On the stability condition of a precedence-based queueing discipline

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ON THE STABILITY CONDITION OF A PRECEDENCE-BASED QUEUEING DISCIPLINE

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Sur la condition de stabilité d'une discipline d'attente définie par des contraintes de précédence

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Abstract

On considère le modèle de file d'attente avec contraintes de précédence proposé par J.N. Tsitsiklis, C.H. Papadimitriou et P. Humblet dans [3]. Ce système possède une infinité de serveurs. Pour tout couple de clients i et j tel que i arrive dans le système après j, on retarde le début de l'exécution de i jusqu'à la fin de l'execution de j avec la probabilité p. Ces auteurs ont déterminé la condition de stabilité du système dans le cas particulier où les durées de service sont deterministes et le processus d'arrivée est un processus de Poisson et ont émis la conjecture suivante: Sous des hypothèses statisitiques de type renouvellement, la condition de stabilité dépend de la distribution complète des durées de service mais seulement du premier moment des inter-arrivées.

Dans cet article, nous étudions ce modèle de file d'attente sous des hypothèses statistiques générales ne supposant que la stationnarité et l'ergodicité des processus d'arrivée et de service. Les hypothèses statistiques sur les contraintes de précédence sont elles aussi généralisées pour permettre de faire dépendre la probabilité que le client i ait à attendre le client j de la "distance" $i - j$.

On établit une expression générale pour la condition de stabilité qui permet de prouver la conjecture precédemment mentionnée. Ces résultats sont obtenus à partir d'équations d'évolution stochastiques décrivant les trajectoires du système et à partir d'un schéma généralisant le schéma de Loynes pour la file $G/G/1$.

On the Stability Condition of a Precedence-Based Queueing Discipline

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Abstract

The queueing model, called *Precedence-Based Queueing Discipline*, proposed in [3] by J.N. Tsitsiklis, C.H. Papadimitriou and P. Humblet is addressed. In this queueing model, there are infinitely many servers and for any pair of customers i and j such that i arrived later than j, there is a fixed probability p that i will have to wait for j's execution to terminate before i starts its execution. In the case where the customer service times are deterministic and the arrival process is Poisson, these authors have derived the stability condition which determines the maximum arrival rate that keeps the system stable. For more general statistics, they conjectured that the stability condition would depend on the complete service time distribution functions but only on the first moment of the inter-arrival times.

In this paper, we consider this queueing model and relax the restrictive statistical assumptions mentioned above by only assuming that the service times and the inter-arrival times are stationary and ergodic sequences, so that these variables can receive general distribution functions and be correlated. The assumptions on the precedence relationship are also generalized in that the probability for a new customer to have a precedence relation with a previously arrived customer may depend upon the difference between the two customers indices.

We derive a general expression for the stability condition which in turn proves the above conjectures. The results are obtained by establishing pathwise evolution equations for these queueing systems, and then a schema which, in certain sense, generalizes the schema of Loynes for the $G/G/1$ queues.

Categories and Subject Descriptors: C.4 [**Performance of Systems**]: *modeling techniques, design studies*; D.4.8 [**Operating Systems**]: *Performance–queueing theory*; D.4.8 [**Database Management**]: Physical Design.

General Terms: Design, Performance, Theory, Verification.

Additional Key Words and Phrases: Database concurrency control, ergodic theory, queueing theory, stability condition, static locking, throughput

1 Problem Description

In [3], J.N. Tsitsiklis, C.H. Papadimitriou and P. Humblet considered a queueing system with infinitely many servers, and with the following queueing discipline: For any pair of customers i and j such that i arrived later than j, there is a fixed probability p that i will have to wait for j to terminate its execution before i starts its execution. This queueing system is a very simple model for database concurrency control via "static" locking, as well as for parallel execution of programs consisting of several interdependent processes.

In the case where the customer service times are deterministic and the arrival process is Poisson, these authors have derived the stability condition which determines the maximum arrival rate for which the system is stable. For more general statistical assumptions they conjectured that the stability condition would depend on the complete service time distribution functions but only on the first moment of the inter-arrival times.

In this short paper, we relax these restrictive statistical assumptions by assuming only the stationarity and ergodicity for the service times and the inter-arrivals times. The model for precedence relationship is also generalized in that the precedence relations between two customers are assumed to be established with a probability that depends on the "distance" between them. We derive a general expression for the stability condition which in turn proves the above conjectures.

Consider the following queueing system. Customers $C_0, C_1, C_2, \cdots$ arrive at dates $t_0 = 0 < t_1 < t_2 < \cdots$, respectively, which define the inter-arrival sequence $\{\tau_n = t_{n+1} - t_n\}_{n=0}^{\infty}$. Customer C_n $(n \geq 0)$ requires service time $\sigma_n > 0$. There are infinitely many servers in the system so that we can assign each incoming customer to a different server. However, there exist certain precedence constraints between the customers which are indicated by the random variables $b_{n,n-j}$ $(n \geq 0,\ 1 \leq j \leq n)$ in the following way: $b_{n,n-j} = 0$ indicates that there is no precedence relation between customers C_n and C_{n-j}; $b_{n,n-j} = 1$ indicates that C_n has to wait for the end of the service of C_{n-j} before it starts service.

Define w_n $(n \geq 0)$ to be the waiting time of customer C_n. In the sequel, the above queueing system is said to be stable iff w_n converges weakly to a finite Random Variable (RV) w_∞ when n goes to ∞.

It is not difficult to check that the evolution of the system is captured by the following set of recursive equations.

Theorem 1

Assume that the queueing system is empty at time 0. Then, for all $n \geq 0$,

$$
\begin{aligned}
w_0 &= 0, && (1.1)\\
w_n &= \max(0, \max_{0 \leq j \leq n-1} b_{n,j}(w_j + \sigma_j - \delta_{n,j})) \\
&= \max(0, \max_{1 \leq j \leq n} b_{n,n-j}(w_{n-j} + \sigma_{n-j} - \delta_{n,n-j})) && (1.2)
\end{aligned}
$$

where

$$\delta_{n,j} = \tau_j + \tau_{j+1} + \cdots + \tau_{n-1}\ , \quad 0 \leq j \leq n-1.$$

2 Preliminary results

The basic idea for analyzing the stability conditions of the queueing system consists in generalizing the schema of Loynes for the response time of a $G/G/1$ queue [2], to the waiting time w_n in the precedence-based queueing system. For this, we will assume without loss of generality that the sequences defined above are subsequences of three bi-infinite sequences of RV's $\{\tau_n\}_{-\infty}^{\infty}$, $\{\sigma_n\}_{-\infty}^{\infty}$, and $\{b_{n,n-j}, j \geq 1\}_{n=-\infty}^{\infty}$ with values in $I\!R^+$, $I\!R^+$ and $\{0,1\}^\infty$ respectively, being defined on a common probability space (Ω, F, P). Without loss of generality, (Ω, F, P) will be assumed to be the canonical space of these sequences. θ will denote the leftshift operator on this canonical space. For more details on this formalism, see [1].

Within this framework, the hypothesis that these constituting sequences are jointly stationary and ergodic translates into Assumption A_1 bellow:

A_1: *P is θ-invariant (stationarity) and θ-ergodic.*

As for the precedence relations, the following additional statistical assumptions will be made:

A_2: The $\{0,1\}^\infty$-valued sequences $\{b_{n,n-j},\ j \geq 1\}$, $n = \cdots,-1,0,1,\cdots$, are mutually independent, and independent of the sequences $\{\tau_n\}_{n=-\infty}^{\infty}$ and $\{\sigma_n\}_{n=-\infty}^{\infty}$; For all $n \in \mathbb{Z}$, the sequence $\{b_{n,n-j}\}_{n=0}^{\infty}$ is made of independent RV's on $\{0,1\}$; The mean values $E[b_{n,n-j}] = p_j$ satisfy the property $\bar{p} = \inf_{j\geq 1} p_j > 0$.

Let $\tau = \tau_0$, $\sigma = \sigma_0$, $\delta_i = \delta_{0,-i}$ $(i \geq 1)$, and $b_i = b_{0,-i}$ $(i \geq 1)$. Consider the schema $\{M_n\}_{n=0}^{\infty}$ defined by

$$M_0 = 0, \tag{2.1}$$
$$M_n = \max(0, \max_{1\leq j\leq n} b_j(M_{n-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)) \tag{2.2}$$

It can easily be proved that w_n and M_n are equivalent in law: $w_n =_{st} M_n$.

Lemma 1

For all $n \geq 0$,

$$w_n = M_n \circ \theta^n. \tag{2.3}$$

Proof

The relation can be proved by induction using the facts that for all $n \geq 0$ and $i \geq 1$, $\tau_n = \tau\circ\theta^n$, $\sigma_n = \sigma\circ\theta^n$, $\delta_{n,n-i} = \delta_i\circ\theta^n$, $b_{n,n-i} = b_i\circ\theta^n$. □

Lemma 2

The sequence $\{M_n\}_{n=0}^{\infty}$ is increasing in n.

Proof

The proof proceeds by induction on n. It is trivial that $M_1 \geq 0 = M_0$. Assume the assertion is true for all $m \leq n$ where $n \geq 0$. Then

$$\begin{aligned} M_{n+1} &= \max(0, \max_{1\leq j\leq n+1} b_j(M_{n+1-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)) \\ &\geq \max(0, \max_{1\leq j\leq n} b_j(M_{n+1-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)) \\ &\geq \max(0, \max_{1\leq j\leq n} b_j(M_{n-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)) \\ &= M_n \end{aligned}$$

so that the assertion holds for all $n \geq 0$. □

More precisely, we can rewrite M_n as

Lemma 3

For all $n \geq 0$,

$$M_n = \max_{1\leq m\leq n} H_m \tag{2.4}$$

where

$$H_m = \max_{1\leq k\leq m} D_m^k \tag{2.5}$$

$$D_m^k = \max(0, \max_{J_m^k} B(J_m^k)\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})) \tag{2.6}$$

$$= \max(0, \max_{J_m^k} B(J_m^k)\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{m}\tau\circ\theta^{-i})) \tag{2.7}$$

and

$$\mathcal{J}_m^k = \{J_m^k = (j_1, \cdots, j_k) \in I\!N^k | 1 \leq j_1, \cdots, j_k \leq m, \sum_{i=1}^{k} j_i = m\}$$

$$B(J_m^k) = \prod_{i=1}^{k} b_{j_i} \circ \theta^{-s_i}$$

$$s_i = \sum_{h=1}^{i-1} j_h \quad , \quad l_i = s_i + j_i$$

with $s_1 = 0$ *by definition.*

Proof

The proof proceeds by induction on n. The assertion trivially holds for $n = 0$ and $n = 1$. Suppose it holds for all $n \leq v$, where $v \geq 1$.

$$\begin{aligned} M_{v+1} &= \max\{0, \max_{1\leq j\leq v+1} b_j(M_{v+1-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)\} \\ &= \max\{0, \ \max[b_{v+1}(\sigma\circ\theta^{-(v+1)} - \delta_{v+1}), \ \max_{1\leq j\leq v} b_j(M_{v+1-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)]\} \\ &= \max\{\ \max[0, b_{v+1}(\sigma\circ\theta^{-(v+1)} - \delta_{v+1})], \ \max[0, \max_{1\leq j\leq v} b_j(M_{v+1-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)]\} \\ &= \max\{D_{v+1}^1, \max[0, K_v]\} \end{aligned} \tag{2.8}$$

where

$$K_v = \max_{1\leq j\leq v} b_j(M_{v+1-j}\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j)$$

Owing to the inductive assumption,

$$\begin{aligned} K_v &= \max_{1\leq j\leq v} b_j(\max_{1\leq m\leq v+1-j,\ 1\leq k\leq m} D_m^k\circ\theta^{-j} + \sigma\circ\theta^{-j} - \delta_j) \\ &= \max_{1\leq j\leq v}\ \max_{1\leq m\leq v+1-j}\ \max_{1\leq k\leq m} b_j D_m^k\circ\theta^{-j} + b_j(\sigma\circ\theta^{-j} - \delta_j) \\ &= \max_{1\leq m\leq v}\ \max_{1\leq j\leq v+1-m}\ \max_{1\leq k\leq m} b_j D_m^k\circ\theta^{-j} + b_j(\sigma\circ\theta^{-j} - \delta_j) \\ &= \max_{1\leq m\leq v}\ \max_{1\leq k\leq m}\ \max_{1\leq j\leq v+1-m} b_j D_m^k\circ\theta^{-j} + b_j(\sigma\circ\theta^{-j} - \delta_j) \\ &= \max_{1\leq k\leq v}\ \max_{k\leq m\leq v}\ \max_{1\leq j\leq v+1-m} b_j D_m^k\circ\theta^{-j} + b_j(\sigma\circ\theta^{-j} - \delta_j) \end{aligned}$$

Adopting the notation

$$R(k,m,j) = b_j D_m^k\circ\theta^{-j} + b_j(\sigma\circ\theta^{-j} - \delta_j)$$

the last equation reads

$$K_v = \max_{1\leq k\leq v}\ \max_{k\leq m\leq v}\ \max_{1\leq j\leq v+1-m} R(k,m,j) \tag{2.9}$$

Using (2.6) we get

$$\begin{aligned} R(k,m,j) &= b_j\{\max[0, \ \max_{\mathcal{J}_m^k} B(J_m^k)\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})]\circ\theta^{-j}\} + b_j(\sigma\circ\theta^{-j} - \delta_j) \\ &= b_j\max[0, \ \max_{\mathcal{J}_m^k}(B(J_m^k)\circ\theta^{-j})\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-j}] + b_j(\sigma\circ\theta^{-j} - \delta_j) \end{aligned}$$

Since $b_j = 0$ or 1, we obtain

$$\begin{aligned}
R(k,m,j) &= \max[0, \ \max_{J_m^k} b_j(B(J_m^k)\circ\theta^{-j})\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-j}] + b_j(\sigma\circ\theta^{-j}-\delta_j) \\
&= \max[b_j(\sigma\circ\theta^{-j}-\delta_j), \\
&\qquad \max_{J_m^k} b_j(B(J_m^k)\circ\theta^{-j})\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-j} + b_j(\sigma\circ\theta^{-j}-\delta_j)]
\end{aligned}$$

The last equation together with the fact that $B(J_m^k)\circ\theta^{-j} = 0$ or 1 immediately entail

$$\begin{aligned}
&\max[0, R(k,m,j)] \\
&= \max[0, \ b_j(\sigma\circ\theta^{-j}-\delta_j), \\
&\qquad \max_{J_m^k} b_j(B(J_m^k)\circ\theta^{-j})\cdot(\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-j} + b_j(B(J_m^k)\circ\theta^{-j})\cdot(\sigma\circ\theta^{-j}-\delta_j)] \\
&= \max[0, \ b_j(\sigma\circ\theta^{-j}-\delta_j), \ S(k,j,m)\ ] \qquad (2.10)
\end{aligned}$$

where

$$S(k,m,j) = \max_{J_m^k}\ [\ b_j(B(J_m^k)\circ\theta^{-j})\cdot((\sigma\circ\theta^{-j}-\delta_j) \ + (\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-j})\ ]$$

Equation (2.6) together with a direct decomposition of sets of the type $J_u^{k+1} = \{(j,j_1,\cdots,j_k)\in \mathbb{N}^{k+1} |\ j+j_1+\cdots j_k = u\}$, $u\geq k+1$, based on the value of the first element j yield the relation

$$\begin{aligned}
&\max_{k+1\leq u\leq v+1} D_u^{k+1} \\
&= \max[0, \max_{k+1\leq u\leq v+1}\ \max_{1\leq j\leq u-k}\ \max_{(j_1,\cdots,j_k)\in J_{u-j}^k} \\
&\qquad b_j\ (B(j_1,\cdots,j_k)\circ\theta^{-j})[(\sigma\circ\theta^{-j}-\delta_j) \ + \ (\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-j}\ )] \\
&= \max[0, \max_{k+1\leq u\leq v+1}\ \max_{k\leq m\leq u-1}\ \max_{(j_1,\cdots,j_k)\in J_m^k} \\
&\qquad b_{u-m}\ (B(j_1,\cdots,j_k)\circ\theta^{-u+m})[(\sigma\circ\theta^{-u+m}-\delta_{u-m}) \ + \ (\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-u+m}\ )] \\
&= \max[0, \max_{k\leq m\leq v}\ \max_{m+1\leq u\leq v+1}\ \max_{(j_1,\cdots,j_k)\in J_m^k} \\
&\qquad b_{u-m}\ (B(j_1,\cdots,j_k)\circ\theta^{-u+m})[(\sigma\circ\theta^{-u+m}-\delta_{u-m}) \ + \ (\sum_{i=1}^{k}\sigma\circ\theta^{-l_i} - \sum_{i=1}^{k}\delta_{j_i}\circ\theta^{-s_i})\circ\theta^{-u+m}\ )] \\
&= \max[0, \max_{k\leq m\leq v,\ 1\leq j\leq v+1-m} S(k,m,j)] \qquad (2.11)
\end{aligned}$$

Using successively (2.9), (2.10) and (2.11), we then get

$$\begin{aligned}
&\max[0, \ K_v] \\
&= \max_{1\leq k\leq v}\ \max_{k\leq m\leq v,\ 1\leq j\leq v+1-m} \max(0, \ R(k,m,j)) \\
&= \max_{1\leq k\leq v}\ \max_{k\leq m\leq v,\ 1\leq j\leq v+1-m} \max[\max(0, \ b_j(\sigma\circ\theta^{-j}-\delta_j))\ , \ \max(0, \ S(k,m,j))] \\
&= \max_{1\leq k\leq v}\ \max_{k\leq m\leq v,\ 1\leq j\leq v+1-m} \max[D_j^1, \ \max(0, \ S(k,m,j))] \\
&= \max[\max_{1\leq j\leq v} D_j^1, \ \max_{1\leq k\leq v}\ \max_{k\leq m\leq v,\ 1\leq j\leq v+1-m} \max(0, S(k,m,j))] \\
&= \max[\max_{1\leq j\leq v} D_j^1, \ \max_{1\leq k\leq v}\ \max_{k+1\leq u\leq v+1} D_u^{k+1}] \qquad (2.12)
\end{aligned}$$

Combining (2.8) and (2.12) finally yields

$$\begin{aligned}
M_{v+1} &= \max\{D^1_{v+1}, \max[0,\ K_v]\} \\
&= \max\{D^1_{v+1},\ \max[\max_{1\le j\le v} D^1_j,\ \max_{1\le k\le v}\ \max_{k+1\le u\le v+1} D^{k+1}_u]\} \\
&= \max\{\max_{1\le m\le v+1} D^1_m,\ \max_{2\le k+1\le v+1}\ \max_{k+1\le m\le v+1} D^{k+1}_m\} \\
&= \max_{1\le m\le v+1}\ \max_{1\le k\le m} D^k_m
\end{aligned}$$

By induction, the lemma holds for all $n \geq 0$. □

Lemma 4

Assume that A_1 and A_2 hold, then the event $\{M_\infty = \infty\}$ is of probability 0 or 1.

Proof

We prove that if $P[M_\infty = \infty] > 0$ then $P[M_\infty = \infty] = 1$.

1) *Under the assumption that $P[M_\infty = \infty] > 0$, there almost surely (a.s.) exists an infinite sequence of non-negative integers $0 \leq k_1 < k_2 < \cdots < k_n < \cdots$ such that for all $n \geq 1$, $M_\infty \circ \theta^{-k_n} = \infty$.*

The pointwise ergodic theorem implies

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} I[M_\infty \circ \theta^{-i} = \infty] = P[M_\infty = \infty] \qquad a.s.$$

which together with the assumption $P[M_\infty = \infty] > 0$ readily imply the existence of the above sequence.

2) *Under the assumption that $P[M_\infty = \infty] > 0$, then, $P[M_\infty = \infty] = 1$.*

First of all, observe that the sequences of RV's $\{b_j\}_{j=1}^{\infty}$ and $\{M_\infty \circ \theta^{-j}\}_{j=1}^{\infty}$ are mutually independent. Indeed, from Lemma 3, we readily get

$$M_\infty = \lim_{n\to\infty} \max_{1\le m\le n}\ \max_{1\le k\le m} \max(0, \max_{J^k_m} B(J^k_m) \cdot (\sum_{i=1}^{k} \sigma \circ \theta^{-l_i} - \sum_{i=1}^{m} \tau \circ \theta^{-i}))$$

so that for all $j \in \mathbb{Z}$,

$$M_\infty \circ \theta^{-j} = \lim_{n\to\infty} \max_{1\le m\le n}\ \max_{1\le k\le m} \max(0, \max_{J^k_m}(B(J^k_m) \circ \theta^{-j}) \cdot (\sum_{i=1}^{k} \sigma \circ \theta^{-l_i - j} - \sum_{i=1}^{m} \tau \circ \theta^{-i-j}))$$

In view of the definition of J^k_m and $B(J^k_m)$ in Lemma 3, it immediately follows from this and assumption A_2 that the RV's $\{M_\infty \circ \theta^{-j}\}_{j=1}^{\infty}$ and the RV's $\{b_j\}_{j=1}^{\infty}$ are mutually independent.

In addition to that, we get from step 1 that almost surely the sequence $\{M_\infty \circ \theta^{-j}\}_{j=1}^{\infty}$ is equal to ∞ for infinitely many values of the index $j \geq 1$. Since the RV's $\{b_j\}_{j=1}^{\infty}$ are mutually independent and receive value 1 with a probability greater than or equal to $\bar{p} > 0$, it immediately follows that there almost surely exists a finite integer $j \geq 1$ such that simultaneously $M_\infty \circ \theta^{-j} = \infty$ and $b_j = 1$. This together with the equation

$$M_\infty \quad = \quad \max(0, \sup_{j\ge 1} b_j(M_\infty \circ \theta^{-j} + \sigma \circ \theta^{-j} - \delta_j))$$

entail that $M_\infty = \infty$ $a.s.$. The proof of the lemma is thus completed. □

3 Stability Condition

We are now in a position to prove the main result of the present paper concerning the stability conditions of the precedence-based queueing systems which is stated in the following theorem.

Theorem 2

Assume A_1 and A_2 hold, and let

$$\alpha = E[\limsup_{n\to\infty} \max_{1\le k\le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^k \sigma\circ\theta^{-l_i}}{n}] \tag{3.1}$$

where

$$\mathcal{J}_n^k = \{J_n^k = (j_1,\cdots,j_k) \in I\!N^k | 1 \le j_1,\cdots,j_k \le n, \sum_{i=1}^k j_i = n\}$$

$$B(J_n^k) = \prod_{i=1}^k b_{j_i} \circ \theta^{-s_i}$$

$$s_i = \sum_{h=1}^{i-1} j_h \quad , \quad l_i = s_i + j_i$$

with $s_1 = 0$ by definition.
i) *If $\alpha < E[\tau]$, then $M_\infty < \infty$ a.s., so that w_n converges weakly to a finite RV when n goes to ∞.*
ii) *If $\alpha > E[\tau]$, then $M_\infty = \infty$ a.s. and w_n diverges.*

Proof

First of all, observe that the RV

$$\limsup_{n\to\infty} \max_{1\le k\le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^k \sigma\circ\theta^{-l_i}}{n}$$

is integrable under assumption A_1. Indeed

$$\max_{1\le k\le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^k \sigma\circ\theta^{-l_i}}{n} \le \frac{\sum_{i=1}^n \sigma\circ\theta^{-i}}{n}$$

Using now the ergodicity assumption on θ, we get

$$\limsup_{n\to\infty} \max_{1\le k\le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^k \sigma\circ\theta^{-l_i}}{n} \le \lim_{n\to\infty} \frac{\sum_{i=1}^n \sigma\circ\theta^{-i}}{n} = E[\sigma] < \infty \tag{3.2}$$

The left hand side of (3.2) is hence integrable since it is bounded by a constant. Therefore α is well defined by (3.1) and finite.

The proof of the theorem consists of two steps.

1) Under assumptions A_1 and A_2, Lemma 4 indicates that the event $\{M_\infty = \infty\}$ is of probability 0 or 1. Assume it is of probability 1. It follows from (2.4) that the fact that $M_n \uparrow \infty$ is equivalent to

$$\limsup_{n\to\infty} H_n = \infty.$$

which in turn implies

$$\limsup_{n\to\infty} \max_{1\le k\le n} \max_{J_n^k} \{B(J_n^k)(\sum_{i=1}^k \sigma\circ\theta^{-l_i} - \sum_{i=1}^n \tau\circ\theta^{-i})\} = \infty. \tag{3.3}$$

For all $n = 1, 2, .., $ let $G(n)$ be the indicator function :

$$G(n) = \begin{cases} 1, & \text{if } \exists k_0,\ 1 \le k_0 \le n,\ J_n^{k_0} = (u_1, \cdots, u_{k_0})\ :\ B(J_n^{k_0}) = 1 \\ 0, & \text{otherwise} \end{cases} \tag{3.4}$$

and $L(n)$ be defined by

$$L(n) = \max_{1 \le k \le n} \max_{J_n^k} \{B(J_n^k) \sum_{i=1}^{k} \sigma \circ \theta^{-l_i}\} - \sum_{i=1}^{n} \tau \circ \theta^{-i} \tag{3.5}$$

Since for every J_n^k, $B(J_n^k) = 0$ or 1, we obtain the relation

$$\max_{1 \le k \le n} \max_{J_n^k} \{B(J_n^k)(\sum_{i=1}^{k} \sigma \circ \theta^{-l_i} - \sum_{i=1}^{n} \tau \circ \theta^{-i})\}^{+} = G(n)L(n)^{+} \tag{3.6}$$

Equations (3.3) and (3.6) readily yield

$$\limsup_{n \to \infty} \max_{1 \le k \le n} \max_{J_n^k} \{B(J_n^k)(\sum_{i=1}^{k} \sigma \circ \theta^{-l_i} - \sum_{i=1}^{n} \tau \circ \theta^{-i})\} = \limsup_{n \to \infty} G(n)L(n) = \infty \tag{3.7}$$

Observe that $G(n)$ is an indicator function that only takes values 1 and 0, and that for all $n \ge 1$, $L(n) > 0$ implies $G(n) = 1$. Hence (3.7) implies

$$\limsup_{n \to \infty} L(n) = \infty$$

so that

$$\limsup_{n \to \infty} (\max_{1 \le k \le n} \max_{J_n^k} \{B(J_n^k) \frac{\sum_{i=1}^{k} \sigma \circ \theta^{-l_i}}{n}\} - \frac{\sum_{i=1}^{n} \tau \circ \theta^{-i}}{n}) \ge 0 \tag{3.8}$$

Owing to the ergodicity assumption on θ ,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \tau \circ \theta^{-i} = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \tau \circ \theta^{-i} = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \tau \circ \theta^{-i} = E[\tau] \quad a.s.$$

so that (3.8) entails

$$\begin{aligned}
&\limsup_{n \to \infty} (\max_{1 \le k \le n} \max_{J_n^k} \{B(J_n^k) \frac{\sum_{i=1}^{k} \sigma \circ \theta^{-l_i}}{n}\} - \frac{\sum_{i=1}^{n} \tau \circ \theta^{-i}}{n}) \\
&= \limsup_{n \to \infty} \max_{1 \le k \le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^{k} \sigma \circ \theta^{-l_i}}{n} - \lim_{n \to \infty} \frac{\sum_{i=1}^{n} \tau \circ \theta^{-i}}{n} \\
&= \limsup_{n \to \infty} \max_{1 \le k \le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^{k} \sigma \circ \theta^{-l_i}}{n} - E[\tau] \\
&\ge 0
\end{aligned}$$

We hence get

$$\limsup_{n \to \infty} \max_{1 \le k \le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^{k} \sigma \circ \theta^{-l_i}}{n} \ge E[\tau]$$

which entails that

$$\alpha = E[\limsup_{n \to \infty} \max_{1 \le k \le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^{k} \sigma \circ \theta^{-l_i}}{n}] \ge E[\tau] \tag{3.9}$$

In other words, $M_n \uparrow \infty$ when n tends to ∞ entails that $\alpha \ge E[\tau]$. Taking the contrapositive of (3.9) and using Lemma 4, we immediately get that the assumption $\alpha < E[\tau]$ entails $M_\infty < \infty$ a.s. . The first part of the theorem is thus proved.

2) Assume now that $\alpha > E[\tau]$. Then there exists a subset $\Omega_0 \subset \Omega$ with $P[\Omega_0] > 0$, such that for all $\omega \in \Omega_0$:

$$\limsup_{n\to\infty} \max_{1\le k\le n} \max_{J_n^k} B(J_n^k) \frac{\sum_{i=1}^k \sigma \circ \theta^{-l_i}}{n} \ge \alpha > E[\tau] > 0 \tag{3.10}$$

Therefore for all $\omega \in \Omega_0$,

$$\limsup_{n\to\infty} (\max_{1\le k\le n} \max_{J_n^k} \{B(J_n^k) \frac{\sum_{i=1}^k \sigma \circ \theta^{-l_i}}{n}\} - \frac{\sum_{i=1}^n \tau \circ \theta^{-i}}{n}) > 0$$

so that

$$\limsup_{n\to\infty} L(n) = \limsup_{n\to\infty} (\max_{1\le k\le n} \max_{J_n^k} \{B(J_n^k) \sum_{i=1}^k \sigma \circ \theta^{-l_i}\} - \sum_{i=1}^n \tau \circ \theta^{-i}) = \infty \tag{3.11}$$

Since $L(n) > 0$ implies $G(n) = 1$, (3.11) implies the existence of a sequence of integers $1 \le n_1 < n_2 < \ldots$ such that $L(n_k) \uparrow \infty$ when k goes to ∞ and $G(n_k) = 1$ for all $k = 1, 2, \cdots$. Hence

$$\limsup_{n\to\infty} H_n = \limsup_{n\to\infty} G(n)L(n) = \infty$$

This implies that with a positive probability

$$\lim_{n\to\infty} M_\infty = \infty \tag{3.12}$$

so that Lemma 4 entails that (3.12) holds almost surely. The proof of the theorem is hence completed. □

Remark: Theorem 2 indicates explicitly that the critical value of the the stability conditions of the precedence-based queueing systems depend on the first moment of the inter-arrival times alone, not on the other moments of the inter-arrivals; On the contrary, the expression of α defined by (3.1) is effectively a function of the whole distribution of the service times. The conjecture of J.N. Tsitsiklis, C.H. Papadimitriou and P. Humblet is therefore proved.

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