On a class of stochastic evolution equations

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Sur une classe d'équations d'évolution stochastiques

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Résumé

Cet article étudie une classe d'équations d'évolution stochastiques que l'on rencontre dans plusieurs branches de la théorie des réseaux de files d'attente. On démontre que, quelque soit la condition initiale, la solution de cette équation converge faiblement vers un processus stochastique stationnaire défini de manière unique.

Mots clés: Équations d'évolution, processus stochastiques, réseaux de files d'attente, théorie ergodique, processus stationnaires.

Abstract

This paper is concerned with a stochastic evolution equation that arises in various branches of queueing theory. We show that regardless of the initial condition, the solution of this equation converges weakly to a uniquely defined stationary stochastic process.

Keywords: Evolution Equation, Stochastic Processes, Queueing Theory, Ergodic Theory, Stationary Processes.

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Abstract

This paper is concerned with a stochastic evolution equation that arises in various branches of queueing theory. We show that regardless of the initial condition, the solution of this equation converges weakly to a uniquely defined stationary stochastic process.

Keywords: Evolution Equation, Stochastic Processes, Queueing Theory, Ergodic Theory, Stationary Processes.

1 Introduction

All the random variables that will be considered will be defined on a common probability space $(\Omega, I\!F, P, \theta)$, where θ is an ergodic shift that leaves P invariant. Let $J \in I\!N$ be a positive integer, and $\vec{Y} \in I\!R^{+J}$ be an arbitrary non-negative real random vector. J and $\vec{Y}$ will be respectively referred to as the *size* and *initial condition* of the equation. The basic random data of the equation are $l^{j,k} \in I\!R^+$, $1 \leq j,k \leq J$, $\tau \in I\!R^+$ and $\{\pi(k)\} \in 2^{\{1,\cdots,J\}}$, $1 \leq k \leq J$, where 2^S denotes the set of all subsets of set S. For all $1 \leq k \leq J$, it is assumed that $k \in \pi(k)$ a.s.. It will also be assumed that $l^{j,k}$, $1 \leq j,k \leq J$, and τ are integrable. The state variables $W_n^k(\vec{Y})$, $n = 0, 1\cdots$, $1 \leq k \leq J$ are defined by the recursion

$$\begin{aligned} \vec{W}_0(\vec{Y}) &= \vec{Y} \\ W_{n+1}^k(\vec{Y}) &= \max_{\{j \in \pi(k) \circ \theta^n\}} (W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n - \tau \circ \theta^n)^+ \end{aligned} \tag{1.1}$$

The aim of this paper is to analyze both the transient and the stationary solutions of (1.1). After giving a couple of motivation examples, we introduce a pathwise increasing schema that is equivalent in law to $\vec{W}_n(\vec{0})$ which generalizes the schema initially proposed by Loynes for $G/G/1$ queues [5]. This schema is then used to investigate the weak convergence properties of $\vec{W}_n(\vec{Y})$ when n goes to ∞ and to prove the existence of a unique stationary solution to (1.1).

2 Examples

2.1 $G/G/1$ queues

For $J = 1$, we obtain the classical Lindley equations for $G/G/1$ queueing systems, with $l^{1,1}$ representing the service time variable and τ the interarrival time variable.

2.2 First Come First Serve Queueing Networks

Consider a network of J single-server queues. Several variables are defined on the probability space $(\Omega, I\!F, P, \theta)$: $\tau \in I\!R^+$ is the interarrival variable, $m \in I\!N$ represents the number of queues visited by customer 0, $\{r_1, \cdots, r_m\}$, where $r_i \in \{1, \cdots, J\}$, $1 \leq i \leq m$, is the route followed by customer 0 and σ^i is the service time of customer 0 in the i-th queue of its route. The queueing discipline is FCFS in the sense that in each queue, the service requirements brought by customer n must all be completed before any attention is given to those brought by customer $n+1$.

Tandem queueing networks are particular cases of such FCFS networks, where the route is the sequence $(1, 2, \cdots, J)$ for all customers. However, more complex systems can be contemplated where the length and the structure of the route may be random, with possible loops.

For $1 \leq k \leq J$, let $b(k)$ (resp. $e(k)$) be the the first visit (resp. last visit) to queue k by customer 0:

$$b(k) = \min_{1 \leq i \leq m, r_i = k} i$$

$$e(k) = \max_{1 \leq i \leq m, r_i = k} i$$

with, by convention, $b(k) = 0$ and $e(k) = 0$ if $k \notin \{r_1, \cdots, r_m\}$. For k such that $e(k) \neq 0$, let

$$\pi(k) = \{r_i, 1 \leq i \leq e(k)\}$$

and for $j \in \pi(k)$, let

$$l^{j,k} = \sum_{i=b(j)}^{e(k)} \sigma^i$$

If $e(k) = 0$, take $\pi(k) = \{k\}$ and $l^{k,k} = 0$, by convention.

Let $W_n^j(\vec{Y})$ be the residual workload in queue k as seen by customer n at its arrival in the network, where $\vec{Y}$ denotes the initial workload in the queues.

The workload vectors $\vec{W}_n(\vec{Y})$, $n \geq 0$, satisfy the equations

$$\begin{aligned} \vec{W}_0(\vec{Y}) &= \vec{Y} \\ W_{n+1}^k(\vec{Y}) &= \max_{\{j \in \pi(k) \circ \theta^n\}} (W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n - \tau \circ \theta^n)^+ \end{aligned} \tag{2.1}$$

Proof. Let t_n, $n \geq 0$, be the n-th arrival instant to the network. These variables are defined by the relations $t_0 = 0$ and $\tau \circ \theta^n = t_{n+1} - t_n$. Similarly, let $T_n^k(\vec{Y})$, $n \geq 0$, $1 \leq k \leq J$, be the instant at which queue k completes the last service brought by customer n. The proof of the theorem is by induction. Assume the property holds up to rank $n-1$.

If $e(k) \circ \theta^n = 0$,

$$W_{n+1}^k(\vec{Y}) = (W_n^k(\vec{Y}) - \tau \circ \theta^n)^+$$

and (2.1) then holds for n too.

If $e(k) \circ \theta^n \neq 0$, then, for all $j \in \pi(k) \circ \theta^n$, we have

$$T_n^k(\vec{Y}) \geq t_n + W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n$$

This relation follows from the fact that the first service requirement of customer n that is addressed to queue j is attended at the earliest at time $t_n + W_n^j(\vec{Y})$. Since the additional delay due to the migration of customer n along the route $r_{b(j)} \circ \theta^n, \cdots, r_{e(k)} \circ \theta^n$ cannot take place in less than $l^{j,k} \circ \theta^n$, queue k cannot have completed servicing the last requirement of customer n before that time. We get hence

$$T_n^k(\vec{Y}) \geq \max_{\{j \in \pi(k) \circ \theta^n\}} (t_n + W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n)$$

On the other hand, observe that, owing to the service discipline, for all i, $1 \leq i \leq m \circ \theta^n$, either customer n is serviced immediately at the i-th queue of its route (i.e. queues $r_1 \circ \theta^n, \cdots, r_m \circ \theta^n$), or this service is delayed by the completion of service requirements brought by customers $n-1, n-2, \ldots$ to this queue. Let $p(k)$ be the largest j, $1 \leq j \leq e(k)$ such that the service of customer n is delayed on queue r_j, with $p(k) = 1$, if no such delay takes place on the route $\{r_1, \cdots, r_{e(k)}\}$. Then, obviously

$$T_n^k(\vec{Y}) = t_n + W_n^{p(k) \circ \theta^n}(\vec{Y}) + l^{p(k),k} \circ \theta^n$$

so that

$$T_n^k(\vec{Y}) \leq \max_{\{j \in \pi(k) \circ \theta^n\}} (t_n + W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n)$$

We finally obtain

$$T_n^k(\vec{Y}) = \max_{\{j \in \pi(k) \circ \theta^n\}} (t_n + W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n)$$

which implies

$$\begin{aligned} W_{n+1}^k(\vec{Y}) &= (T_n^k(\vec{Y}) - t_{n+1})^+ \\ &= \max_{\{j \in \pi(k) \circ \theta^n\}} (W_n^j(\vec{Y}) + l^{j,k} \circ \theta^n - \tau \circ \theta^n)^+ \end{aligned}$$

The proof is therefore completed. ∎

2.3 Further Examples

This type of Stochastic Evolution Equation was also encountered in various queueing systems involving *synchronization constraints.* For instance, the class of *Synchronized Queueing Networks* that was introduced in [2] leads to an equation of this class where the functions $\pi(k)$, $1 \leq k \leq J$, are deterministic. A simpler case, where these functions reduce to $\pi(k) = k$, $1 \leq k \leq J$, can also be found in the class of *Fork-Join Queues* that was analyzed in [3].

3 A Loynes' Schema

The basic idea for analyzing (1.1) consists in associating to this equation another recursive schema that generalizes in a sense the schema that was originally proposed by Loynes [5] for analyzing $G/G/1$ queues.

3.1 Definition

Consider the variables $\{M_n^k\}_{n=0}^{\infty}$, $1 \leq k \leq J$, defined by

$$\begin{aligned} M_0^k &= 0 \\ M_{n+1}^k \circ \theta &= \max_{j \in \pi(k)} (M_n^j + l^{j,k} - \tau)^+ \end{aligned} \tag{3.1}$$

Lemma 3.1
For every $1 \leq k \leq J$, the sequence M_n^k is increasing in n.
Proof. It is clear that for every k, $1 \leq k \leq J$, $M_1^k \geq 0 = M_0^k$. Assume now that for some $n \geq 1$, $M_n^k \geq M_{n-1}^k$ holds for every k, $1 \leq k \leq J$. Then for any k, $1 \leq k \leq J$,

$$\begin{aligned} M_{n+1}^k \circ \theta &= (\max_{j \in \pi(k)} M_n^j + l^{j,k} - \tau)^+ \\ &\geq (\max_{j \in \pi(k))} (M_{n-1}^j + l^{j,k} - \tau)^+ \;=\; M_n^k \circ \theta \end{aligned}$$

which completes the proof of the assertion. ■

Let M_∞^k be the limiting value of the increasing sequence M_n^k when n goes to ∞. Classical continuity arguments yield

$$M_\infty^k \circ \theta = (\max_{j \in \pi(k)} M_\infty^j + l^{j,k} - \tau)^+ \tag{3.2}$$

From this we get
Lemma 3.2
For each k, $1 \leq k \leq J$, the event $\{M_\infty^k = \infty\}$ is of probability either 0 or 1.

Proof. For all $1 \leq k \leq J$, we have $k \in \pi(k)$. Hence (3.2) entails

$$M_\infty^k \circ \theta \geq M_\infty^k + l^{k,k} - \tau$$

Therefore, $M_\infty^k = \infty$ implies $M_\infty^k \circ \theta = \infty$. Let $I = 1_{\{M_\infty^k = \infty\}}$. Then the function $I \circ \theta^n$ is a.s. increasing in n, so that for all $n \geq 1$,

$$I \leq \sum_{i=1}^{n} I \circ \theta^i / n \quad a.s.$$

Letting n tend to ∞ and using the ergodic assumption yields

$$I \leq \lim_{n \to \infty} \sum_{i=1}^{n} I \circ \theta^i / n = E[I] \quad a.s.$$

Hence $I = E[I]$ *a.s.*. Since I is an indicator function, $E[I]$ takes its values in $\{0, 1\}$. ■

The following expansion of the Loynes' schema M_n^k will be needed later on.

Lemma 3.3

For every n and k, $n \geq 1$, $1 \leq k \leq J$,

$$M_n^k = \max(0, \max_{1\leq m\leq n}(H_m^k - \sum_{i=1}^{m} \tau \circ \theta^{-i})) \tag{3.3}$$

where

$$H_m^k = \max_{\{1\leq v_s\leq J,\ s=0,\cdots,m,\ v_0=k,\ v_s\in\pi(v_{s-1})\circ\theta^{-s}} (\sum_{s=1}^{m} l^{v_s,v_{s-1}} \circ \theta^{-s}) \tag{3.4}$$

Proof. The proof proceeds by induction on n. For $n = 1$, (3.3) is simply a restatement of (3.1). Suppose it holds for some $n \geq 1$. Then, we get from equation (3.1) that

$$M_{n+1}^k = \max(0, \max_{j\in\pi(k)\circ\theta^{-1}}(M_n^j \circ \theta^{-1} + l^{j,k} \circ \theta^{-1} - \tau \circ \theta^{-1}))$$

Using the inductive assumption, we obtain

$$\begin{aligned}
M_{n+1}^k &= \max(0, \max_{j\in\pi(k)\circ\theta^{-1}} \max(0, \max_{1\leq m\leq n} \\
&\quad (H_m^j \circ \theta^{-1} - \sum_{i=2}^{m+1} \tau \circ \theta^{-i})) + l^{j,k} \circ \theta^{-1} - \tau \circ \theta^{-1})) \\
&= \max(0, \max_{j\in\pi(k)\circ\theta^{-1}} \max(0, \max_{1\leq m\leq n} \\
&\quad (H_m^j \circ \theta^{-1} + l^{j,k} \circ \theta^{-1} - \sum_{i=2}^{m+1} \tau \circ \theta^{-i} - \tau \circ \theta^{-1}), l^{j,k} \circ \theta^{-1} - \tau \circ \theta^{-1}) \\
&= \max(0, \max_{1\leq m\leq n} \max_{j\in\pi(k)\circ\theta^{-1}} \\
&\quad (H_m^j \circ \theta^{-1} + l^{j,k} \circ \theta^{-1} - \sum_{i=1}^{m+1} \tau \circ \theta^{-i}), l^{j,k} \circ \theta^{-1} - \tau \circ \theta^{-1}) \\
&= \max(0, \max_{1\leq m\leq n}(H_{m+1}^k - \sum_{i=1}^{m+1} \tau \circ \theta^{-i}), H_1^k - \tau \circ \theta^{-1}) \\
&= \max(0, \max_{1\leq m\leq n+1}(H_m^k - \sum_{i=1}^{m} \tau \circ \theta^{-i}))
\end{aligned}$$

Therefore the equation holds for $n + 1$, which proves the lemma. ∎

3.2 Decomposition of the equation

In this subsection, we decompose equation (3.1) into a set of simpler equations of the same class that are endowed with some *irreducibility* property and prove that the stability condition of (3.1) reduces to the stability conditions for these *irreducible equations*.

Define the *communication graph* of the equation to be the directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where

$$\begin{aligned}
\mathcal{V} &= \{1, 2, \cdots, J\} \\
\mathcal{E} &= \{(j,k) | P[j \in \pi(k)] > 0\}
\end{aligned}$$

Obviously, $\mathcal{G}$ can have cycles.

Decompose $\mathcal{G}$ into its *communicating classes*, namely the maximal strongly connected subgraphs of $\mathcal{G}$:

$$\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1), \ \cdots, \ \mathcal{G}_g = (\mathcal{V}_g, \mathcal{E}_g).$$

such that if there is a path from j to k in $\mathcal{G}_i$, there is also a path from k to j in $\mathcal{G}_i$. It is obvious that this decomposition satisfies the properties

$$\mathcal{V}_1 \bigcup \cdots \bigcup \mathcal{V}_g = \mathcal{V}, \quad \text{and} \quad \mathcal{E}_1 \bigcup \cdots \bigcup \mathcal{E}_g \subseteq \mathcal{E},$$

and, for all $1 \leq i < j \leq g$,

$$\mathcal{V}_i \bigcap \mathcal{V}_j = \emptyset, \quad \text{and} \quad \mathcal{E}_i \bigcap \mathcal{E}_j = \emptyset.$$

Furthermore, define the *class graph*, which is denoted by $\tilde{\mathcal{G}}$, to be the graph that describes the oneway relations that may exist between the communicating classes: $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$, where

$$\begin{aligned} \tilde{\mathcal{V}} &= \{1, 2, \cdots, g\}, \\ \tilde{\mathcal{E}} &= \{(e, f) | e, f \in \{1, 2, \cdots, g\}, \ e \neq f, \ \exists (j, k) \in \mathcal{E}, \ j \in \mathcal{V}_e, \ k \in \mathcal{V}_f\}, \end{aligned}$$

It follows from the very definition of strong connectedness that $\tilde{\mathcal{G}}$ is acyclic.

We associate now g equations to (3.1), one per communicating class. Equation i has for dimension $J_i = |\mathcal{V}_i|$, for state variables $M_n^{k,i}$ ($k \in \mathcal{V}_i$), and for evolution equation

$$\begin{aligned} M_0^{k,i} &= 0 \\ M_{n+1}^{k,i} \circ \theta &= \max_{j \in \pi(k) \bigcap \mathcal{V}_i} (M_n^{j,i} + l^{j,k} - \tau)^+ \end{aligned} \tag{3.5}$$

By analogy with the Theory of Markov Chains, we will say that the system (3.5) is *irreducible*. Our main result of this subsection is stated in the following theorem.

Theorem 3.4

Equation (3.1) *is stable iff for all* $1 \leq i \leq g$, *equation* (3.5) *is stable .*

Proof. The proof requires a couple of technical lemmas. The first one consists in the following *solidarity* property:

Lemma 3.5

For every i, $1 \leq i \leq g$, either $M_\infty^k < \infty$ a.s. for all $k \in \mathcal{V}_i$, or $M_\infty^k = \infty$ a.s. for all $k \in \mathcal{V}_i$

Proof. Owing to Lemma 3.2, either $M_\infty^k < \infty$ a.s. for all $k \in \mathcal{V}_i$, or there is some $k \in \mathcal{V}_i$ such that $M_\infty^k = \infty$ a.s.. If we are in the second case, then for all h, $h \in \mathcal{V}_i$, $P[k \in \pi(h)] > 0$. Therefore the relation

$$M_\infty^h \circ \theta = \max_{j \in \pi(h)} (M_\infty^j + l^{j,h} - \tau)^+ \geq (M_\infty^k + l^{k,h} - \tau)^+$$

holds with a positive probability, so that $M_\infty^h \circ \theta = \infty$ occurs with a positive probability, and hence, according to Lemma 3.2, is an almost sure event, which concludes the proof. ■

Lemma 3.6
Let i, $1 \leq i \leq g$ be fixed. Assume that for all $k \in \mathcal{V}_i$ and for all j, such that $P[j \in \pi(k) - \mathcal{V}_i] > 0$, $M^j_\infty < \infty$ a.s.. If in addition $M^{k,i}_\infty < \infty$ a.s., for all $k \in \mathcal{V}_i$, then $M^k_\infty < \infty$ a.s..

Proof. Let $\mathcal{A}_k = \{j | P[j \in \pi(k)] > 0\}$. If $\mathcal{A}_k - \mathcal{V}_i = \emptyset$, the conclusion is obvious since the relation $M^k_n = M^{k,i}_n$ holds for all $n \geq 0$ and for all $k \in \mathcal{V}_i$. Assume now $\mathcal{A}_k - \mathcal{V}_i \neq \emptyset$. We prove the property by reduction to absurdity.

Suppose $M^k_\infty = \infty$ a.s. for some $k \in \mathcal{V}_i$. By Lemma 3.5 $M^h_n \uparrow \infty$ a.s. holds true for all h, $h \in \mathcal{V}_i$. On the other hand, by assumption, $M^j_n \uparrow M^j_\infty < \infty$ holds for all j, $j \in \mathcal{A}_k - \mathcal{V}_i$. Therefore there exists a Z such that for all $n \geq Z$, and all $j_0, h \in \mathcal{V}_i$,

$$M^j_n + l^{j,h} < M^{j_0}_n + l^{j_0,h}, \quad j \in \mathcal{A}_k - \mathcal{V}_i$$

Thus for every $h \in \mathcal{V}_i$, and $n \geq Z$,

$$M^h_{n+1} \circ \theta \;=\; (\max_{j \in \pi(h)} M^j_n + l^{j,h} - \tau)^+ \;=\; (\max_{j \in \pi(h) \bigcap \mathcal{V}_i} M^j_n + l^{j,h} - \tau)^+ \qquad a.s.$$

Let

$$U = \max_{h \in \mathcal{V}_i} (M^h_Z - M^{h,i}_Z)$$

For all $n \geq Z$, we get

$$\begin{aligned} M^h_{n+1} \circ \theta &= (\max_{j \in \pi(h) \bigcap \mathcal{V}_i} M^j_n + l^{j,h} - \tau)^+ \\ &\leq (\max_{j \in \pi(h) \bigcap \mathcal{V}_i} M^{j,i}_n + U + l^{j,h} - \tau)^+ \\ &\leq U + (\max_{j \in \pi(h) \bigcap \mathcal{V}_i} M^{j,i}_n + l^{j,h} - \tau)^+ \\ &= U + M^{h,i}_{n+1} \circ \theta \qquad a.s. \end{aligned}$$

It follows that for all $n \geq 0$ and $h \in \mathcal{V}_i$,

$$M^h_{Z+n} \circ \theta^n \leq U + M^{h,i}_{Z+n} \circ \theta^n \qquad a.s.$$

so that for all $X > 0$, according to the θ-invariance assumption on P,

$$P[M^h_{Z+n} > X] \;\leq\; P[U + M^{h,i}_{Z+n} \circ \theta^n > X] \;\leq\; P[U > \frac{X}{2}] + P[M^{h,i}_{Z+n} > \frac{X}{2}]$$

Letting n go to ∞ in the preceding relation yields

$$P[M^h_\infty > X] \leq P[U > \frac{X}{2}] + P[M^{h,i}_\infty > \frac{X}{2}]$$

where we have used the increasingness of the Loynes' schemas to permute the limits and the expectations. Owing to the assumption that $M^h_\infty = \infty$ a.s., it follows that $\lim_{X \to \infty} P[M^h_\infty > X] = 1$. Similarly, the finiteness of the variable U entails $\lim_{X \to \infty} P[U > \frac{X}{2}] = 0$, so that taking the limit in X in the last relation yields

$$P[M^{h,i}_\infty = \infty] = \lim_{X \to \infty} P[M^{h,i}_\infty > X] = 1$$

which contradicts our assumption that $M_n^{k,i} \uparrow M_\infty^{k,i} < \infty$ *a.s.*. Hence we reach the conclusion that $M_\infty^k < \infty$ for every $k \in \mathcal{V}_i$. ■

We now turn back to the proof of Theorem 3.4. We show first the following two facts

1) $M_\infty^{k,i} < \infty$ for all k, $1 \le k \le J$, and i, $1 \le i \le g$, entails $M_\infty^k < \infty$ for all k, $1 \le k \le J$.

Since $\tilde{\mathcal{G}}$ is acyclic, we can label the nodes of $\tilde{\mathcal{G}}$ as $1, \cdots, g_0, g_0+1, \cdots, g$ in such a way that $i \to j \in \tilde{\mathcal{E}}$ implies $i < j$, and nodes $1, \cdots, g_0$ have no predecessor in $\tilde{\mathcal{G}}$. The proof of this property is by induction on i, $1 \le i \le g$. Consider all i, $1 \le i \le g_0$. Since i has no predecessor in $\tilde{\mathcal{G}}$, for every $k \in \mathcal{V}_i$, k has thus no other predecessor than the elements of $\mathcal{V}_i$ in the processor graph $\mathcal{G}$: $\pi(k) = \pi(k) \bigcap \mathcal{V}_i$ *a.s.*. Therefore it follows from Lemma 3.6 that $M_\infty^k < \infty$.

Consider now i, $g_0 < i \le g$, assume that for all j, $j < i$, $M_\infty^k < \infty$ is true for all $k \in \mathcal{V}_j$. Then the fact that for all $k \in \mathcal{V}_i$, $M_\infty^k < \infty$ is an immediate consequence of Lemma 3.6 due to the fact that $\pi(k) - \mathcal{V}_i \subseteq \{1, 2, \cdots, i-1\}$. The property is hence established.

2)$M_\infty^k < \infty$, for every k, $1 \le k \le J$, entails $M_\infty^{k,i} < \infty$, for all k, $1 \le k \le J$, and i, $1 \le i \le g$.

This follows immediately from the relation $M_n^k \ge M_n^{k,i}$ for all $n \ge 0$ and $k \in \mathcal{V}_i$, $1 \le i \le g$, which are due to the fact that $\pi(k) - \mathcal{V}_i \subseteq \pi(k)$.

In view of the above properties $M_\infty^k < \infty$, for every k, $1 \le k \le J$, and $M_\infty^{k,i} < \infty$, for all k, $1 \le k \le J$, and i, $1 \le i \le g$, are equivalent. The assertion of Theorem 3.4 is therefore proved. ■

3.3 Stability condition in the irreducible case

For every i, $1 \le i \le g$, we define

$$Q_n^i = \max_{\{v_1,\cdots,v_{n+1} \in \mathcal{V}_i \mid v_{s+1} \in \pi(v_s) \circ \theta^{-s},\ s=1,\cdots,n\}} \sum_{s=1}^{n} l^{v_{s+1},v_s} \circ \theta^{-s} \qquad n = 1, 2, \cdots \tag{3.6}$$

Lemma 3.7
For all i, $1 \le i \le g$, there exists a constant γ_i such that

$$\lim_{n\to\infty} \frac{Q_n^i}{n} = \lim_{n\to\infty} \frac{E[Q_n^i]}{n} = \gamma_i \qquad a.s. \tag{3.7}$$

Proof. Observe first that the finiteness of $E[Q_n^i]$ follows from the integrability assumption on the service times (use the fact that $\max(a,b) \le a+b$). Now let $U_{m,m+n}^i = Q_n^i \circ \theta^{-m}$, $m \in \mathbb{Z}$, $n \ge 1$. Then for all $n \ge 1$, and all $p, q \ge 1$ such that $p+q=n$, we have

$$\begin{aligned} U_{m,m+n}^i &= \max_{\{v_1,\cdots,v_{n+1} \in \mathcal{V}_i \mid v_{s+1} \in \pi(v_s) \circ \theta^{-s},\ s=1,\cdots,n\}} \left(\sum_{s=1}^{n} l^{v_{s+1},v_s} \circ \theta^{-s}\right) \circ \theta^{-m} \\ &\le \max_{\{v_1,\cdots,v_{p+1} \in \mathcal{V}_i \mid v_{s+1} \in \pi(v_s) \circ \theta^{-s},\ s=1,\cdots,p\}} \left(\sum_{s=1}^{p} l^{v_{s+1},v_s} \circ \theta^{-s}\right) \circ \theta^{-m} \end{aligned}$$

$$
\begin{aligned}
&+ \max_{\{v_{p+1},\cdots,v_{n+1}\in\mathcal{V}_i \ \mid \ v_{s+1}\in\pi(v_s)\circ\theta^{-s}, \ s=p+1,\cdots,n\}} \Big(\sum_{s=p+1}^{n} l^{v_{s+1},v_s}\circ\theta^{-s}\Big)\circ\theta^{-m} \\
&= \max_{\{v_1,\cdots,v_{p+1}\in\mathcal{V}_i \ \mid \ v_{s+1}\in\pi(v_s)\circ\theta^{-s}, \ s=1,\cdots,p\}} \Big(\sum_{s=1}^{p} l^{v_{s+1},v_s}\circ\theta^{-s}\Big)\circ\theta^{-m} \\
&+ \max_{\{v_1,\cdots,v_{q+1}\in\mathcal{V}_i \ \mid \ v_{s+1}\in\pi(v_s)\circ\theta^{-s}, \ s=1,\cdots,q\}} \Big(\sum_{s=p+1}^{n} l^{v_{s+1},v_s}\circ\theta^{-s}\Big)\circ\theta^{-p}\circ\theta^{-m} \\
&= U^i_{m,m+p} + U^i_{m+p,m+p+q}
\end{aligned}
$$

Therefore

$$U^i_{m,m+n} \le U^i_{m,m+p} + U^i_{m+p,m+n}$$

and $U^i_{m,m+n}$ is hence a sub-additive process. Applying Kingman's Theorem on sub-additive ergodic processes ([4]) readily yields

$$\lim_{n\to\infty} \frac{U^i_{0,n}}{n} = \lim_{n\to\infty} \frac{E[U^i_{0,n}]}{n} = \gamma_i \qquad a.s.$$

which concludes the proof. ■

Define the workload factors of the subsystems to be

$$\rho_i = \frac{\gamma_i}{E[\tau]} \tag{3.8}$$

Theorem 3.8
For all i, $1 \le i \le g$, if $\rho_i < 1$, then $M^{k,i}_\infty < \infty$ a.s. for all $k \in \mathcal{V}_i$. If $\rho_i > 1$, then $M^{k,i}_\infty = \infty$ a.s. for all $k \in \mathcal{V}_i$.

Proof. The proof proceeds in two steps.
1) It follows from Lemma 3.5 that the event $\forall k \in \mathcal{V}_i \ : \ M^{k,i}_\infty = \infty$ is of probability 0 or 1. Assume it is of probability 1. Then $\max_{k\in\mathcal{V}_i} M^{k,i}_\infty = \infty$ a.s.. Let

$$H^{k,i}_n = \max_{\{v_s\in\mathcal{V}_i, \ s=0,\cdots,n, \ v_0=k, \ v_s\in\pi(v_{s-1})\circ\theta^{-s}} \Big(\sum_{s=1}^{m} l^{v_s,v_{s-1}}\circ\theta^{-s}\Big) \tag{3.9}$$

In view of (3.3), $\max_{k\in\mathcal{V}_i} M^{k,i}_n \uparrow \infty$ a.s. is equivalent to

$$\limsup_{n\to\infty} \max_{k\in\mathcal{V}_i} H^{k,i}_n - \sum_{s=1}^{n} \tau\circ\theta^{-s} = \infty \quad a.s..$$

Using the identity $Q^i_n = \max_{k\in\mathcal{V}_i} H^{k,i}_n$ in the last relation yields

$$\limsup_{n\to\infty} \frac{Q^i_n}{n} - \frac{\sum_{s=1}^n \tau\circ\theta^{-s}}{n} \ge 0 \quad a.s.$$

Owing to the ergodic assumption and to Lemma 3.7, this entails

$$\gamma_i = \lim_{n\to\infty} \frac{Q^i_n}{n} \ge \lim_{n\to\infty} \frac{\sum_{s=1}^n \tau\circ\theta^{-s}}{n} = E[\tau] \tag{3.10}$$

Therefore the fact that $\forall k \in \mathcal{V}_i : M_\infty^{k,i} = \infty$ *a.s.* entails that $\gamma_i \geq E[\tau]$. Taking the contrapositive of the above inference, we get that $\rho_i < 1$ entails $M_\infty^{k,i} < \infty$ *a.s.*. for all $k \in \mathcal{V}_i$. The first part of the theorem is thus proved.
2) Assume now that $\rho_i > 1$. Let $\delta = \gamma_i - E[\tau] > 0$. Then

$$\lim_{n\to\infty} \frac{Q_n^i}{n} - \frac{\sum_{s=1}^{n} \tau \circ \theta^{-s}}{n} = \delta > 0 \quad a.s.$$

which implies

$$\lim_{n\to\infty} Q_n^i - \sum_{s=1}^{n} \tau \circ \theta^{-s} = \infty \quad a.s.$$

so that $\max_{k \in \mathcal{V}_i} M_n^{k,i} \uparrow \infty$. Owing to Lemma 3.5, the last fact entails that $M_\infty^{k,i} = \infty$ *a.s.* for all $k \in \mathcal{V}_i$. The proof is hence completed. ∎

Now define

$$Q_n = \max_{\{v_1,\cdots,v_{n+1} \in \mathcal{V} \mid v_{s+1} \in \pi(v_s)\circ\theta^{-s},\ s=1,\cdots,n\}} \sum_{s=1}^{n} l^{v_{s+1},v_s} \circ \theta^{-s} \qquad n = 1, 2, \cdots \tag{3.11}$$

Using the same proof as in Lemma 3.7 allows one to establish the convergence

$$\lim_{n\to\infty} \frac{Q_n}{n} = \lim_{n\to\infty} \frac{E[Q_n]}{n} = \gamma \qquad a.s. \tag{3.12}$$

where γ is a constant. Define the workload factor of the system to be

$$\rho = \frac{\gamma}{E[\tau]} \tag{3.13}$$

It follows from the definition of irreducibility that

$$\gamma = \max_{1 \leq i \leq g} \gamma_i \tag{3.14}$$

so that

$$\rho = \max_{1 \leq i \leq g} \rho_i \tag{3.15}$$

From Theorem 3.4 and Theorem 3.8, we immediately obtain
Corollary 3.9
If $\rho < 1$, then $M_\infty^k < \infty$ a.s. for all k, $1 \leq k \leq J$. If $\rho > 1$, then there exists some k, $1 \leq k \leq J$, such that $M_\infty^k = \infty$ a.s..

4 Existence and Uniqueness of Stationary solutions

We are now in a position to study the stationary solutions of Equation (1.1). We first examine the conditions under which the solution of Equation (1.1) converges weakly. Then we show that (1.1) has a unique finite stationary solution.

4.1 Stability of the Evolution Equation

As usual, we shall understand by stability of (1.1) the weak convergence of the state vector $\vec{W}_n(\vec{Y})$ when n tends to infinity.

Using the results of Section 3 it is easy to establish the stability condition of $\vec{W}_n(\vec{0})$. Indeed, it can readily be checked that for all $n \geq 0$,

$$W_n^k(\vec{0}) = M_n^k \circ \theta^n \tag{4.1}$$

Consequently, the almost sure convergence of the schema M_n^k to a finite limit when n goes to ∞ translates into the weak convergence of the state variables $W_n^k(\vec{0})$.

Theorem 4.1

If $\rho < 1$, then, for all k, $1 \leq k \leq J$, $W_n^k(\vec{0})$ converges weakly to a finite RV $W_\infty^k(\vec{0})$ when n tends to ∞. If $\rho > 1$, then there exists some k, $1 \leq k \leq J$, such that $W_n^k(\vec{0})$ converges a.s. to ∞ when n tends to ∞.

The following lemma will be the basis for extending the above result to the case with arbitrary finite initial condition $\vec{Y} \in I\!R^{+J}$.

Lemma 4.2

Assume that $\rho < 1$. Then for any $\vec{Y} \in I\!R^{+J}$, there exists an positive integer $N(\vec{Y})$ such that for all $n \geq N(\vec{Y})$, $\vec{W}_n(\vec{Y}) = \vec{W}_n(\vec{0})$.

Proof. It can easily be checked by induction on n that for all $n \geq 0$, $\vec{W}_n(\vec{Y}) \geq \vec{W}_n(\vec{0}) \geq \vec{0}$. Assume that the statement of the theorem does not hold. Then $\vec{W}_n(\vec{Y}) > \vec{W}_n(\vec{0})$ for all $n \geq 0$.

For any fixed $n \geq 1$, let $k_n \in \{1, \cdots, J\}$ be an index such that $W_n^{k_n}(\vec{Y}) > W_n^{k_n}(\vec{0}) \geq 0$. In view of (1.1), there exists an index k_{n-1} such that

$$\begin{aligned} W_n^{k_n}(\vec{Y}) &= \max_{\{j \in \pi(k_n)\circ\theta^{n-1}\}} (W_{n-1}^j(\vec{Y}) + l^{j,k_n} \circ \theta^{n-1} - \tau \circ \theta^{n-1})^+ \\ &= W_{n-1}^{k_{n-1}}(\vec{Y}) + l^{k_{n-1},k_n} \circ \theta^{n-1} - \tau \circ \theta^{n-1} \end{aligned}$$

It is easy to see that necessarily $W_{n-1}^{k_{n-1}}(\vec{Y}) > W_{n-1}^{k_{n-1}}(\vec{0}) \geq 0$ If this were not true, we would then have

$$\begin{aligned} W_n^{k_n}(\vec{Y}) &= W_{n-1}^{k_{n-1}}(\vec{Y}) + l^{k_{n-1},k_n} \circ \theta^{n-1} - \tau \circ \theta^{n-1} \\ &\leq W_{n-1}^{k_{n-1}}(\vec{0}) + l^{k_{n-1},k_n} \circ \theta^{n-1} - \tau \circ \theta^{n-1} \\ &\leq \max_{\{j \in \pi(k_n)\circ\theta^{n-1}\}} (W_{n-1}^j(\vec{0}) + l^{j,k_n} \circ \theta^{n-1} - \tau \circ \theta^{n-1})^+ = W_n^{k_n}(\vec{0}) \end{aligned}$$

and hence, $W_n^{k_n}(\vec{Y}) \leq W_n^{k_n}(\vec{0})$, which would contradict the definition of k_n.

Similarly, there exists an index k_{n-2} such that

$$\begin{aligned} W_{n-1}^{k_{n-1}}(\vec{Y}) &= \max_{\{j \in \pi(k_{n-1})\circ\theta^{n-2}\}} (W_{n-2}^j(\vec{Y}) + l^{j,k_{n-1}} \circ \theta^{n-2} - \tau \circ \theta^{n-2})^+ \\ &= W_{n-2}^{k_{n-2}}(\vec{Y}) + l^{k_{n-2},k_{n-1}} \circ \theta^{n-2} - \tau \circ \theta^{n-2} \end{aligned}$$

and $W_{n-2}^{k_{n-2}}(\vec{Y}) > W_{n-2}^{k_{n-2}}(\vec{0}) \geq 0$. More generally, one can find a series of indices k_{n-i}, $i = 1, 2, \cdots, n$, which satisfy the relations

$$W_{n-i+1}^{k_{n-i+1}}(\vec{Y}) = W_{n-i}^{k_{n-i}}(\vec{Y}) + l^{k_{n-i},k_{n-i+1}} \circ \theta^{n-i} - \tau \circ \theta^{n-i}$$

Therefore,

$$W_n^{k_n}(\vec{Y}) = Y^{k_0} + \sum_{i=1}^{n} l^{k_{i-1},k_i} \circ \theta^{n-i} - \sum_{i=1}^{n} \tau \circ \theta^{n-i}$$

Obviously $\sum_{i=1}^{n} l^{k_{i-1},k_i} \circ \theta^{n-i} \leq Q_n \circ \theta^n$, where Q_n is defined by (3.11). Hence

$$W_n^{k_n}(\vec{Y}) \ \leq \ Y^{k_0} + Q_n - \sum_{i=1}^{n} \tau \circ \theta^{n-i} \tag{4.2}$$

Owing to Lemma 3.3, $(Q_n \circ \theta^n)/n \to \gamma$ when $n \to \infty$. Similarly $(\sum_{i=1}^{n} \tau \circ \theta^{n-i})/n \to E[\tau]$ when $n \to \infty$. Therefore, under the assumption $\rho < 1$, (4.2) readily entails that $W_n^{k_n} \to -\infty$ when $n \to \infty$, where comes the contradiction. The assertion of the theorem is thus proved. ∎

The stability condition of (1.1) with arbitrary initial condition $\vec{Y}$ is a direct consequence of Theorem 4.1 and Lemma 4.2.

Theorem 4.3

Let $\vec{Y}$ be an arbitrary non-negative real vector in $I\!R^{+J}$. If $\rho < 1$, then, for all k, $1 \leq k \leq J$, $W_n^k(\vec{Y})$ converges weakly to the finite RV $W_\infty^k(\vec{0})$ when n tends to ∞. If $\rho > 1$, then there exists some k, $1 \leq k \leq J$, such that $W_n^k(\vec{Y})$ converges a.s. to ∞ when n tends to ∞.

4.2 Existence and Uniqueness of Stationary Solutions

A sequence of finite non-negative random variables $\vec{V}_n$, $n \in \mathbb{Z}$, is said to be a stationary solution of (1.1) if $\vec{V}_n = \vec{V}_0 \circ \theta^n$ for all $n \in \mathbb{Z}$ and if $\vec{V} = \vec{V}_0$ satisfies the relation

$$V^k \circ \theta \ = \ \max_{\{j \in \pi(k)\}} (V^j + l^{j,k} - \tau)^+ \tag{4.3}$$

Corollary 3.9 together with Equation 3.2 show that the stochastic process $\vec{M}_\infty \circ \theta^n$ is such a solution when $\rho < 1$.

It can easily be shown that it is the smallest stationary solution. The proof is the same as the one proposed in [1], p. 37, for the corresponding property in the $G/G/1$ queue and will hence be omitted.

This existence result is completed by the following uniqueness property:

Theorem 4.4

Assume that $\rho < 1$, then, $\vec{M}_\infty \circ \theta^n$ is the unique stationary solution of (1.1).

proof Assume there is another solution V_n. From Lemma 4.2, there exists a finite integer $N(\vec{V}) > 0$ such that for all $n \geq N(\vec{V})$,

$$\vec{V} \circ \theta^n = \vec{W}_n(\vec{V}) = \vec{W}_n(\vec{0}) \quad a.s.$$

Using again Lemma 4.2 to obtain another finite integer $N(\vec{M}_\infty) > 0$ such that for all $n \geq N(\vec{M}_\infty)$,

$$\vec{M}_\infty \circ \theta^n = \vec{W}_n(\vec{M}_\infty) = \vec{W}_n(\vec{0}) \quad a.s.$$

Hence for all $n \geq N = \max(N(\vec{V}), N(\vec{M}_\infty))$,

$$\vec{M}_\infty \circ \theta^n = \vec{V} \circ \theta^n \quad a.s.$$

which immediately entail that $V = M_\infty$ a.s.. ∎

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