On random walks arising in queueing systems : ergodicity and transience via quadratic forms as Lyapounov functions -part I-

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Abstract : A simple and seemingly general approach is proposed to derive criteria for transience and ergodicity of a certain class of irreducible N-dimensional Markov chains in $\mathbb{Z}_+^N$ assuming a boundedness condition on the second moment of the jumps. The method consists in constructing convenient smooth supermartingales outside some compact set. The Lyapounov functions introduced belong to the set of quadratic forms in $\mathbb{Z}_+^N$ and do not always have a definite sign. Existence and construction of these forms is shown to be basically equivalent to finding vectors satisfying a system of linear inequalities.

Part I is restricted to $N = 2$, in which case a complete characterization is obtained for the type of random walks analyzed by Malyshev and Mensikov, thus relaxing their condition of boundedness of the jumps. The motivation for this work is partly from a large class of queueing systems that give rise to random walks in $\mathbb{Z}_+^N$.

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- August 1988 -

PAPIER RÉCUPÉRÉ ET RECYCLÉ

SUR DES MARCHES ALEATOIRES LIEES A DES SYSTEMES DE FILES D'ATTENTE : ERGODICITE ET TRANSIENCE PAR UTILISATION DE FORMES QUADRATIQUES COMME FONCTIONS DE LYAPOUNOV - PARTIE I -

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Résumé

On propose une approche simple et assez générale pour obtenir les conditions d'ergodicité et de transience pour certaines classes de chaînes de Markov irreductibles à valeurs dans $\mathbb{Z}_+^N$, en supposant au plus une condition d'existence des deuxièmes moments des sauts.

La méthode consiste à construire des surmartingales en dehors d'un ensemble compact. Les fonctions de Lyapounov ainsi introduites sont choisies dans l'ensemble des formes quadratiques et elles peuvent êtres signées. Leur détermination équivaut essentiellement à trouver des vecteurs satisfaisant un système d'inégalités linéaires.

La partie I de cette étude est consacrée au cas N = 2. On obtient alors une caractérisation complète des marches analysées par Malyshev et Mensikov, sans supposer les sauts bornés.

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1. Introduction

In many situations of practical interest-for instance queueing networks-it is important to have general criteria to decide wether a given stochastic system is stable or not. Moreover the rule of the game imposes to avoid an explicit computation of some hypothetical invariant measure, since it is even for "small" Markovian systems a very difficult and rarely tractable problem. [e.g. joining or serving the shorter queue, coupled processors, etc...].

The main goal of this paper is to give general ergodicity and transience conditions for random walks or Markov chains with state space the discrete positive lattice

$$\mathbb{Z}_+^N = \left\{(z_1, z_2, \ldots z_n), z_i \in \mathbb{Z}_+, i = 1, \ldots N\right\},$$

where $\mathbb{Z}_+$ denotes the set of non negative integers. The Markov chain of interest will be describded by the vector

$$z(t) = (z_1(t), z_2(t), \ldots z_n(t)), \qquad t = 0, 1, \ldots,$$

in discrete time, where, using the notation $u^+ = \max(u, 0)$,

$$z(t+1) = [z(t) + \theta(z(t))]^+, \qquad t \geq 0, \tag{1.1}$$

the distribution of the vector $\theta(z)$ depending only on the point z and ensuring the irreducibility of the process Z(t).

We shall denote by (Ω, F, P) the probability space underlying the stochastic process z(t) and by F_t the increasing family of σ algebras $\sigma\left\{z(s), 0 \leq s \leq t,\right\}$, so that z(t) is F_t measurable.

The pioneering works concerning the classification of a certain class of positive random walks in $\mathbb{Z}_+^N$, $N \geq 2$, can be found in Malyshev [5], Mensikov [7], Malyshev and Mensikov [6]. In these three nice studies, the authors establish conditions for recurrence (null or positive) and transience. In particular, explicit conditions are given for $\mathbb{Z}_+^2$ and $\mathbb{Z}_+^3$, although one may regret that no detailed proof be adduced in the above references for the case $\mathbb{Z}_+^3$.

Define the following vectors

$$\theta(z) = (\theta_1(z), \theta_2(z), \dots \theta_n(z)), \tag{1.2}$$

$$\Delta(z) = E(\theta(z)) = E\left[z(t+1) - z(t) / z(t) = z\right] = (\Delta_1(z), \Delta_2(z), \dots \Delta_n(z)), \quad z \in \mathbb{Z}_+^N.$$

Malyshev and Mensikov in [6] considered bounded and partially homogeneous random walks in $\mathbb{Z}_+^N$, i.e. random walks satisfying the two following conditions

M1 : <u>"Limited state dependency"</u> :

There exist constants $K_1, K_2, \dots K_n$, such that $\theta(z)$ is independent of the coordinate z_i, for $z_i > K_i$, $i = 1, \dots N$. In [5] one has simply $K_i = 0$, $\forall\, i \geq 1$.

For $N = 2$, a model with such a limited state dependency was introduced independently in [2] Fayolle et al.

M2 : <u>Boundedness of the jumps</u> :

$\| \theta(z) \| < D$, for all $z \in \mathbb{Z}_+^N$,

where D is a fixed positive constant.

$\Delta(z)$ is piecewise constant and depends only on the position of z with respect to the various faces of $\mathbb{Z}_+^N$.

For instance, when $N = 3$, and $K_1 = K_2 = K_3 = 0$, $\Delta(z)$ takes essentially 7 different values corresponding to the three two dimensional planes, the three positive axes and the interior of the octant.

Note that we may have an eighth value $\Delta(0)$ at the origin which is indeed of no importance for our concern as a rule, the homogeneity condition may be assumed to be satisfied only outside some compact set including the origin. The values of $\Delta(z)$ inside this compact set can be arbitrary and have no influence on the stationary behavior of the process z(t).

For $N = 2$, the criteria obtained in [5], for transience and positive or null recurrence are listed in the appendix.

Condition M2 is very restrictive, especially for queueing systems where many embedded Markov chains are encountered, which do not have bounded jumps. In a very recent study [9] for $N = 2$, Rosenkrantz extended partly Malyshev's results. In particular he does replace M2 by the weaker condition

R2 : i) (lower boundedness) $\theta(z) > - C$,
  ii) $E \| \theta^2(z) \| < \infty$, $\forall z \in \mathbb{Z}_+^2$.

But he could not decide between positive and null recurrence, which might be embarrassing in typical queueing applications. His approach consists in constructing Lyapounov functions in polar coordinates of the form $\Phi(r,\theta) = r^{\alpha} \cos(\alpha\theta - \theta_1)$, also used in [11] by Varadhan and Williams to solve a submartingale problem.

We propose here a direct method to find necessary and sufficient conditions for transience and ergodicity of a wide class of irreducible random walks under the assumptions M2 and F2, where

F2 : Second Moment condition

$$E \| \theta^2(z) \| < \infty, \ \forall z \in \mathbb{Z}_+^N. \qquad (1.3)$$

Note that F2 is much weaker than M2 and R2.

We also introduce

F3 : First moment condition

$$E \|\theta(z)\| < \infty, \quad \forall z \in \mathbb{Z}_+^N. \qquad (1.4)$$

2. *The two dimensional case with maximal homogeneity*

To underline the key, ideas, we first consider the case $N = 2$, under the assumptions F2 and G1, where G1 is adapted from M1 as follows :

Condition G1

i) $K1 = K2 = 0$ (The constants K_i have been introduced in F2)

$$\text{ii) } \Delta(z) = \left\{ \begin{array}{ll} (\alpha, \beta) & , \text{for } a > 0, b > 0, \\ (\alpha_1, \beta_1) & , \text{for } a < 0, b = 0, \\ (\alpha_2, \beta_2) & , \text{for } a = 0, b > 0, \end{array} \right\}$$

where, for the sake of brevity, *we shall write throughout this section* $z(t) = (a(t), b(t))$ and

$z = (a,b)$.

Note that the constraints here are imposed only on $\Delta(z)$ and are thus slightly weaker than in M1. For less stringent homogeneity conditions, i.e. $K_i > 0$, we have to go back to condition M1, as will be shown in section 3.

The problem of ergodicity and transience is tackled thereafter via the construction of convienient super (or sub) martingales, i.e. Liapounov functions.

Indeed, after mature consideration, it does not seen supernatural to seek these functions in the class of quadratic forms, which, as soon as they are positive definite, determine a metric in the L_2 space. A similar approach for a two dimensional birth and death process has been proposed in [1].

Theorem 2.1 Assume conditions G1 and F2 are satisfied.

a) If $\alpha < 0$, $\beta < 0$, then the process $z(t)$ is

i) - ergodic if

$$\begin{aligned} \alpha\beta_1 - \alpha_1\beta < 0, \\ \beta\alpha_2 - \beta_2\alpha < 0; \end{aligned} \tag{2.1}$$

ii) - transient if either

$$\alpha\beta_1 - \alpha_1\beta > 0 \quad \text{or} \quad \beta\alpha_2 - \beta_2\alpha > 0, \tag{2.2}$$

or

$$\alpha\beta_1 - \alpha_1\beta = \beta\alpha_2 - \beta_2\alpha = 0. \tag{2.3}$$

b) If $\alpha \geq 0$, $\beta < 0$, then the process $z(t)$ is

i) - ergodic if

$$\alpha\beta_1 - \alpha_1\beta < 0\ ; \tag{2.4}$$

ii) - transient if

$$\alpha\beta_1 - \alpha_1\beta > 0. \tag{2.5}$$

and also if $\alpha > 0$ and $\alpha\beta_1 - \alpha_1\beta = 0$

c) If $\alpha \geq 0$, $\beta \geq 0$

The process z(t) is transient.

d) is symetric to case b)

Moreover the results of par b), c), d) hold under the sole condition (1.4)

Note : to ensure the irreducibility of the process we shall assume $\beta_1 > 0$ and $\alpha_2 > 0$.

Proof

Introduce the following real function on $\mathbb{Z}_+^2$

$$f(a,b) = \frac{ua^2}{2} + \frac{vb^2}{2} + w\,ab, \quad \text{for all } (a,b) \in \mathbb{Z}_+^2, \tag{2.7}$$

where u,v,w are unspecified constants to be properly chosen later. Most of the time, f(a(t), b(t)) will be rewritten as f_t and f(a,b) as f(z).

We have

$$[f_{t+1} - f_t / F_t] = h_t + D_t\ , \tag{2.8}$$

where, using definition (1.2),

$$h_t = a(t)\,[u\,\Delta_1(z(t)) + w\,\Delta_2(z(t))] + b(t)\,[v\Delta_2(z(t)) + w\,\Delta_1(z(t))],$$

$$D_t = E\,[f(\Delta_1(z(t)), \Delta_2(z(t))]. \tag{2.9}$$

The game now will consist in choosing properly the constants u,v,w to be in a position to apply one of the two theorems quoted thereafter and which hold in more general contexts e.g. Φ - irreducible chains as defined by Orey [8]. See also and Lamperti [4], Tweedie [10].
Let $\{X_n\}$, $n \geq 0$,be a Markov chain with a countable state space S.

Theorem 2.2 Foster's criterion see [3].

In order that X_n be positive recurrent, it is sufficient that there exist a finite set U, an irreducible $\varepsilon > 0$ and a positive function $f : S \to \mathbb{R}_+$ such that

$$E\,[f(X_{n+1}) - f(X_n) / X_n = i] \ < \varepsilon, \ \forall\, i \in S - U,$$

$$E[f(X_{n+1}) / X_n = i] < \infty, \ \forall\, i \in U. \tag{2.10}$$

Theorem 2.3

In order that $\{X_n\}$ be transient, it is sufficient that there exists a positive function $f : S \to \mathbb{R}_+$ and a set U such that

$$E[f(X_{n+1}) - f(X_n) / X_n = i] \leq 0, \ \forall i \notin U, \tag{2.11}$$

$$\inf_{i \notin U} f(i) < f(j), \quad \forall\, j \in U. \tag{2.12}$$

Proof of 2.3

Let τ be the stopping time

$$\tau = \inf \{n > 0, X_n \in U / X_0 \notin U\}.$$

It is readily verified that $(X_{n\wedge\tau}, F_n)$ $n \geq 0$, form a bounded supermartingale such that

$$E(f(X_{n\wedge\tau})) \leq E(f(X_0)) \ , \ X_0 \notin U, \text{ for all } n \geq 0.$$

Assume $P(\tau < \infty) = 1$. Then, from Fatou's Lemma, we have

$$E(f(X_\tau)) \leq E(f(X_0)) \ , \ X_0 \notin U.$$

But on $(\tau < \infty)$, $X_\tau \in U$ so that it is possible to choose $X_0 = i$ satisfying (2.12), which yields a contradiction. Thus $P(\tau < \infty) < 1$ and transience follows.

An alternative and less direct proof has been given in [10]. Theorem 2.2 can be established in a similar way.

We also will need the following

Lemma 2.1

Define τ as in theorem 2.3.
Suppose there exists a positive function $f : S \to R_+$, such that

$$E\,[f(X_{n+1}) - f(X_n) / F_n] < -\varepsilon, \qquad \text{on } \{X_n \notin U\}.$$

Then, $E(\tau) < \infty$.

The result holds for any sequence of random variables (X_n), $n \geq 0$, (i.e. not necessarily forming a Markov chain).

Proof

Let $Y_n = f(X_{n \wedge \tau})$ and $\mathbb{1}(A)$ be the indicator function of the set A.

We have

$$E[Y_{n+1} - Y_n / F_n] \leq -\varepsilon\,(\tau > n), \qquad n \geq 0,$$

which yields

$$0 \leq E(Y_{n+1} / F_0) \leq Y_0 - \varepsilon \sum_{k=0}^{n} P(\tau > k).$$

Using the positivity of Y_n and letting $n \to \infty$ yields the result.

We now proceed to prove part a) of theorem 2.1. Assumption F2, i.e. equation (1.3), ensures that D_t, defined in (2.9), is uniformly bounded in the positive quarter plane.
Thus, for some positive D,

$$\sup_{z \in \mathbb{Z}_+^2} E[f(\Delta_1(z), \Delta_2(z))] < D. \qquad (2.13)$$

i) Ergodicity

Theorem 2.2 and equations (2.8), (2.9) show that it suffices to achieve the following system of inequalities

$$u\alpha + w\beta < 0 \quad , \tag{2.14 a}$$
$$u\alpha_1 + w\beta_1 < 0 \quad ; \tag{2.14 b}$$

$$u\alpha + w\beta < 0 \quad , \tag{2.15 a}$$
$$u\beta_2 + w\alpha_2 < 0 \quad ; \tag{2.15 b}$$

u, v, w being selected to render the quadratic form f(a,b) definite positive, that is

$$\begin{gathered} u > 0\,, v > 0, \\ w > 0 \quad \text{or,} \quad \text{if } w < 0, \;\; w^2 < uv. \end{gathered} \tag{2.16}$$

Since $\alpha > 0$, $\beta < 0$, (2.14) and (2.15) are equivalent respectively to

$$\frac{-u\alpha}{\beta} < w < \frac{-u\alpha_1}{\beta_1} \,, \tag{2.17}$$

and

$$\frac{-v\beta}{\alpha} < w < \frac{-v\beta_2}{\alpha_2} \,, \tag{2.18}$$

which in turn entail the inequalities $u(\alpha\beta_1 - \alpha_1\beta) < 0$ and $v(\beta\alpha_2 - \alpha\beta_2) < 0$.

Suppose (2.1) holds. It follows immediately that (2.17) and (2.18) can be satisfied for all $u > 0$, $v > 0$.

Moreover, the two left inequalities in (2.17) and (2.18) allows one to find w such that $w^2 < uv$.

Hence, from (2.8), $h_t + D_t$ can be rendered strictly negative for all z(t) outside a compact set U, where

$$U = \{z = (a, b) : a < A\,, b < B\}, \tag{2.19}$$

where the positive constants A and B are taken large enough. In other words, we have achieved (2.10) with a positive f and ergodicity follows.

ii) Transience

Assume for instance that

$$\alpha\beta_1 - \alpha_1\beta > 0 \text{ and } \beta\alpha_2 - \beta_2\alpha < 0$$

Then (2.17) entails $u < 0$. It follows that (2.14), (2.15) and (2.10) can be achieved by means of a quadratic form f(z) which now, in the usual terminology, is not *definite* :

$$f(a,b) = \frac{v}{2}[b - m_1 a][b + m_2 a],$$

where m_1 and m_2 are positive constants. (2.20)

We have

$$E[f_{t+1}/F_t] = f_t - \varepsilon, \qquad \forall z(t) \notin U$$

for some U as in (2.1.9), but here f(z) does not have a constant sign.
Upon writing now

$$f_t = f_t^+ - f_t^-,$$

where

$$f_t^+ = \max(f_t, 0) \text{ and } f_t^- = -\min(f_t, 0)$$

we obtain, from (2.20),

$$-E[f_{t+1}^-/F_t] \leq E[f_{t+1}/F_t] \leq f_t - \varepsilon \leq f_t^+ - f_t^-, \quad \text{for } z(t) \notin U,$$

which yields

$$-E[f_{t+1}^-/F_t] \leq -f_t^-, \text{ for } z(t) \notin U.$$

Then $-(f_t^- + C)$ a negative supermartingale, for some strictly positive constant C.

By Jensen's inequality, we have, after setting

$$\tilde{f}_t = \frac{-1}{f_t^- + C},$$

$$E[\tilde{f}_{t+1}/\mathcal{F}_t] \geq \frac{-1}{E[f_t^- + C/\mathcal{F}_t]} \geq \tilde{f}_t, \quad \text{for } z(t) \notin U \tag{2.21}$$

On $\mathbb{Z}_+^2 - V$, $\tilde{f}_t$ is a negative submartingale.

The application of theorem 2.3 is now immediate, by means of the positive function $-\tilde{f}_t$ constructed above. It suffices to choose the original point z(o) in the wedge where f(a,b), as given (2.20), is negative and such that

$$0 < -f_0 = -f(z(0)) < \varepsilon_1,$$

where ε_1 satisfies

$$\varepsilon_1 < -\tilde{f}_t, \text{ for all } z(t) \in U.$$

Thus, when (2.2) holds the system is transient.

– *Assume now that (2.3) holds*

Then it is possible to satisfy the following system

$$\begin{matrix} u\alpha + w\beta = 0, & & v\beta + w\alpha = 0 \\ & \text{and} & \\ u\alpha_1 + w\beta_1 = 0, & & v\beta_2 + w\alpha_2 = 0 \end{matrix}, \tag{2.22}$$

by taking $u < 0$, $v < 0$, $w > 0$ and $w^2 = uv$.

In this case, the function f(a,b) takes the form

$$f(a,b) = \frac{v}{2}\left(b + \frac{ua}{w}\right)^2,$$

and (2.8) entails, since $D_t \leqslant 0$,

$$E[f_{t+1}/\mathcal{F}_t] = f_t + D_t \leq f_t, \ \forall\, z(t) \in \mathbb{Z}_+^2 \tag{2.23}$$

Arguing exactly as above yields the transience of z(t).

* <u>*Proof of part c)*</u>

Here $\alpha \geq 0$ and $\beta \geq 0$.

The derivation of the result mimics exactly the proof of transience in part a (ii). In fact, (2.14) and (2.15) are satisfied as soon as

$u = v = 0$, $w = -1$, in which case the function

$f(a,b) = -(ab+C)$, for some positive constant C, is a negative supermartingale...

* *Proof of part b)*

i) First we proceed to the derivation of the transience criterion
- Assume the first statement of (2.5) holds, i.e. $\alpha\beta_1 - \alpha_1\beta > 0$.

Then, equations (2.14) and (2.15) yield

$$\begin{aligned} &\frac{-\alpha u}{\beta} < w < \frac{-\alpha_1 u}{\beta_1}, \\ &w < \min\left(\frac{-\beta v}{\alpha}, \frac{-\beta_2 v}{\alpha_2}\right). \end{aligned} \tag{2.24}$$

It is straightforward to check that system (2.24) admits a set of solutions (u, v, w), with

$u < 0, v < 0, w < 0$.

We have constructed a strictly negative supermartingale in the whole positive quarter plane since $D_t < 0$ and, again, the result follows.

When $\alpha > 0$ and $\alpha\beta_1 - \alpha\beta_2 = 0$, there exist real numbers $u < 0$, $v < 0$, $w < 0$,

which are solution of the system

$$\begin{aligned} &\frac{-\alpha u}{\beta} = w = \frac{-\alpha_1 u}{\beta_1} \\ &w < \min\left(\frac{-\beta v}{\alpha}, \frac{-\beta_2 v}{\alpha_2}\right), \end{aligned} \tag{2.25}$$

obtained upon replacing the inequalities in (2.14) by equalities. Consequently the process z(t) is transient.

ii) Ergodicity

Assume (2.4) holds.

The situation here is different and somehow more complicated since, as can easily be checked, the inequations (2.14 a) and (2.15 b) are incompatible for $\beta_2 > 0$.

The case $\beta_2 < 0$ is solved by repeating the argument of part a(i) : there exists a function f(z) given in (2.7), positive and satisfying (2.10), so that z(t) is ergodic. The details are omitted.

If one does not mortgage the sign of β_2, it is necessary to resort to a more general method extensively used in the next section and involving the construction of "embedded" supermartingales.

Introduce the region

$$V = (\mathbb{Z}^2_+ - U) \cap (a > 0).$$

For all $z \in V$, the system formed by (2.14) and (2.15 b) admits solutions $u > 0$, $v > 0$, $w > 0$.

Hence, for some ε_1 positive,

$$E[f_{t+1}/F_t] \leq f_t - \varepsilon_1, \quad \text{for all } z(t) \in V. \tag{2.26}$$

According to (2.9), define the following quantities

$$\begin{aligned} r(t) &= u\,\Delta_1(z(t)) + w\,\Delta_2(z(t)), \\ s(t) &= v\,\Delta_2(z(t)) + w\,\Delta_1(z(t)). \\ R_{k,t} &= r(k)\,E[a(k)/F_t], \quad k \geq t, \\ S_{k,t} &= s(k)\,E[b(k)/F_t], \quad k \geq t. \end{aligned} \tag{2.27}$$

From (2.9) and (2.27) we have, for any positive integer T and t ,

$$E(f_{t+T}/F_t) = f_t + \sum_{k=t}^{t+T-1} (R_{k,t} + S_{k,t} + D_k). \tag{2.28}$$

Take (2.28) for all sample paths

$$\{z\,(t+k) \in U_2,\ 0 \le k \le T\}\ , \tag{2.29}$$

where

$$U_2 = \{z = (a,b)\ ,\ a \le A,\ b > B\}\ .$$

Then

$$S_{k,t} = [S_2 P\,(a(k) = 0) + S\,(1\text{-}P\,(a(k) = 0))]\ E(b(k)\,/\,F_{\ t})\ ,$$

with

$$\begin{aligned} S_2 &= u\alpha_2 + w\beta_2, \\ S &= u\alpha + w\beta\ . \end{aligned} \tag{2.30}$$

We remark now that the component a(t) evolves, in the region U_2, as a one dimensional random walk with a positive drift α. This implies (uniformly for all $b(t) > B$ and $a(t) \le A$) the existence of t_0 and $\delta > 0$ such that, assuming (2.14 a), [i.e. $S < 0$] ,

$$S_2 P\,(a_k = 0\,) + S(1 - P\,(a_k = 0)) < -\,\delta,\ \text{for all } k \ge t_C\ .$$

It follows that, for $T > t_C$

$$\frac{1}{T}\sum_{k=t}^{t+T-1} S_{k,t} = \frac{1}{T}\sum_{k=t}^{t+t_0-1} S_{k,t} + \frac{1}{T}\sum_{k=t+t_0}^{t+T-1} S_{k,t}$$

$$\le \frac{1}{T}\,[Mt_0\,|\rho_{t_{0,t}}| + B\,\rho_{T,t}]\ , \tag{2.31}$$

with

$$M = \sup_{z(t)}\ r(t)\ ,$$

$$\rho_{s,t} = \sum_{k=t}^{t+s} r(k)\ .$$

Now the ergodic theorem shows that, when (2.29) is satisfied, $\frac{1}{T}\rho_{T,t}$ behaves as $S < 0$, for large T. Thus, (2.31) and (2.28) ensure the existence of bounded constants T and B, such that

$$E[f_{t+T}/F_{\ t}] \le f_t - \varepsilon_2 T\quad ,\qquad \text{for all } z(t) \in U_2, \tag{2.32}$$

for some positive ε_2.

Hence, by (2.26) and (2.32) we have constructed an increasing sequence of stopping times τ_n, such that

$$E[f_{\tau_{n+1}}/F_{t}] \leq f_{\tau_n} - \varepsilon(\tau_{n+1} - \tau_n) \quad , \quad \text{for all } z(\tau_n) \notin U, \qquad (2.33)$$

where $\tau_{n+1} - \tau_n$ takes only the values 1 and T and U is the ubiquitous compact set defined in (2.19).

We shall prove in section 3 the theorem 3.2 valid for more general random sequences τ_n, which says that, whenever (2.33) holds, the Markov chain z(t) is positive recurrent.

A careful examination the proof of (2.26) and (2.32) shows that all the arguments used would indeed have been valid if we had choosen a linear Liapounov function of the form $g(a, b) = ua + wb$. Hence, when $\alpha \geq 0$, $\beta < 0$, we have the sharper result announced in the statement of part b) of theorem 2.1 : Under the sole assumption (1.4), the process z(t) is positive recurrent when (2.14) holds.

The proof of theorem 2.1 is terminated.

3 The two dimensional case with limited state dependency

We assume in this section that conditions M1 and F2 of section 1 are fulfilled.

The notation and the argument will be essentially the same as in the proof of part b) of theorem 2.1 above. In particular, we shall encounter quite naturally the important notion of "induced Markov chain" introduced in [5] and [6] for any dimension N, which amounts to remark that, on $\{a > K_1\}$ [resp $\{b > K_2\}$], the component a(t) (resp. b(t)) evolves as a one-dimensional random walk. Let us denote its corresponding stationary distribution by

$$\gamma = (\gamma_1 , \gamma_2 ,\ldots) ,$$

respectively (3.1)

$$\psi = (\psi_1 , \psi_2 ,\ldots) .$$

Introduce the following quantities :

$$E[\Delta_1(z) / z = (a, n)] = \gamma_n, \quad \forall\, a > K_1,$$
$$\gamma_n = \gamma, \ \text{for } n > K_2,$$

$$E[\Delta_2(z) / z = (m, b)] = \delta_m, \quad \forall\, b > K_2,$$
$$\delta_m = \delta, \text{ for } m > K_1,$$

$$\rho_1 = \sum_{k \geq 0} \psi_k \gamma_k, \tag{3.2}$$

$$\rho_2 = \sum_{k \geq 0} \varphi_k \delta_k.$$

It will also be convenient to define the regions, for $A \geq K_1$ and $B \geq K_2$,

$$\begin{aligned} U &= \{z = (a, b) / a \leq A, b \leq B\}, \\ U_1 &= \{z = (a, b) / a > A, b \leq B\}, \\ U_2 &= \{z = (a, b) / a \leq A, b > B\}, \\ U_3 &= \{z = (a, b) / a > A, b > B\}, \end{aligned} \tag{3.3}$$

Theorem 3.1

a) If $\gamma < 0$, $\delta < 0$, then the process z(t) is

– i) ergodic if
$\rho_1 < 0$ or $\rho_2 < 0$;

– ii) transient if either
$\rho_1 > 0$ or $\rho_2 > 0$;

b) If $\gamma \geq 0$, $\delta < 0$, then the process z(t) is

i) ergodic if
$\rho_1 < 0$;

ii) transient if
$\rho_1 > 0$,
and also

if $\gamma > 0$ and $\rho_1 = 0$;

c) If $\gamma \geq 0$, $\delta \geq 0$,

The process z(t) is transient ;

d) Symetric to case b).

Proof

We shall first prove the assertion mentioned at the end of section 2 [see equation (2.33)].

Theorem 3.2

Let (Y_n), $n \geq 0$, be an increasing sequence of positive F_n measurable random variables.
Let (σ_n), $n \geq 0$, be an increasing sequence of positive random variables with values in $\mathbb{Z}_+ - 0$, such that

$$\overline{\lim_{n \to \infty}} \frac{E(\sigma_n)}{n} \leq \theta < \infty . \tag{3.4}$$

Define also the F_n stopping time
$\tau = \inf\{n > 0 , Y_n < C\}$.
If there exists a constant C such that, for very $n \geq 0$,

$$E[Y_{\sigma_{n+1}} / F_{\sigma_n}] < Y_{\sigma_n} - \varepsilon(\sigma_{n+1} - \sigma_n) , \text{ on } \{Y_{\sigma_n} \geq C\}, \tag{3.5}$$

then $E(\tau) < \infty$.

Proof

Let $\zeta_n = Y_{\sigma_n} \mathbb{1}(\tau > \sigma_n)$ and $\tilde{F}_n = F_{\sigma_n}$.

Then, since $\mathbb{1}(\tau > \sigma_n) \geq \mathbb{1}(\tau > \sigma_{n+1})$,

$$E[\zeta_{n+1} / F_n] \leq \zeta_n - \varepsilon(\sigma_{n+1} - \sigma_n) \mathbb{1}(\tau > \sigma_n), \quad \text{on } (\tau > \sigma_n)$$

On the other hand, $\zeta_{n+1} = \zeta_n = 0$ on $(\tau \leq \sigma_n)$.

Thus $(\zeta_n, \tilde{F}_n)$ form a non negative supermartingale.

We get

$$0 \leq E(\zeta_{n+1}) \ldots \leq E(\zeta_0) - \varepsilon \sum_{k=1}^{n} E[(\sigma_{k+1} - \sigma_k)\, 1\, (\tau > \sigma_k)] .$$

Hence, fixing $\zeta_0 = Y_0$ constant,

$$\sum_{k=1}^{n} E[(\sigma_{k+1} - \sigma_k)\, 1\, (\tau > \sigma_k)] < \frac{Y_0}{\varepsilon}, \qquad \forall n > 0 \tag{3.6}$$

Since $\sigma_{k+1} - \sigma_k \geq 1$, by using (3.4) and Tchebycheff inequalities, we have

$$\sum_{k=n_0}^{n} E[(\sigma_{k+1} - \sigma_k)]\, 1\, (\tau > \sigma_k)] > \sum_{k=n_0}^{n} P(\tau > \sigma_k) \geq \sum_{k=n_0}^{n} P(t > kD)\, P(\sigma_k < kD)$$

$$\geq (1 - \frac{\theta}{D}) \sum_{k=n_0}^{n} P(\tau > kD), \quad \text{for } n_0 \text{ sufficiently large and any positive constant D.} \tag{3.7}$$

Choosing $D > \theta$, using (3.6), and letting $n \to \infty$ in (3.7) yields $E(\tau) < \infty$.

The proof of theorem 3.2 is completed.

In the Markovian case, theorem 3.2 and relation (2.33) entail the ergodicity of z(t) since U is a finite set. The lines of force in the proof of theorem 3.1 reside in the construction of a convenient sequence of bounded times in order to get (2.33).

We shall only sketch the proof of part a), as the other cases can be treated in a similar manner.

<u>Proof of part a</u>

By assumption, $\gamma < 0$ and $\delta < 0$.

Using (2.27) and (3.2), we have

$$S_{k,t} = \sum_{i=0}^{A} [(v\delta_i + w\gamma_i)\, P(a(k) = i)]\, E[b(k) / F_t] \text{, for all } z(k) \in U_2,\ t \leq k \leq t + T_2,$$

Now, as in the derivation of (2.31), we can apply the ergodic theorem to the one dimensional random walk a(k), for $b(k) > B$, $t \leq k < t + T_2$, where T_2 is bounded our selected so that

$$\frac{1}{T_2}\rho_{T_{2,t}} \approx \sum_{i=0}^{A} (v\delta_i + w\gamma_i)\,\varphi_i\,, \qquad (3.8)$$

with $\rho_{s,t}$ defined in (2.31).
In the region U_3, it suffices to satisfy the system

$$\begin{aligned} &u\gamma + w\delta < 0, \\ &v\delta + w\gamma < 0, \end{aligned} \qquad (3.9)$$

to ensure

$$E[f_{t+1}/F_t] = f_t - \varepsilon \quad , \qquad z(t) \in U_3$$

Finally, from (2.28), (3.8) and (3.9), we get (2.32), uniformly in U_2 for some B and A, provided that holds the following relation

$$v\,F_A + w\,G_A < 0\,,$$

where

$$F_j = \sum_{i=0}^{j} \delta_i \varphi_i\,, \qquad G_j = \sum_{i=0}^{j} \gamma_i \varphi_i \quad .$$

Analogous relations and conclusions could be derived for $R_{k,t}$, $\{z(k) \in U_1, t \le k < t + T_1\}$, for some bounded T_1.

Putting together the pieces of the puzzle leads to the following assertion : Relation (2.33) holds, or, equivalently, the system is ergodic, if there exist u, v, w such that, for some $A > K_1$ and $B > K_2$,

$$\begin{aligned} &vF_A + wG_A < 0, \\ &v\delta + w\gamma < 0 \end{aligned} \qquad (3.10)$$

and

$$\begin{aligned} &uH_B + wL_B < 0\,, \\ &u\gamma + w\delta < 0\,, \end{aligned} \qquad (3.11)$$

where, in (3.11), we have set

$$H_j = \sum_{k=0}^{j} \gamma_k \psi_k\,, \qquad L_j = \sum_{k=0}^{j} \delta_k \psi_k\,,$$

Multiplying the second equation of (3.10) by $[1 - \sum_{i=A+1}^{\infty} \varphi_i]$ and summing with (3.12) yields $\rho_2 < 0$,

since $\sum_{i=0}^{\infty} \varphi_i \gamma_i = 0$.

Arguing similarly, we get $\rho_1 < 0$.

The transience as well as parts (b) and (c) can be shown along the lines of section 2.

The proof of theorem 3.1 is terminated.

4. More dimensions and conclusions of part I

The method proposed in the preceding sections applies to the case $N \geq 3$.

Introduce the quadratic form

$$f(z) = \frac{1}{2} z^* Q z \quad , \quad z \in \mathbb{Z}_+^N ,$$

where z is taken as a column vector, z^* is the transpose of z, and Q denotes a symetric matrix of size N.

Upon setting

$$Y_t = \frac{1}{2} z^*(t) Q z(t) .$$

we get, with the notation of section 1,

$$E[Y_{t+1} / F_t] = Y_t + z^*(t) Q \Delta (z(t)) + \frac{1}{2} E[\theta^* z(t) Q \theta z(t)] . \qquad (4.1)$$

Assume conditions M1 and F2 hold. In order to render the second term in the right member of (4.1) negative outside some compact set, it is necessary to satisfy $N\,2^{N-1}$ linear inequalities, where the unknowns are the $\frac{N(N+1)}{2}$ coefficients of the matrix Q.

Checking the "sign" of f(z) involves the (easy) computation of the N principal "minors" of Q.

According to Jacobi's results these determinants should be all positive if the form is definite positive.

When $N = 3$, the criteria, for transience and ergodicity have been given in [7], assuming the boundedness of the jumps. In part II of our study, we will show that these criteria are still. A formal computer program will be provided for $N \geqslant 4$.

In the case of more general random walk i.e. when the input sequence is assumed to be only stationary or ergodic, it is possible to proceed along the lines of section 2 and 3, by constructing sequences of "empty points" and using respectedly the maximal ergodic lemma. Part II will also consider briefly this situation.

It is worth mentioning that many of the results obtained so far can easily be extended to more a general state, for instance $\mathbb{R}_+^N$, space under adequate conditions of irreducibility.

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APPENDIX

Assume that conditions M1 and M2 hold with $K_1 = K_2 = 0$.
The following theorem was proved by Malyshev [5], [6].
[The notation is the one of section 2].

A) If $\alpha > 0$, $\beta \geq 0$, the process z(t) transient;
B) If $\alpha < 0$, $\beta < 0$, the process z(t) is

i) ergodic if and only if

$$\alpha\beta_1 - \alpha_1\beta < 0 \ , \ \beta\alpha_2 - \alpha\beta_2 < 0 \ ;$$

ii) null recurrent if

$$\alpha\beta_1 - \alpha_1\beta \leq 0 \ , \ \beta\alpha_2 - \alpha\beta_2 \leq 0 \ ;$$

iii) transient in the remaining cases ;

C) if $\alpha \geq 0$, $\beta < 0$, the process z(t) is

i) ergodic if and only if

$$\alpha\beta_1 - \alpha_1\beta < 0 \ ;$$

ii) null recurrent if

$$\alpha\beta_1 - \alpha_1\beta = 0 \ ;$$

iii) transient in the remaining cases .

D) is symetric to case C .

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