_2, \lambda_2, \mu_2))$ *belongs to the set* $\mathbf{E}_S$.

(iii) V_{st}, *the smallest solution of the Stoneley's equation associated with* $((\rho_1, \lambda_1, \mu_1), (\rho_2, \lambda_2, \mu_2))$, *is strictly smaller than* $\left(\frac{\mu_\infty}{\rho_\infty}\right)^{1/2}$.

Remarks

. In fact, it will be clear in the proof of theorem that the interface Γ only needs to be locally of class C^1.

. We shall see that the set $\mathbf{E}_S$ has the following symmetry property:

$((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty)) \in \mathbf{E}_S \Rightarrow ((\rho_\infty, \lambda_\infty, \mu_\infty), (\rho_0, \lambda_0, \mu_0)) \in \mathbf{E}_S$

This property allows us to construct a medium for which $(\mu/\rho)_- = \mu_\infty/\rho_\infty$ and for which guided waves exist.

. The guided waves that we point out in theorem can be considered as generalized Stoneley waves. These waves are interface waves (as the Rayleigh wave is a surface wave). Indeed it can be proved that, for large β, their energy concentrates exponentially near the interface Γ.

Proof of theorem 3.6

Let us recall that:

$$(\rho(x), \lambda(x), \mu(x)) = \begin{cases} (\rho_0, \lambda_0, \mu_0) \text{ if } x \in \boldsymbol{O} \\ (\rho_\infty, \lambda_\infty, \mu_\infty) \text{ if } x \in \Omega = \mathrm{R}^2 \setminus \overline{\boldsymbol{O}} \end{cases}$$

and that the interface Γ between $\boldsymbol{O}$ and Ω is supposed to be C^1. So, there exist locally:

$$\begin{cases} \blacklozenge \text{ a system of orthonormal coordinates } (0, x_1, x_2) \\ \blacklozenge \text{ a function } f \in C^1(\mathrm{R}) \,/\, f(0) = f'(0) = 0 \\ \blacklozenge \text{ a neighborhood } V \text{ of the origin } 0 \end{cases}$$

such that, for some a > 0:

$$\Gamma \cap V = \{\, (x_1, x_2) \in \mathrm{R}^2 \,/\, x_2 = f(x_1), x_1 \in \,]-a, a[\,\}$$
$$\boldsymbol{O} \cap V = \{\, (x_1, x_2) \in V \,/\, x_2 < f(x_1), x_1 \in \,]-a, a[\,\}$$
$$\Omega \cap V = \{\, (x_1, x_2) \in V \,/\, x_2 > f(x_1), x_1 \in \,]-a, a[\,\}$$

Then it is clear that there exists $\delta > 0$ such that

$$\boldsymbol{O} \cap V = \{\, (x_1, x_2) \in \mathrm{R}^2 \,/\, |x_1| < a, f(x_1) - \delta < x_2 < f(x_1) \,\}$$
$$\Omega \cap V = \{\, (x_1, x_2) \in \mathrm{R}^2 \,/\, |x_1| < a, f(x_1) < x_2 < f(x_1) + \delta \,\}$$

We choose ϕ in $H_0^1\,(]-a, a[)$ and ψ_δ in $C_0^\infty(\mathrm{R})$, $0 \le \psi_\delta \le 1$ such that $\psi_\delta(x_2) = 0$ if $|x_2| > \delta$ and $\psi_\delta(x_2) = 1$ if $|x_2| < \delta/2$. We shall consider in the sequel test displacement fields u in the form:

$$(3.6) \qquad \begin{cases} u(x_1, x_2) = \phi(x_1)\, u^\delta(x_2 - f(x_1)) \\ u^\delta(x_2) = \psi_\delta(x_2)\, u^{st}(x_2) \end{cases}$$

where u^{st} has been defined in appendix by (A.4) and (A.8). We have supposed implicitly that the first coordinate u_1 is identically equal to 0. Let us note that the support of u is contained in:

$$\Omega_0 = \{(x_1, x_2) \in \mathrm{R}^2 \,/\, |x_1| < a, f(x_1) - \delta < x_2 < f(x_1) + \delta\}$$

For functions u of the form (3.6) we can calculate that:

$$(3.7)\quad \left| \begin{aligned} a'(\beta; u, u) &= \Big(\int_{-a}^{a} |\phi(x_1)|^2\, dx_1\Big)\, a'(\beta; u^\delta, u^\delta) \\ &+ \Big(\int_{-a}^{a} |\phi'(x_1)|^2\, dx_1\Big)\Big(\int_{-\infty}^{+\infty} \mu(x_2)\, |u^\delta(x_2)|^2\, dx_2\Big) \\ &+ \Big(\int_{-a}^{a} (\phi(x_1))^2\, (f'(x_1))^2\, dx_1\Big)\Big(\int_{-\infty}^{+\infty} \mu(x_2)\, \Big|\frac{du^\delta}{dx_2}(x_2)\Big|^2\, dx_2\Big) \\ &-2\Big(\int_{-a}^{a} \phi(x_1)\, \phi'(x_1)\, f'(x_1)\, dx_1\Big)\Big(\int_{-\infty}^{+\infty} \mu(x_2)\, u^\delta(x_2).\, \frac{du^\delta}{dx_2}(x_2)\, dx_2\Big) \end{aligned} \right.$$

where $a'(\beta;.,.)$ is defined by (A.2). When β becomes large, the function u^{st} is concentrated in a neighborhood of $x_2 = 0$. We shall explain later why this permits us to write:

$$(3.8)\quad \left| \begin{aligned} & a'(\beta; u^\delta, u^\delta) = a'(\beta; u^{st}, u^{st})\, (1+O(\beta^2\, e^{-\eta\beta\delta})) \\ & \int_{-\infty}^{+\infty} \mu(x_2)\, |u^\delta(x_2)|^2\, dx_2 = \Big(\int_{-\infty}^{+\infty} \mu(x_2)\, |u^{st}(x_2)|^2\, dx_2\Big)\, (1+O(e^{-\eta\beta\delta})) \\ & \int_{-\infty}^{+\infty} \mu(x_2)\, \Big|\frac{du^\delta}{dx_2}(x_2)\Big|^2\, dx_2 = \Big(\int_{-\infty}^{+\infty} \mu(x_2)\, \Big|\frac{du^{st}}{dx_2}(x_2)\Big|^2\, dx_2\Big)\, (1+O(\beta e^{-\eta\beta\delta})) \\ & \int_{-\infty}^{+\infty} \mu(x_2)\, u^\delta(x_2).\, \frac{du^\delta}{dx_2}(x_2)\, dx_2 = \Big(\int_{-\infty}^{+\infty} \mu(x_2)\, u^{st}(x_2).\frac{du^{st}}{dx_2}(x_2)\, dx_2\Big)\, (1+O(\beta e^{-\eta\beta\delta})) \end{aligned} \right.$$

where η denotes a strictly positive constant. In the way, one easily checks that u, defined by (3.6), satisfies:

$$(3.9)\qquad \int_{\mathbb{R}^2} \rho\, |u|^2 dx = \Big(\int_{-a}^{a} |\phi(x_1)|^2\, dx_1\Big)\, (1+O(e^{-\eta\beta\delta}))$$

It suffices to join this result to (3.7) and (3.8) and apply the Cauchy-Schwartz inequality and identity (A.9) to obtain:

$$\frac{1}{\beta^2}\frac{a(\beta;\, u,\, u)}{\|u\|^2} \le V_{st}^2\, (1+O(\beta^2\, e^{-\eta\beta\delta/2})) + \frac{1}{\beta^2}\, q(a;\, \phi, f)\, (1+O(\beta e^{-\eta\beta\delta}))$$

where we have set:

$$(3.10)\quad \left|\begin{array}{l} q(a;\phi,f) = \left[\displaystyle\int_{-\infty}^{+\infty} \mu \left(\left|\frac{du^{st}}{dx_2}(x_2)\right|^2 + u^{st}.\frac{du^{st}}{dx_2}(x_2)\right) dx_2 \right] \displaystyle\sup_{|x_1|<a} |f'(x_1)|^2 \\ \qquad + \left[\displaystyle\int_{-\infty}^{+\infty} \mu \left(|u^{st}|^2 + u^{st}.\frac{du^{st}}{dx_2}\right) dx_2 \right] \dfrac{\displaystyle\int_{-a}^{a} |\phi'(x_1)|^2\, dx_1}{\displaystyle\int_{-a}^{a} |\phi(x_1)|^2\, dx_1} \end{array}\right.$$

It is then easy to see that there exists a positive constant C such that, for β large enough:

$$\frac{1}{\beta^2} q(a;\phi,f) \le C \sup_{|x_1|<a} |f'(x_1)|^2 + \frac{C}{\beta} \frac{\displaystyle\int_{-a}^{a} |\phi'(x_1)|^2\, dx_1}{\displaystyle\int_{-a}^{a} |\phi(x_1)|^2\, dx_1}$$

Now, let $(\phi_1, \phi_2,..., \phi_m)$ be functions associated to the m first eigenvalues of the operator $-d^2/dx^2$ in $H_0^1(]-a, a[)$ and V_m the subspace of V generated by $(\phi_1 u^\delta, \phi_2 u^\delta,..., \phi_m u^\delta)$. By definition of $s_m(\beta)$ (cf. (3.2)) we have :

$$\frac{s_m(\beta)}{\beta^2} \le V_{st}^2 (1+O(\beta^2 e^{-\eta\beta\delta/2})) + C (1+O(\beta^2 e^{-\eta\beta\delta})) \left(\sup_{|x_1|<a} |f'(x_1)|^2 + \frac{\lambda_m(a)}{\beta} \right)$$

$\lambda_m(a) = m^2\pi^2/4a^2$ denoting the mth eigenvalue of $-d^2/dx^2$ in $H_0^1(]-a, a[)$. We obtain easily:

$$(3.11)\quad \limsup_{\beta\to+\infty} \frac{s_m(\beta)}{\beta^2} \le V_{st}^2 + C \sup_{|x_1|<a} |f'(x_1)|^2$$

Now as $f'(0) = 0$ and f' is continuous, we know that:

$$\forall\, \varepsilon < 0, \exists\, a > 0, C \sup_{|x_1|<a} |f'(x_1)|^2 < \varepsilon$$

So by choosing a small enough, which is always possible, we see that lim sup $(s_m(\beta)/\beta^2)$ is smaller than $V_{st}^2 + \varepsilon$ for all $\varepsilon > 0$, which finally gives (3.5). To be complete we simply have to prove the relations (3.8) and (3.9). We shall content ourselves to establish the third equality of (3.8). The other ones can be obtained by analogous calculations.

From the equality:

$$\frac{du_2^\delta}{dx_2} = \frac{du_2^{st}}{dx_2}\psi_\delta + u_2^{st} \cdot \frac{d\psi_\delta}{dx_2}$$

We use (A.10) (see appendix) to deduce ($\psi'_\delta(x_2) = 1$ if $|x_2| < \delta/2$) :

(3.12)

$$\int_{-\infty}^{+\infty} \mu(x_2)\left|\frac{du_2^\delta}{dx_2}(x_2)\right|^2 dx_2 \geq \int_{-\frac{\delta}{2}}^{\frac{\delta}{2}} \mu(x_2)\left|\frac{du_2^{st}}{dx_2}(x_2)\right|^2 dx_2$$

$$\geq \left(\int_{-\infty}^{+\infty} \mu(x_2)\left|\frac{du_2^{st}}{dx_2}(x_2)\right|^2 dx_2\right)(1 - C\beta^2 e^{-\eta\beta\delta})$$

where $\eta = [1-V_{st}^2(\mu/\rho)_-]^{1/2}\,\delta$.

On the other hand we write :

$$\left|\frac{du_2^\delta}{dx_2}\right|^2 = \left|\frac{du_2^{st}}{dx_2}\right|^2 \psi_\delta^2 + 2u_2^{st}\,\psi_\delta\,\frac{du_2^\delta}{dx_2}\frac{d\psi_\delta}{dx_2} + |u_2^{st}|^2\left|\frac{d\psi_\delta}{dx_2}\right|^2$$

which gives:

(3.13) $$\int_{-\infty}^{+\infty} \mu\left|\frac{du_2^\delta}{dx_2}\right|^2 dx_2 \leq$$

$$\int_{-\infty}^{+\infty} \mu\left|\frac{du_2^{st}}{dx_2}\right|^2 dx_2 + 2\int_{|x|>\frac{\delta}{2}} \mu\,|u_2^{st}|\left|\frac{du_2^{st}}{dx_2}\right|\left|\frac{d\psi_\delta}{dx_2}\right| dx_2 + \int_{|x|>\frac{\delta}{2}} \mu\,|u_2^{st}|^2\left|\frac{d\psi_\delta}{dx_2}\right|^2 dx_2$$

The Cauchy-Schwartz inequality permits us to estimate the two last terms of the right hand side of (3.13) in terms of $C\beta\exp(-\eta\beta\delta)$. It suffices to regroup this result with (3.12) to obtain the third relation of (3.8).∎

3.2 Study of the thresholds

Our two main existence results (sections 3.1.1 and 3.1.2) lead us naturally to introduce the quantities (for $m \geq 1$)

(3.14)
$$\left|\begin{array}{l} \beta_m^0 = \sup\{\beta_m;\ \forall\,\beta \leq \beta_m,\ s_m(\beta) = \beta^2 \frac{\mu_\infty}{\rho_\infty}\} \\ \beta_m^* = \inf\{\beta_m;\ \forall\,\beta > \beta_m,\ s_m(\beta) < \beta^2 \frac{\mu_\infty}{\rho_\infty}\} \end{array}\right.$$

By definition, β^0_m is the m[th] lower threshold (or cut off wave number) and β^*_m is the m[th] upper threshold.

It is clear that the sequences (β_m) and (β^*_m) are increasing and that:

$$0 \le \beta^0_m \le \beta^*_m \le +\infty , \quad m = 1,2,.....$$

Of course one can have $\beta^0_m = +\infty$, if $\boldsymbol{A}(\beta)$ has no eigenvalue. Otherwise, results of theorems (3.5) and (3.6) express that, under correct assumptions, the numbers β^*_m are finite.

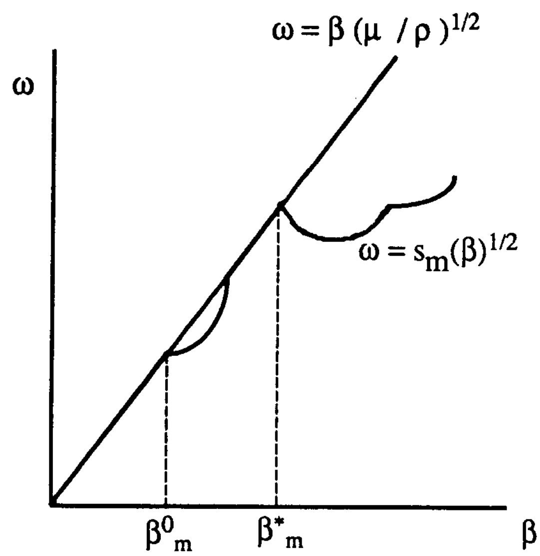
We can interpret graphically the meaning of the quantities β^0_m and β^*_m by considering, in the plane (β, ω), the curve $\omega = s_m(\beta)^{1/2}$. We know that this curve is located under the line $\omega = \beta(\mu_\infty/\rho_\infty)^{1/2}$.

Moreover:

-as long as $\beta < \beta^0_m$, the curve $\omega = s_m(\beta)^{1/2}$ coïncides with the line $\omega = \beta(\mu_\infty/\rho_\infty)^{1/2}$,

-as soon as $\beta > \beta^0_m$, the curve $\omega = s_m(\beta)^{1/2}$ lies strictly under the line $\omega = \beta(\mu_\infty/\rho_\infty)^{1/2}$ and represents the graph of the dispersion relation of the m[th] guided mode.

A priori, the numbers β^0_m and β^*_m can be finite and different, as illustrated below.



Our first result concern the asymptotic behaviour of the sequence β^0_m.

Theorem 3.8

The sequence $(\beta^0_m)_{m\ge1}$ *goes to infinity. More precisely, there exists a positive constant* C *(depending on* ρ, λ, μ*) such that:*

(3.15) $\quad \beta^0_{3m-2} \ge C\,(\lambda^N_m)^{1/2}$, m = 1, 2, 3,...,

where $(\lambda^N_m)_{m\ge1}$ *is the sequence of the eigenvalues of the operator* $-\Delta$ *in the disc of radius* R *with Neumann boundary condition.*

Proof:

As the sequence (β_m^0) is increasing it is sufficient to prove (3.15). For this, we shall evaluate the quantity $a(\beta; u, u) - \beta^2 (\mu_\infty/\rho_\infty) \|u\|^2$ and use the Max-Min characterization (3.1) of the numbers $s_m(\beta)$. With the help of (1.10) and (1.13), we can write:

$$a(\beta, u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \geq \lambda_- \int_{R^2} \left|\mathrm{div}^\beta u\right|^2 dx + \mu_- \int_{R^2} \left|\nabla u_3\right|^2 dx$$
$$+ \mu_- \int_{R^2} \left[\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 + 2 \sum_{j=1}^{2} \left| \frac{\partial u_j}{\partial x_j} - \frac{\beta}{2} u_3 \right|^2 \right] dx + p(\beta, u, u).$$

By integration by parts, we eliminate the term $\int \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} dx$ to prove that:

$$\int_{R^2} \left[\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)^2 + 2 \sum_{j=1}^{2} \left| \frac{\partial u_j}{\partial x_j} - \frac{\beta}{2} u_3 \right|^2 \right] dx = \int_{R^2} \left|\mathrm{div}^\beta u\right|^2 dx + \sum_{j=1}^{2} \int_{R^2} \left|\nabla u_j\right|^2 dx$$

and then to deduce the lower bound

$$(3.16) \quad \left| \begin{array}{l} a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \geq (\lambda_- + \mu_-) \int_{R^2} |\mathrm{div}^\beta u|^2 dx + \mu_- \int_{R^2} |\nabla u|^2 dx \\ \qquad + p(\beta; u, u). \end{array} \right.$$

We have now to bound $p(\beta; u, u)$ from below. For this, we use the inequality ($\varepsilon \geq 0$):

$$2\beta (\mu - \mu_\infty) \left| \frac{\partial}{\partial x_1} (u_1 u_3) + \frac{\partial}{\partial x_2} (u_2 u_3) \right| \leq \mu_+ \sum_{j=1}^{3} (2 \frac{\beta^2}{\varepsilon} |u_j|^2 + \varepsilon \left|\nabla u_j\right|^2)$$

Choosing ε small enough, we can find positive constants C_1 and C_2 such that:

$$(3.17) \quad a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \geq C_1 \int_{|x| \leq R} |\nabla u|^2 dx - \beta^2 C_2 \int_{|x| \leq R} |u|^2 dx$$

Let B be the open disc of center 0 and radius R and $-\Delta_N$ the selfadjoint operator associated to the Laplacian in B with Neumann boundary condition. If $(\lambda_m^N)_{m \geq 1}$ denotes the sequence of the eigenvalues of $-\Delta_N$ and (w_n^N) is a corresponding sequence of eigenfunctions we know by the Max-Min principle that:

$$\int_B \left|\nabla v\right|^2 dx \geq \lambda_m^N \int_B |v|^2 dx \ , \ \forall v \in \left\{ v \in H^1(B); \ (v, w_j^N) = 0, \ 1 \leq j \leq m \right\}$$

Now, we define 3(m-1) elements of H by:

$$v_j^{(1)}(x) = \begin{cases} (w_j^N(x), 0, 0) & \text{if } |x| \le R \\ (0, 0, 0) & \text{if } |x| > R \end{cases}$$

$$v_j^{(2)}(x) = \begin{cases} (0, w_j^N(x), 0) & \text{if } |x| \le R \\ (0, 0, 0) & \text{if } |x| > R \end{cases}$$

$$v_j^{(3)}(x) = \begin{cases} (0, 0, w_j^N(x)) & \text{if } |x| \le R \\ (0, 0, 0) & \text{if } |x| > R \end{cases}$$

Let V_{3m-3} be the subspace of $\boldsymbol{H}$ of dimension (3m-3) generated by these elements. If $u \in V_{3m-3}^{\perp}$, we note that on **B** the restrictions of u_j belong to $\boldsymbol{H}^1(\mathbf{B})$ and are orthogonal to $w_1^N, \ldots, w_{m-1}^N$. Then, we have:

$$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \ge (C_1 \lambda_m^N - C_2\beta^2) \int_B |u|^2 \, dx \quad , \forall\ u \in V_{3m-3}^{\perp}.$$

When β^2 is smaller than $(C_1/C_2)\lambda_m^N$, the right hand side of this inequality is positive which, with (3.1), proves that $s_{3m-2}(\beta) \ge \beta^2 \mu_\infty/\rho_\infty$. Thus, by theorem 3.2, we know that $s_{3m-2}(\beta) = \beta^2 \mu_\infty/\rho_\infty$ for $\beta \le C\ (\lambda_m^N)^{1/2}$, $C = (C_1/C_2)^{1/2}$, which completes the proof.■

Remark

The classical results about eigenvalues of the Laplacian operator with Neumann boundary condition [R.S.2] permits us to say that there exists a positive constant C_1 such that:

$$\beta_{3n-2}^0 \ge C_1\, n^{1/2} \qquad (n \ge 2)$$

Corollary 3.9

(i) The sequence β_m^* tends to $+\infty$.

(ii) $\beta_4^0 > 0$

(iii) For any $\beta \ge 0$, *the number of eigenvalues of* $\boldsymbol{A}(\beta)$ *in the interval* $(\frac{\mu_-}{\rho_+}\beta^2, \frac{\mu_\infty}{\rho_\infty}\beta^2)$ *is finite.*

Proof:

(i) is immediate and (ii) comes from (3.15) for m = 2 since $\lambda_2^N > 0$. Finally, if $\boldsymbol{A}(\beta)$ had an infinite number of eigenvalues, one would have $\beta_m^0 \le \beta$ for any m which contradicts the fact that β_m^0 goes to infinity.■

We can describe precisely the behaviour of the numbers β_m^0 and β_m^* when the assumptions of theorem 3.5 are made very slightly stronger.

Theorem 3.10
If there exist an open $U \subset R^2$ *and a real number* $\delta > 0$ *such that:*

(3.18) $$\frac{\mu(x)}{\rho(x)} + \delta < \frac{\mu_\infty}{\rho_\infty} \quad \text{a.e. } x \in U$$

then there exist two positive constants C_1 and C_2 *such that, for large* m :

$$C_1\, m^{1/2} \le \beta_m^0 \le \beta_m^* \le C_2\, m^{1/2}$$

Proof :
We have only to prove that $\beta_m^* \le C_2\, m^{1/2}$. We keep the principle of the proof of theorem 3.5. For test functions u in the form $u = (u_1, 0, 0)$, we have

(3.19) $$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \le (\lambda_+ + 2\mu_+) \int_{R^2} |\nabla u_1|^2 dx \ - \beta^2 \int_{R^2} \left(\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho}\right) \rho\, |u_1|^2\, dx$$

Let us consider the Laplacian operator with Dirichlet boundary condition in the open set U. Let $(\lambda_m^D)_{m\ge1}$ be the increasing sequence of corresponding eigenvalues and $(v_m)_{m\ge1}$ a sequence of associated eigenfunctions. One has

$$\int_U |\nabla v|^2\, dx \le \lambda_m^D \int_U |v|^2\, dx\,, \quad \forall\, v \in [v_1, \ldots, v_m]$$

Let us set

$$\begin{cases} u^k(x) = (\, v_k(x), 0, 0) & \text{if } x \in U \\ u^k(x) = (\, 0, 0, 0) & \text{if } x \notin U \end{cases}$$

Clearly the sequence (u^k) belongs to V. Now, if u is a linear combination of $u^1, u^2, \ldots, u^m$, it is clear that the right hand side member (3.19) is strictly negative as soon as $\beta > C(\lambda_m^D/\delta)^{1/2}$, where the constant C only depends on Lamé's coefficients. Then using the Max-Min principle, we deduce the bound:

$$\beta_m^* \le C \left(\frac{\lambda_m^D}{\delta}\right)^{1/2}$$

which taking into account the classical properties of the sequence (λ_m^D) gives the result.■
From corollary 3.9, we already know that the fourth threshold β_4^0 is strictly positive. We are now going to prove that it is also thrue for the third threshold β_3^0.

Theorem 3.11
We have $\beta_3^0 > 0$ *which means that, for* β *small enough,* $\boldsymbol{A}(\beta)$ *has at most two guided modes.*

Proof :

Let ε and η be real numbers with $0 \le \varepsilon \le 1$ and $0 < \eta$, and B' be the disc of center 0 and radius R' > R (we shall fix R' later in the proof). We have:

$$\lambda_- \int_{R^2} |\mathrm{div}^\beta u|^2\, dx \ge \varepsilon\, \lambda_- \int_{R^2} |\mathrm{div}^\beta u|^2\, dx$$
$$\ge \varepsilon\, \lambda_- \left[\beta^2 (1-2\eta) \int_{B'} |u_3|^2 dx - \frac{1}{\eta} \int_{B'} \left(\sum_{j=1}^{2} |\nabla u_j|^2\right) dx \right]$$

We plug this result into (3.16) to obtain:

$$(3.20)\quad \left| \begin{aligned} a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \ge \sum_{j=1}^{2} \Big[(\mu_- - \varepsilon \frac{\lambda_-}{\eta}) \int_{B'} |\nabla u_j|^2 dx + \beta^2 \int_{B'} (\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty})\, \rho\, u_j^2\, dx \Big] \\ + \mu_- \int_{B'} |\nabla u_3|^2 dx + \beta^2 \int_{B'} \left[(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty})\, \rho + \varepsilon \lambda_- (1-2\eta) \right] u_3^2\, dx \\ + 2\beta \int_{B'} (\mu - \mu_\infty) \left[\frac{\partial}{\partial x_1} (u_1 u_3) + \frac{\partial}{\partial x_2} (u_2 u_3) \right] dx\, . \end{aligned} \right.$$

Let us set $P(\Omega) = \{ v \in H^1(\Omega)\, ;\ \int_\Omega v\, dx = 0 \}$. If Ω is bounded, there exists a constant $C(\Omega) > 0$ such that:

$$\int_\Omega |v|^2 dx \le C(\Omega) \int_\Omega |\nabla v|^2 dx\, , \quad \forall v \in P(\Omega)$$

Let us denote by χ the characteristic function of B' and let us consider the two elements of H:

$$\begin{cases} v^1 = (\chi,\ 0,\ 0) \\ v^2 = (0,\ \chi,\ 0) \end{cases}$$

If an element u of V is orthogonal to v^1 and v^2, the two first components u_1 and u_2 belong to P(B'). Thus:

$$(3.21)\quad \left| \int_{B'} (\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty})\, \rho\, |u_j|^2\, dx \right| \le C(B') \left\| (\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty})\, \rho \right\|_\infty \int_{B'} |\nabla u_j|^2 dx$$

Let us decompose $H^1(B')$ as $H^1(B') = P(B') + P(B')^\perp$, with:
$P(B')^\perp = \{ v \in H^1(B');\ v \text{ is constant in } B' \}$.
If we set:
$u_3 = u_{3P} + u_{3P}^\perp$, $\qquad u_{3P} \in P(B'),\ u_{3P}^\perp \in P(B')^\perp$
we can write (we identify $u_{3P}^\perp$ and its constant value in the ball B')

$$(3.22)\quad \int_{B'}\left[\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right]u_3^2\,dx=\left(u_{3_P}^1\right)^2\left(\int_{B'}\left[\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right]dx\right)+D$$

where D can be estimated as follows (γ is an arbitrary positive number)

$$(3.23)\quad |D|\le\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right\|_\infty\left[\gamma\pi R'^2\left(u_{3_P}^1\right)^2+C(B')\left(1+\frac{1}{\gamma}\right)\int_{B'}|\nabla u_3|^2\,dx\right]$$

We regroup (3.21), (3.22) and (3.23) in (3.20) to obtain for any u in $\boldsymbol{V}$, orthogonal to $[v^1, v^2]$:

$$(3.24)\quad \left|\begin{aligned}
a(\beta;u,u)-\beta^2\frac{\mu_\infty}{\rho_\infty}\|u\|^2\ge\left(\mu_--\varepsilon\frac{\lambda_-}{\eta}-\beta^2C(B')\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho\right\|_\infty\right)\sum_{j=1}^2\int_{B'}|\nabla u_j|^2dx\\
+\left[\mu_--\beta^2C(B')\left(1+\frac{1}{\gamma}\right)\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right\|_\infty\right]\int_{B'}|\nabla u_3|^2dx\\
+\beta^2\left|u_{3_P}^1\right|^2\left\{\int_{B'}\left[\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right]dx-\gamma\pi(R')^2\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right\|_\infty\right\}\\
+2\beta\int_{B'}(\mu-\mu_\infty)\left[\frac{\partial}{\partial x_1}(u_1u_3)+\frac{\partial}{\partial x_2}(u_2u_3)\right]dx
\end{aligned}\right.$$

Suppose for the moment that $\mu=\mu_\infty$. If we fix $\eta=1/4$ and $\varepsilon<\eta\mu_-/\lambda_-$, both quantities $\mu_- - \varepsilon\lambda_-/\eta$ and $\varepsilon\lambda_-(1-2\eta)$ are strictly positive. In particular, as $(\mu/\rho)-(\mu_\infty/\rho_\infty)=0$ when $|x|\ge R$, we can choose R' large enough that

$$\int_{B'}\left[\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right]dx>0$$

Now, we can choose γ, small enough, that

$$(3.25)\quad \gamma\pi R'^2\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right\|_\infty<\int_{B'}\left[\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right]dx$$

Finally, if β is small enough, we shall have:

$$(3.26)\quad \left\{\begin{aligned}
&\mu_--\varepsilon\frac{\lambda_-}{\eta}-\beta^2C(B')\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho\right\|_\infty>0\\
&\mu_--\beta^2C(B')\left(1+\frac{1}{\gamma}\right)\left\|\left(\frac{\mu}{\rho}-\frac{\mu_\infty}{\rho_\infty}\right)\rho+\varepsilon\lambda_-(1-2\eta)\right\|_\infty>0
\end{aligned}\right.$$

and consequently

$$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} ||u||^2 \geq 0, \forall u \in \{u \in V ; (u, v^j) = 0, j = 1, 2\}$$

By characterization (3.1), we deduce that $s_3(\beta) = \beta^2 \mu_\infty / \rho_\infty$ which means that $\beta_3^0 \geq \beta > 0$.
In the general case $\mu \neq \mu_\infty$ we have to estimate the last term of (3.24). For this, we write ($\alpha > 0$):

$$2\beta |\mu - \mu_\infty| \left|\frac{\partial u_j}{\partial x_j} u_3\right| \leq \alpha \left|\frac{\partial u_j}{\partial x_j}\right|^2 + \frac{\beta^2}{\alpha} |\mu - \mu_\infty|^2 |u_3|^2$$

$$2\beta |\mu - \mu_\infty| \left|\frac{\partial u_3}{\partial x_j} u_j\right| \leq \alpha |\mu - \mu_\infty| \left|\frac{\partial u_3}{\partial x_j}\right|^2 + \frac{\beta^2}{\alpha} |\mu - \mu_\infty| |u_j|^2$$

The terms obtained with $\alpha\, |\partial u_j / \partial x_j|^2$ and $\alpha|\mu - \mu_\infty|\, |\partial u_3 / \partial x_j|^2$ can be estimated with the two first terms of the right hand side of (3.24) by choosing α small enough. The terms obtained with $(\beta^2/\alpha)\, |u_j|^2$ do not pose any problem because of (3.21).
It remains to treat the term in $|u_3|^2$. For this we use again the decomposition $u_3 = u_{3P} + u_{3P}^{\perp}$ and we have then to choose R' large enough in order that

$$\int_{B'} \left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho - \frac{1}{\alpha} |\mu - \mu_\infty|^2 + \varepsilon\lambda_- (1 - 2\eta) \right] dx > 0$$

Then the end of the proof is exactly the same as in the case $\mu = \mu_\infty$.■

We now examine the case of the two first thresholds.

Theorem 3.12

As soon as one of the two following conditions is satisfied:

(*i*) $\displaystyle\int_{R^2} \left(\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho}\right) \rho \; dx > 0$

(*ii*) $\displaystyle\int_{R^2} \left(\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho}\right) \rho \; dx = 0$ *and* $\displaystyle\left(\frac{\mu}{\rho}\right)_- < \frac{\mu_\infty}{\rho_\infty}$

one has:

$$\beta_1^* = \beta_2^* = 0$$

which means that the operator $\boldsymbol{A}(\beta)$ *has at least two eigenvalues for any positive value of* β.

Proof:
Let us consider an integer $n \geq R$. We can define the function v^n in $H^1(R^2)$ by

$$\begin{cases} v^n(x) = 1 & \text{if } |x| \leq R \\ v^n(x) = \mathrm{Log}(|x|/R)/\mathrm{Log}(n/R) & \text{if } R \leq |x| \leq n \\ v^n(x) = 0 & \text{if } |x| \geq n \end{cases}$$

It is easy to check that $||\nabla v^n||$ tends to 0 when n goes to infinity.

Assume that (i) holds. We consider test functions in the form $u = \alpha_1 u^{1,n} + \alpha_2 u^{2,n}$, $(\alpha_1, \alpha_2) \in R^2$

where: $\begin{cases} u^{1,n} = (v^n, 0, 0) \\ u^{2,n} = (0, v^n, 0) \end{cases}$

For such functions we have:

$$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty}||u||^2 = b_0(u, u) - (\alpha_1^2 + \alpha_2^2)\,\beta^2 \int_{R^2} (\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho})\,\rho\ dx \tag{3.27}$$

where $b_0(u, u)$ is defined by (1.11) and satisfies:

$$b_0(u, u) \le C(\alpha_1^2 + \alpha_2^2) \int_{R^2} |\nabla v^n|^2\ dx$$

Thus, it suffices to choose n large enough, namely such that

$$\int_{R^2} |\nabla v^n|^2\ dx \ \le \frac{\beta^2}{C} \int_{R^2} (\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho})\,\rho\ dx$$

in order that the right hand side of (3.27) be strictly negative. The max-min principle proves then that $\beta_2^* = 0$.

When (ii) holds we use the idea of [Bo.]. We now set:

$$\begin{cases} u^{1,n} = (v^n + \gamma w, 0, 0) \\ u^{2,n} = (0, v^n + \gamma w, 0) \end{cases}$$

where w is a function of $H^1(R^2)$ and γ a real number.

One easily checks that $u = \alpha_1 u^{1,n} + \alpha_2 u^{2,n}$ implies one has:

$$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty}||u||^2 = b_0(u, u) - (\alpha_1^2 + \alpha_2^2)\,\beta^2 \int_{R^2} (\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho})\,\rho\,(2\gamma w + \gamma^2 w^2)\ dx$$

We choose now w such that

$$\int_{R^2} (\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho})\,\rho\, w\ dx \ > 0$$

which is possible because of the inequality $(\mu/\rho)_- < \mu_\infty/\rho_\infty$. Let us remark that

$$b_0(u, u) \le C(\alpha_1^2 + \alpha_2^2) \int_{R^2} \left(|\nabla v^n|^2 + \gamma^2 |\nabla w|^2\right) dx$$

Then it suffices to choose γ small enough, that the terms in γ^2 be negligible compared to the term in γ, and then to take n large enough.■

Now we obtain a necessary condition in order that the first threshold be strictly positive.

Theorem 3.13

Under the assumption

$$\int_{R^2} \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho \, dx > \frac{1}{\mu_-} \int_{R^2} (\mu - \mu_\infty)^2 \, dx$$

the first threshold β_1^0 *is strictly positive.*

Proof:

(i) For simplicity, we first give the proof when $\mu = \mu_\infty$. From (3.16) we deduce the inequality:

$$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \geq \mu_- \int_B |\nabla u|^2 dx + \beta^2 \int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho |u|^2 \, dx$$

Using the decomposition $H^1(B) = P(B) + P(B)^\perp$ introduced in the proof of theorem 3.11 (B is the open ball of radius R), we can write:

$$u = u_P + u_P^\perp, \quad u_P \in P(B), u_P^\perp \in P(B)^\perp$$

Then, we have, as $u_P^\perp$ is constant in B,

$$\int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho |u|^2 \, dx = \left(\int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho \, dx\right) |u_P^\perp|^2 + 2 \int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho \, (u_P \cdot u_P^\perp) \, dx$$
$$+ \int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho |u_P|^2 \, dx$$

Using the inequality

$$2 \left| u_P \cdot u_P^\perp \right| \leq \varepsilon \left| u_P^\perp \right|^2 + \frac{1}{\varepsilon} |u_P|^2$$

and choosing ε small enough, $\varepsilon > 0$, we see that there exists a positive constant C (depending only on μ, ρ and ε) such that:

$$\int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho |u|^2 \, dx \geq \frac{1}{2} \left(\int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho \, dx\right) |u_P^\perp|^2 - C \int_B |u_P|^2 \, dx$$

Thus, we obtain

$$a(\beta; u, u) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2 \geq \frac{\beta^2}{2} \left(\int_B \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho \, dx\right) |u_P^\perp|^2 + \mu_- \int_B |\nabla u_P|^2 dx - C\beta^2 \int_B |u_P|^2 \, dx$$

But there exists C(B) > 0 such that

$$C(B) \int_B |\nabla u_P|^2 dx \geq \int_B |u_P|^2\, dx$$

Therefore as soon as $\beta^2\, C/C(B) < \mu_-$, the quantity $a(\beta;\, u,\, u) - \beta^2\, \mu_\infty/\rho_\infty \|u\|^2$ is positive for any u in V, which proves that $s_1(\beta) = \beta^2\, \mu_\infty/\rho_\infty$ and consequently that $\beta_1^0 > 0$.

(ii) In the general case the proof is technically more complicated. As in the proof of theorem 3.11, we introduce a ball B' of radius R' ≥ R and use the inequality (3.20), with ε, η positive constants. ε, η and R' will be determined later in the proof. For any $\alpha > 0$, we can write that, for any x in R^2:

$$\left| 2\beta\ (\mu-\mu_\infty)\left[\frac{\partial}{\partial x_1}(u_1 u_3) + \frac{\partial}{\partial x_2}(u_2 u_3)\right]\right| \leq \alpha\left(\left|\frac{\partial u_1}{\partial x_1}\right|^2 + \left|\frac{\partial u_2}{\partial x_2}\right|^2 + |\nabla u_3|^2\right) + \frac{\beta^2}{\alpha}\,|\mu - \mu_\infty|^2\,(|u_1|^2 + |u_2|^2 + 2|u_3|^2)$$

Plugging this inequality into (3.20) leads to the following inequality:

$$a(\beta;\, u,\, u) - \beta^2\frac{\mu_\infty}{\rho_\infty}\|u\|^2 \geq \sum_{j=1}^{2}\left[\left(\mu_- - \alpha - \varepsilon\frac{\lambda_-}{\eta}\right)\int_{B'} |\nabla u_j|^2 dx + \beta^2\int_{B'}\left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right)\rho - \frac{(\mu-\mu_\infty)^2}{\alpha}\right]|u_j|^2\, dx\right]$$
$$+\,\beta^2\int_{B'}\left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right)\rho - \frac{2}{\alpha}(\mu-\mu_\infty)^2 + \varepsilon\lambda_-(1-2\eta)\right]|u_3|^2\, dx + (\mu_- - \alpha)\int_{B'} |\nabla u_3|^2\, dx$$

Because of the assumption of the theorem it is possible to find $\alpha > 0$ such that

$$\alpha < \mu_- \text{ and } \int_B \left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right)\rho - \frac{1}{\alpha}(\mu-\mu_\infty)^2\right] dx > 0$$

We now fix $\eta < 1/2$ and choose ε small enough in order that $\mu_- - \alpha > \varepsilon\,\lambda_-/\eta$. Finally we choose R' large enough that

$$\int_{B'}\left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right)\rho - \frac{2}{\alpha}(\mu-\mu_\infty)^2 + \varepsilon\lambda_-(1-2\eta)\right] dx > 0$$

To prove that $a(\beta;\, u,\, u) - \beta^2\mu_\infty/\rho_\infty \|u\|^2$ is positive for β small enough it suffices then to prove that

$$\begin{cases} \left(\mu_- - \alpha - \varepsilon\frac{\lambda_-}{\eta}\right)\int_{B'} |\nabla u_j|^2 dx + \beta^2\int_{B'}\left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right)\rho - \frac{(\mu-\mu_\infty)^2}{\alpha}\right]|u_j|^2\, dx \geq 0 \\ (\mu_- - \alpha)\int_{B'} |\nabla u_3|^2\, dx + \beta^2\int_{B'}\left[\left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right)\rho - \frac{2}{\alpha}(\mu-\mu_\infty)^2 + \varepsilon\lambda_-(1-2\eta)\right]|u_3|^2\, dx \geq 0 \end{cases}$$

for β small enough. This is easy using the arguments of part (i) of the proof.■

Remark

We don't know what can be said about the two first thresholds in the case where

$$0 < \int_{R^2} \left(\frac{\mu}{\rho} - \frac{\mu_\infty}{\rho_\infty}\right) \rho \, dx \leq \frac{1}{\mu_-} \int_{R^2} (\mu - \mu_\infty)^2 \, dx$$

Finally we give an example for which one can prove that the upper and lower thresholds coincide.

Theorem 3.14

If the function $\mu(x)/\rho(x)$ is, almost everywhere, smaller or equal to μ_∞ / ρ_∞, then

(3.28) $\quad \beta_m^0 = \beta_m^* , \qquad \forall\, m \geq 1.$

Moreover the function $\beta \to \beta^2 \mu_\infty/\rho_\infty - s_m(\beta)$ is increasing.

Proof:

If β and β' are two real positive numbers, we shall denote by $J_{\beta,\beta'}$ the isomorphism in V defined by

$$J_{\beta,\beta'}(u) = (u_1, u_2, \frac{\beta'}{\beta} u_3) \quad , \qquad \forall\, u = (u_1, u_2, u_3) \in V.$$

Using (1.10) one easily checks that:

$$a(\beta; J_{\beta,\beta'}(u), J_{\beta,\beta'}(u)) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|J_{\beta,\beta'}(u)\|^2 = a(\beta'; u, u) - \beta'^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2$$
$$- (\beta^2 - \beta'^2) \left[\int_{R^2} \frac{\mu}{\beta^2} |\nabla u_3|^2 dx + \int_{R^2} \left(\frac{\mu_\infty}{\rho_\infty} - \frac{\mu}{\rho}\right) \rho \, (u_1^2 + u_2^2) \, dx \right]$$

As $\mu/\rho \leq \mu_\infty/\rho_\infty$ a.e., we deduce that:

(3.29) $\quad a(\beta; J_{\beta,\beta'}(u), J_{\beta,\beta'}(u)) - \beta^2 \frac{\mu_\infty}{\rho_\infty} \|J_{\beta,\beta'}(u)\|^2 \leq a(\beta'; u, u) - \beta'^2 \frac{\mu_\infty}{\rho_\infty} \|u\|^2$, if $\beta \geq \beta'$.

Let $m \in N^*$ and $\beta' > \beta_m^0$. There exists V_m, an m-dimensional subspace of V, such that the right hand side of (3.29) is strictly negative when u belongs to V_m. This implies that the left hand side is negative for any $\beta \geq \beta'$ and any u in V_m. As $J_{\beta,\beta'}(V_m)$ is a subspace of V of dimension m, one gets (3.28). Now let β and β' be such that $0 < \beta' < \beta$. If $\beta' \leq \beta_m^0$, $\beta'^2 \mu_\infty/\rho_\infty$ is equal to $s_m(\beta')$ so that, necessarily

(3.30) $\quad \beta'^2 \frac{\mu_\infty}{\rho_\infty} - s_m(\beta') \leq \beta^2 \frac{\mu_\infty}{\rho_\infty} - s_m(\beta).$

If $\beta_m^0 < \beta'$, we rewrite (3.29) in the form :

$$\beta'^2 \frac{\mu_\infty}{\rho_\infty} ||u||^2 - a(\beta'; u, u) \leq \beta^2 \frac{\mu_\infty}{\rho_\infty} ||J_{\beta,\beta'}(u)||^2 - a(\beta; J_{\beta,\beta'}(u), J_{\beta,\beta'}(u))$$

Dividing this inequality by $||u||^2$ and noting that $||J_{\beta,\beta'}(u)|| \leq ||u||$, we get (3.30) again which completes the proof of the theorem.∎

Remarks

•We don't know whether there exists $(\rho(x), \lambda(x), \mu(x))$ for which $\beta_m^0 < \beta_m^*$ for some m.

•If the assumption $\mu(x) / \rho(x) \leq \mu_\infty / \rho_\infty$ a.e. $x \in R^2$ holds, assumptions of theorems 3.5 and 3.12 also hold so that for such a medium we know that:

(i) Guided wave exists

(ii) $\beta_m^0 = \beta_m^* \; (= \beta_m), \quad \forall \; m \in N$

(iii) $\beta_1 = \beta_2 = 0 < \beta_3 < \ldots < \beta_m < +\infty$

(iv) $\beta_m \to +\infty$ when $m \to +\infty$

Conclusion

We have given in this paper a large variety of theorical results concerning guided waves in heterogeneous elastic media. These results illustrate both the richness of the equations of elasticity, in comparaison with the acoustic wave equation or even the Maxwell equations, with respect to this particular phenomena, and the power of the mathematical tools borrowed from the spectral theory of selfadjoint operators.

Nevertheless, some interesting open questions remain to be solved from a purely theorical point of view. Let us quote, without being exhaustive, the questions concerning the regularity and the monotonicity of the dispersion curves, the existence of eigenvalues embedded in the continuous spectrum, the behaviour of corresponding eigenfunctions, continuous dependence of guided waves with respect to the coefficients of the medium, high and low frequencies, comparaison results between two media, and so on...

Moreover, though our results are interesting and rather fine from a qualitative point of view, the quantitative information contained in these results is not yet sufficient. Numerical methods should be a very useful complement to the present work and we intend to develop a strategy for the numerical approximation of the waves we pointed out in this paper.

APPENDIX

Construction of u^{st}

Contrary to what we have done in the preceeding sections we shall consider here the plane (x_2, x_3) instead of the plane (x_1, x_2) and we shall be interested by the 2D linear elastodynamic equations in a two layered medium, defined by:

$$(A.1) \qquad (\rho, \lambda, \mu)(x_2, x_3) = (\rho, \lambda, \mu)(x_2) = \begin{cases} \rho_0, \lambda_0, \mu_0 & \text{if } x_2 < 0 \\ \rho_\infty, \lambda_\infty, \mu_\infty & \text{if } x_2 > 0 \end{cases}$$

The solutions we are looking for are 2D displacement fields with coordinates (u_2, u_3). More precisely, we are interested by the guided modes, that is to say by solutions of the 2D linear elastodynamic equations in the form:

$$(u_2(x_2, x_3, t), u_3(x_2, x_3, t)) = (\tilde{u}_2(x_2), \tilde{u}_3(x_2)) \, exp \; i(\omega t - \beta x_3)$$

where:

$$\int_{-\infty}^{+\infty} (|\tilde{u}_2(x_2)|^2 + |\tilde{u}_3(x_2)|^2) \; \mathrm{d}x_2 < +\infty$$

In practice we shall use the same change of unknown functions as in sections 1.1 and 1.3 by setting:

$$u_2(x_2) = \tilde{u}_2(x_2), \qquad u_3(x_2) = i \, \tilde{u}_3(x_2), \qquad u = (u_2, u_3).$$

Then we consider the positive selfadjoint operator acting in the space $L^2(\mathsf{R})$ (i.e. working with functions of the only variable x_2) defined with the quadratic form:

$$(A.2) \qquad a'(\beta; u; u) = \int_{\mathsf{R}} \{\lambda \, |\frac{\mathrm{d}u_2}{\mathrm{d}x_2} - \beta u_3 \, |^2 + \mu [\, |\frac{\mathrm{d}u_3}{\mathrm{d}x_2} + \beta u_2 \, |^2 + |\frac{\mathrm{d}u_2}{\mathrm{d}x_2}|^2 + \beta^2 \, |u_3|^2] \; \} \; \mathrm{d}x_2$$

To express that $u = (u_2, u_3)$ is a guided mode associated with the eigenvalue ω^2 is equivalent to writing that:

$$(A.3) \qquad a'(\beta; u; v) = \omega^2 \int_{\mathsf{R}} u.v \; \rho(x_2) \; \mathrm{d}x_2 \quad , \qquad \forall \; v \in (H^1(\mathsf{R}))^2.$$

With the aid of assumption (A.1), problem (A.3) can be solved explicitely. In each half space $\{x_2 < 0\}$ and $\{x_2 > 0\}$, one has a second order linear differential system with constant coefficients whose solutions are linear combinations of the two following functions:

$$\begin{pmatrix} 1 \\ \gamma_S \end{pmatrix} e^{\gamma_S \beta x_2} \quad , \quad \begin{pmatrix} \gamma_P \\ 1 \end{pmatrix} e^{\gamma_P \beta x_2}$$

where γ_P and γ_S, and their real parts, satisfy in each half space:

$$\omega^2 = \beta^2 (1-\gamma_S^2) \frac{\mu}{\rho} \quad , \quad \omega^2 = \beta^2 (1-\gamma_P^2) \left(\frac{\lambda+2\mu}{\rho}\right)$$

$$x_2 \operatorname{Re}(\gamma_P) < 0 \ , \qquad x_2 \operatorname{Re}(\gamma_S) < 0.$$

With these conditions, a normalized eigenfunction necessarily will have the following form, if we assume that $\omega^2 = \beta^2 c^2$ and $c^2 < (\mu/\rho)_- = \operatorname{Min}(\mu_0/\rho_0, \mu_\infty/\rho_\infty)$:

$$\text{(A.4)} \quad \begin{pmatrix} u_2(x_2) \\ u_3(x_2) \end{pmatrix} = \sqrt{\beta} \begin{cases} A_P^0 \begin{pmatrix} \alpha_P^0 \\ 1 \end{pmatrix} e^{+\beta\alpha_P^0 x_2} - A_S^0 \begin{pmatrix} 1 \\ \alpha_S^0 \end{pmatrix} e^{+\beta\alpha_S^0 x_2} & \text{if } x_2 < 0 \\ A_P^\infty \begin{pmatrix} -\alpha_P^\infty \\ 1 \end{pmatrix} e^{-\beta\alpha_P^\infty x_2} - A_S^\infty \begin{pmatrix} 1 \\ -\alpha_S^\infty \end{pmatrix} e^{-\beta\alpha_S^\infty x_2} & \text{if } x_2 > 0 \end{cases}$$

where we have set:

$$\text{(A.5)} \quad \alpha_P^i = \left(1 - \frac{\rho_i}{\lambda_i+2\mu_i} c^2\right)^{1/2}, \qquad \alpha_S^i = \left(1 - \frac{\rho_i}{\mu_i} c^2\right)^{1/2}, \qquad i = 0, \infty.$$

In order to ensure that (A.4) defines an eigenfunction, it is then necessary and sufficient to impose the continuity of the displacement u and of the normal stress (i.e. of the functions $\sigma_{22}^\beta(u)$ and $\sigma_{23}^\beta(u)$) at the interface $\{x_2 = 0\}$. This leads to a 4x4 linear system in (A_P^0, A_S^0, A_P^∞, A_S^∞), that we can write:

$$M_S(c)\begin{pmatrix} A_P^0 \\ A_S^0 \\ A_P^\infty \\ A_S^\infty \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

whose matrix $M_S(c)$ is defined by:

$$(A.6) \quad M_S(c) = \begin{pmatrix} 1 & -\alpha_S^0 & -1 & -\alpha_S^\infty \\ \alpha_P^0 & -1 & \alpha_P^\infty & 1 \\ \mu_0\left(2-c^2\frac{\rho_0}{\mu_0}\right) & -2\mu_0\alpha_S^0 & -\mu_\infty\left(2-c^2\frac{\rho_\infty}{\mu_\infty}\right) & -2\mu_\infty\alpha_S^\infty \\ 2\mu_0\alpha_P^0 & -\mu_0\left(2-c^2\frac{\rho_0}{\mu_0}\right) & 2\mu_\infty\alpha_P^\infty & \mu_\infty\left(2-c^2\frac{\rho_\infty}{\mu_\infty}\right) \end{pmatrix}$$

Thus, u will be a guided mode associated with the eigenvalue $\omega^2 = \beta^2 c^2$ if and only if:

$$(A.7) \qquad \det M_S(c) = 0 \qquad 0 < c < \left(\frac{\mu}{\rho}\right)^{1/2}.$$

This equation is called Stoneley's equation (see [Mi.], [E.S.]). We can now define the domain of existence of Stoneley waves, $\mathbf{E}_S$, by:

$$\mathbf{E}_S = \{ ((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty)) \in (\mathsf{R}^+)^3 x (\mathsf{R}^+)^3 \,/\, \text{(A.7) has at least one solution } \}.$$

It is well known that $\mathbf{E}_S$ is not empty and clearly has the symmetry property:

$$((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty)) \in \mathbf{E}_S \Rightarrow ((\rho_\infty, \lambda_\infty, \mu_\infty), (\rho_0, \lambda_0, \mu_0)) \in \mathbf{E}_S.$$

By convention we shall denote by $c = V_{St}$ the smallest solution of (A.7) when the coefficients $((\rho_0, \lambda_0, \mu_0), (\rho_\infty, \lambda_\infty, \mu_\infty))$ belong to the domain $\mathbf{E}_S$. We shall denote by $u = (u_2^{st}, u_3^{st})$ the function defined by (A.4) when the coefficients α_p^0, α_S^0, α_p^∞ and α_S^∞ are evaluated with $c = V_{St}$. In this case the vector (A_P^0, A_S^0, A_P^∞, A_S^∞) belongs to the kernel of the matrix $M_S(c)$ and is entirely determined, as Ker $M_S(c)$ has dimension 1, by the normalization condition:

$$\text{(A.8)} \qquad \int_{-\infty}^{+\infty} |u^{st}(x_2)|^2 \, \rho(x_2) \, dx_2 = 1.$$

Then, equality (A.3) implies that:

$$\text{(A.9)} \qquad a'(\beta; u^{st}, u^{st}) = \beta^2 \, V^2_{st}.$$

By simple calculations it is possible to check that there exists a constant C, depending only on (ρ_0, λ_0, μ_0) and (ρ_∞, λ_∞, μ_∞), such that

$$\text{(A.10)} \qquad \left| \begin{array}{l} \sup \left(\displaystyle\int_{|x_2| > \frac{\delta}{2}} |u^{st}(x_2)|^2 \, dx_2, \frac{1}{\beta^2} \int_{|x_2| > \frac{\delta}{2}} \left| \frac{du^{st}}{dx_2}(x_2) \right|^2 dx_2 \right) \\ \qquad\qquad\qquad \leq C \exp\left(- \delta \left(1 - \dfrac{V^2_{st}}{\left(\frac{\mu}{\rho}\right)_-} \right)^{1/2} \beta \right) \end{array} \right.$$

This inequality expresses the exponential decay of the Stoneley wave $u^{st}(x_2)$ with the distance to the interface $\{x_2 = 0\}$.

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