A transient analysis of the two-node series Jackson network

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ANALYSE TRANSITOIRE D'UN RESEAU DE JACKSON DE DEUX FILES D'ATTENTE EN TANDEM

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RESUME

Cet article donne une représentation de la distribution transitoire de la taille des files d'attente dans un réseau de deux files en tandem au moyen de fonctions de Bessel Laticielles. On obtient par la même approche les distributions jointes des périodes d'activité des deux files d'attente et de l'état du réseau à la fin de ces périodes. Ces derniers résultats sont établis au moyen d'un théorème général sur les processus de Markov qui relie la distribution jointe des temps et lieu de sortie d'un domaine au comportenment du processus avant l'atteinte de sa frontière. Ces résultats sont ensuite étendus au cas d'un réseau à N noeuds.

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ABSTRACT

In this paper, we give the exact solution for the transient queue length distribution of the two-node series Jackson network, in terms of lattice Bessel functions. Exact solutions are also derived for the transient busy period distribution of both queues prior to having a given node become idle. In terms of this solution, we can in turn solve for the associated joint distribution of the queues and the time that this idleness occurs. We obtain this by deriving a general solution to the problem of solving for the joint distribution of where a Markov process exits a region, and its time of departure, in terms of the behavior of the process prior to leaving the region. Finally, we generalize all of these results to the N node series Jackson network.

Keywords: Transient Analysis, Bessel Functions, Markov Processes, Busy Period, Exit Times

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1. Introduction

Let $\mathbf{Q}(t) = (Q_1(t), Q_2(t))$ denote the vector queue length process for a two-node series Jackson network. The first node has Poisson arrivals with rate μ_0, and its service time is exponentially distributed with rate μ_1. Upon completion of that service time, the customer then transfers to the second node whose service time is exponentially distributed with rate μ_2. In this paper, we will concern ourselves with the solution for the following transient distributions of $\mathbf{Q}(t)$:

$$(a, a)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n},\ T > t \mid \mathbf{Q}(0) = \mathbf{m}\} \tag{1.1.a}$$

$$(a, b)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n},\ T_1 > t \mid \mathbf{Q}(0) = \mathbf{m}\} \tag{1.1.b}$$

$$(b, a)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n},\ T_2 > t \mid \mathbf{Q}(0) = \mathbf{m}\} \tag{1.1.c}$$

$$(b, b)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n} \mid \mathbf{Q}(0) = \mathbf{m}\} \tag{1.1.d}$$

where $T_1 = \inf\{t \mid Q_1(t) = 0\}$, $T_2 = \inf\{t \mid Q_2(t) = 0\}$ and $T = \min(T_1, T_2)$. The solution to these four quantities can be given in terms of lattice Bessel functions of rank 2. These functions were introduced in [4] for an arbitrary rank N. For the case $N=2$, the generating function relation for these functions reduces to

$$e^{\frac{y}{3}\left(x_1 + \frac{x_2}{x_1} + \frac{1}{x_2}\right)} = \sum_{\mathbf{n} \in \mathbb{Z}^2} \mathbf{x}^{\mathbf{n}}\, I(\mathbf{n}, y) \tag{1.2}$$

with $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} x_2^{n_2}$. These functions have an associated symmetry group G_2 where

$$G_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \right\}. \tag{1.3}$$

For $g \in G_2$, we define $g(\mathbf{m})$ to be matrix multiplication between g and $\mathbf{m}$ where $\mathbf{m}$ is viewed as a column vector. Generalizing the property of modified Bessel functions that $I_n(y) = I_{-n}(y)$, we have $I(g(\mathbf{m}), y) = I(\mathbf{m}, y)$ for all $g \in G_2$ and $\mathbf{m} \in \mathbb{Z}^2$. A special case of Theorem 1.2 in [4] gives a solution for $(a, a)_t[\mathbf{m}, \mathbf{n}]$, namely

$$(a, a)_t[\mathbf{m}, \mathbf{n}] = e^{-3\alpha t} \, \beta^{\mathbf{n}-\mathbf{m}} \sum_{g \in G_2} \text{ define } I(\mathbf{n} - g(\mathbf{m}), 3\gamma t) \tag{1.4}$$

where $(-1)^g$ is the sign of g, viewed as a permutation, $\alpha = \dfrac{\mu_0 + \mu_1 + \mu_2}{3}$, $\gamma = (\mu_0 \mu_1 \mu_2)^{1/3}$, and $\beta = \left[\dfrac{\mu_0}{\gamma}, \dfrac{\mu_0 \mu_1}{\gamma^2}\right]$. We will now solve for the remaining three quantities by showing that they can be solved in terms of $(a, a)_t[\mathbf{m}, \mathbf{n}]$. By (1.4), we will have then solved for $(a, b)_t[\mathbf{m}, \mathbf{n}]$, $(b, a)_t[\mathbf{m}, \mathbf{n}]$, and $(b, b)_t[\mathbf{m}, \mathbf{n}]$ in terms of lattice Bessel functions of rank 2.

Theorem 1.1. *For all* $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^2$ *and* $t > 0$, *we have*

$$(a, b)_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} (a, a)_t[\mathbf{m}+j\mathbf{e}_2, \mathbf{n}+\mathbf{e}_2] - (a, a)_t[\mathbf{m}+j\mathbf{e}_2, \mathbf{n}]$$

$$(b, a)_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} \left(\frac{\mu_1}{\mu_0}\right)^{j-1} \left((a, a)_t[\mathbf{m}+\mathbf{e}_1, \mathbf{n}+j\mathbf{e}_1] - \frac{\mu_1}{\mu_0}(a, a)_t[\mathbf{m}, \mathbf{n}+j\mathbf{e}_1]\right)$$

$$\begin{aligned}(b, b)_t[\mathbf{m}, \mathbf{n}] = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(\frac{\mu_1}{\mu_0}\right)^{k-1} \Big((a, a)_t[\mathbf{m}+\mathbf{e}_1+j\mathbf{e}_2, \mathbf{n}+k\mathbf{e}_1+\mathbf{e}_2] - (a, a)_t[\mathbf{m}+\mathbf{e}_1+j\mathbf{e}_2, \mathbf{n}+k\mathbf{e}_1] \\ - \frac{\mu_1}{\mu_0}(a, a)_t[\mathbf{m}+j\mathbf{e}_2, \mathbf{n}+k\mathbf{e}_1+\mathbf{e}_2] + \frac{\mu_1}{\mu_0}(a, a)_t[\mathbf{m}+j\mathbf{e}_2, \mathbf{n}+k\mathbf{e}_1]\Big).\end{aligned}$$

In Blanc [2], Riemann-Hilbert methods are employed to determine the Laplace transform of $(b, b)_t[0, 0]$, which in turn is used to derive its asymptotic expansion and relaxation time parameter. We will treat the issue of asymptotic expansions for our distributions in a subsequent paper. In Section 2, we will prove our main theorem. In Section 3, we will solve for the transient distributions related to $(a, b)_t[\mathbf{m}, \mathbf{n}]$ and $(b, a)_t[\mathbf{m}, \mathbf{n}]$. We can solve for the joint density of $\mathbf{Q}(T_1)$ and T_1 in terms of the $(a, b)_t[\mathbf{m}, \mathbf{n}]$ and similarly, the joint density of $\mathbf{Q}(T_2)$ and T_2 in terms of the $(b, a)_t[\mathbf{m}, \mathbf{n}]$. We will do this by solving a related problem for general Markov processes. Using martingales, we can solve for the joint density of a Markov process evaluated at an exit time for leaving some region and the random time of departure, in terms of the stochastic behavior of the process prior to leaving the region.

2. Proving the Main Theorem

Before we begin the proof, the following conventions will be used. Let $\mathbb{Z}_+^2$ denote the quadrant of non-negative integer ordered pairs. Typical elements will be denoted as $\mathbf{m} = (m_1, m_2)$ or $\mathbf{n} = (n_1, n_2)$. The Banach space $l_1(\mathbb{Z}_+^2)$ is the set of absolutely summable real valued functions on $\mathbb{Z}_+^2$. We will also view it as $l_1(\mathbb{Z}_+) \otimes l_1(\mathbb{Z}_+)$ which is the tensor product of absolutely summable real valued functions on $\mathbb{Z}_+$ with itself. Let $(\mathbf{a}, \mathbf{b})_t$ denote the operator that maps $l_1(\mathbb{Z}_+^2)$ into itself where $(a, b)_t[\mathbf{m}, \mathbf{n}]$ is the $\mathbf{e}_{n_1} \otimes \mathbf{e}_{n_2}$ entry for the action of $(\mathbf{a}, \mathbf{b})_t$ on $\mathbf{e}_{m_1} \otimes \mathbf{e}_{m_2}$. Since $(a, b)_t[\mathbf{m}, \mathbf{n}]$ is a transition probability for a sub-Markov process, $(\mathbf{a}, \mathbf{b})_t$ is an operator semigroup. We will let $(\mathbf{a}, \mathbf{b})$ denote its infinitesimal generator. Using the tensor formalism of [3], we can decompose the infinitesimal generators of our four processes in terms of right and left shift operators as follows:

$$\mathbf{R}_1\mathbf{R}_2(\mathbf{a}, \mathbf{a})\mathbf{L}_2\mathbf{L}_1 = \mu_0\mathbf{R}_1 + \mu_1\mathbf{L}_1\mathbf{R}_2 + \mu_2\mathbf{L}_2 - (\mu_0 + \mu_1 + \mu_2)\mathbf{I} \tag{2.1.a}$$

$$\mathbf{R}_1(\mathbf{a}, \mathbf{b})\mathbf{L}_1 = \mu_0\mathbf{R}_1 + \mu_1\mathbf{L}_1\mathbf{R}_2 + \mu_2\mathbf{L}_2 - \mu_0\mathbf{I} - \mu_1\mathbf{I} - \mu_2\mathbf{L}_2\mathbf{R}_2 \tag{2.1.b}$$

$$\mathbf{R}_2(\mathbf{b}, \mathbf{a})\mathbf{L}_2 = \mu_0\mathbf{R}_1 + \mu_1\mathbf{L}_1\mathbf{R}_2 + \mu_2\mathbf{L}_2 - \mu_0\mathbf{I} - \mu_1\mathbf{L}_1\mathbf{R}_1 - \mu_2\mathbf{I} \tag{2.1.c}$$

$$(\mathbf{b}, \mathbf{b}) = \mu_0\mathbf{R}_1 + \mu_1\mathbf{L}_1\mathbf{R}_2 + \mu_2\mathbf{L}_2 - \mu_0\mathbf{I} - \mu_1\mathbf{L}_1\mathbf{R}_1 - \mu_2\mathbf{L}_2\mathbf{R}_2\,. \tag{2.1.d}$$

The last expression for $(\mathbf{b}, \mathbf{b})$ can be recognized as a special case for the operator decomposition of the generator for a Jackson network as introduced in [3]. We will motivate the decomposition for $\mathbf{R}_1(\mathbf{a}, \mathbf{b})\mathbf{L}_1$ as given by (2.1.b), and remark that the formulas for $\mathbf{R}_1\mathbf{R}_2(\mathbf{a}, \mathbf{a})\mathbf{L}_2\mathbf{L}_1$ and $\mathbf{R}_2(\mathbf{b}, \mathbf{a})\mathbf{L}_2$ are achieved in a similar fashion.

Let $\mathbf{Q}^{\#}(t) = (Q_1^{\#}(t), Q_2^{\#}(t))$ be the queue length process for a two node series Jackson network with the following modifications. At $t=0$, an additional tagged customer is added to the first queue. The discipline is such that all the other customers have a preemptive priority over this special customer. Moreover, we let the event of this tagged customer completing service be an absorbing state. Therefore, the process $\mathbf{Q}^{\#}(t)$ "stops" after this event occurs. If $\mathbf{A}^{\#}$ equals the infinitesimal generator of $\mathbf{Q}^{\#}(t)$, we can write it as

$$\mathbf{A}^{\#} = \mu_0\mathbf{R}_1 + \mu_1\mathbf{L}_1\mathbf{R}_2 + \mu_2\mathbf{L}_2 - (\mu_0 + \mu_1)\mathbf{I} - \mu_2\mathbf{L}_2\mathbf{R}_2\,. \tag{2.2}$$

This follows since $\mathbf{Q}^{\#}(t)$ has the same transitions as $\mathbf{Q}(t)$ with one exception. Only when

$Q_1^{\#}(t) = 0$ can an exponential service with rate μ_1 occur for the tagged customer. Consequently, we replace $\mu_1 \mathbf{L}_1 \mathbf{R}_1$ in (b, b) with $\mu_1 \mathbf{I}$.

Let $\mathbf{A}_t^{\#} = \{A_t^{\#}[\mathbf{m}, \mathbf{n}] \mid \mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^2\}$ denote the semigroup of transition probabilities of $\mathbf{Q}^{\#}(t)$. Observing that the transitions of $\mathbf{Q}^{\#}(t)$ are identical to those of $\mathbf{Q}(t) + \mathbf{e}_1$ prior to the stopping time T_1, we have

$$A_t^{\#}[\mathbf{m}, \mathbf{n}] = (a, b)_t[\mathbf{m}+\mathbf{e}_1, \mathbf{n}+\mathbf{e}_1].$$

In terms of operators, this becomes

$$\mathbf{A}_t^{\#} = \mathbf{R}_1(\mathbf{a}, \mathbf{b})_t \mathbf{L}_1 .$$

Taking derivatives on both sides, and setting $t=0$ gives us $\mathbf{A}^{\#} = \mathbf{R}_1(\mathbf{a}, \mathbf{b})\mathbf{L}_1$, and so (2.2) gives us (2.1.b). Let σ be an arbitrary scalar. Since $\mathbf{RL} = \mathbf{I}$, we have the following identities:

$$(\mathbf{I} - \sigma\mathbf{L})\mathbf{R} = [\mathbf{R} + \sigma(\mathbf{I} - \mathbf{LR})](\mathbf{I} - \sigma\mathbf{L}) \tag{2.3.a}$$

$$\mathbf{L}(\mathbf{I} - \sigma\mathbf{R}) = (\mathbf{I} - \sigma\mathbf{R})[\mathbf{L} + \sigma(\mathbf{I} - \mathbf{LR})]. \tag{2.3.b}$$

To proceed further, we must redefine the topology for our operators. Let $\boldsymbol{\rho} = (\rho_1, \rho_2)$ where ρ_1 and ρ_2 are positive real numbers. We define $l_1(\mathbb{Z}_+^2, \boldsymbol{\rho})$ to be the Banach space of real valued functions $\mathbf{f}$ on $\mathbb{Z}_+^2$ such that

$$|\mathbf{f}|_{\rho} = \sum_{\mathbf{n} \in \mathbb{Z}_+^2} |f(\mathbf{n})| \cdot \boldsymbol{\rho}^{\mathbf{n}} < \infty .$$

Letting $|\ \ |_{\rho}$ also denote the induced norm for operators acting on $l_1(\mathbb{Z}_+^2, \boldsymbol{\rho})$, we have $|\mathbf{R}_i|_{\rho} = \rho_i$ and $|\mathbf{L}_i|_{\rho} = \dfrac{1}{\rho_i}$ for $i = 1, 2$. This means that all operators which are in the algebraic closure of $\mathbf{R}_i$ and $\mathbf{L}_i$ are still bounded operators with respect to $|\ \ |_{\rho}$. As a consequence, the set of bounded operators on $l_1(\mathbb{Z}_+^2, \boldsymbol{\rho})$ includes the various generators that we are considering and their semigroups. Moreover, $|\mathbf{e}_{\mathbf{n}}|_{\rho} = \boldsymbol{\rho}^{\mathbf{n}}$ for all $n \in \mathbb{Z}_+^2$, so all vectors of interest to us still belong to $l_1(\mathbb{Z}_+^2, \boldsymbol{\rho})$.

The advantage of using this weighted Banach space becomes clear if we set $\rho_1 < \dfrac{\mu_0}{\mu_1}$ and $\rho_2 < 1$. Now $\left(\mathbf{I} - \dfrac{\mu_1}{\mu_0}\mathbf{L}_1\right)^{-1}$ and $(\mathbf{I} - \mathbf{R}_2)^{-1}$ are bounded operators for $l_1(\mathbb{Z}_+^2, \boldsymbol{\rho})$. Now using (2.3.a) and

(2.3.b), we get the resulting set of operator identities:

$$\left(\mathbf{I}-\frac{\mu_1}{\mu_0}\mathbf{L}_1\right)\mu_0\mathbf{R}_1\left(\mathbf{I}-\frac{\mu_1}{\mu_0}\mathbf{L}_1\right)^{-1} = \mu_0\mathbf{R}_1+\mu_1(\mathbf{I}-\mathbf{L}_1\mathbf{R}_1) \tag{2.4.a}$$

$$(\mathbf{I}-\mathbf{R}_2)^{-1}\mathbf{L}_2(\mathbf{I}-\mathbf{R}_2) = \mathbf{L}_2+\mathbf{I}-\mathbf{L}_2\mathbf{R}_2 \tag{2.4.b}$$

Using our $|\ |_\rho$ norm, as well as (2.4.a) and (2.4.b), we get the following set of "similarity" transformations from $(\mathbf{a},\mathbf{a})$ to $(\mathbf{a},\mathbf{b})$, $(\mathbf{b},\mathbf{a})$, and $(\mathbf{b},\mathbf{b})$.

$$(\mathbf{a},\mathbf{b}) = (\mathbf{I}-\mathbf{R}_2)^{-1}\mathbf{R}_2(\mathbf{a},\mathbf{a})\mathbf{L}_2(\mathbf{I}-\mathbf{R}_2) \tag{2.5.a}$$

$$(\mathbf{b},\mathbf{a}) = \left(\mathbf{I}-\frac{\mu_1}{\mu_0}\mathbf{L}_1\right)\mathbf{R}_1(\mathbf{a},\mathbf{a})\mathbf{L}_1\left(\mathbf{I}-\frac{\mu_1}{\mu_0}\mathbf{L}_1\right)^{-1} \tag{2.5.b}$$

$$(\mathbf{b},\mathbf{b}) = \left(\mathbf{I}-\frac{\mu_1}{\mu_0}\mathbf{L}_1\right)(\mathbf{I}-\mathbf{R}_2)^{-1}\mathbf{R}_1\mathbf{R}_2(\mathbf{a},\mathbf{a})\,\mathbf{L}_2\mathbf{L}_1(\mathbf{I}-\mathbf{R}_2)\left(\mathbf{I}-\frac{\mu_1}{\mu_0}\mathbf{L}_1\right)^{-1} \tag{2.5.c}$$

We will prove (2.5.a). Verifying the other two identities can be done in a similar manner. By (2.1.a), (2.1.b) and (2.4.b), we can show that

$$\mathbf{R}_1(\mathbf{a},\mathbf{b})\mathbf{L}_1 = (\mathbf{I}-\mathbf{R}_2)^{-1}\mathbf{R}_1\mathbf{R}_2(\mathbf{a},\mathbf{a})\mathbf{L}_1\mathbf{L}_2(\mathbf{I}-\mathbf{R}_2)\,.$$

Multiplying both sides on the right by $\mathbf{R}_1$ and on the left by $\mathbf{L}_1$ give us

$$\mathbf{L}_1\mathbf{R}_1(\mathbf{a},\mathbf{b})\mathbf{L}_1\mathbf{R}_1 = (\mathbf{I}-\mathbf{R}_2)^{-1}\mathbf{R}_2\cdot\mathbf{L}_1\mathbf{R}_1(\mathbf{a},\mathbf{a})\mathbf{L}_1\mathbf{R}_1\cdot\mathbf{L}_2(\mathbf{I}-\mathbf{R}_2)\,. \tag{2.6}$$

To derive (2.5.a) from (2.6), it only remains to establish the following two identities:

$$\mathbf{L}_1\mathbf{R}_1(\mathbf{a},\mathbf{a}) = (\mathbf{a},\mathbf{a})\mathbf{L}_1\mathbf{R}_1 = (\mathbf{a},\mathbf{a}) \tag{2.7.a}$$

$$\mathbf{L}_1\mathbf{R}_1(\mathbf{a},\mathbf{b}) = (\mathbf{a},\mathbf{b})\mathbf{L}_1\mathbf{R}_1 = (\mathbf{a},\mathbf{b}) \tag{2.7.b}$$

They follow from showing that (2.7.a) and (2.7.b) also hold if $(\mathbf{a},\mathbf{a})$ and $(\mathbf{a},\mathbf{b})$ are replaced respectively by $(\mathbf{a},\mathbf{a})_t$ and $(\mathbf{a},\mathbf{b})_t$. The resulting identities would then be a simple consequence of the fact that $(\mathbf{a},\mathbf{a})_t[\mathbf{m},\mathbf{n}] = (\mathbf{a},\mathbf{b})_t[\mathbf{m},\mathbf{n}] = 0$ whenever m_1 or n_1 equals zero. Similar arguments of this type are made by noting that $(\mathbf{a},\mathbf{a})$ and $(\mathbf{b},\mathbf{a})$ have analogous commutivity relations with $\mathbf{L}_2\mathbf{R}_2$.

After verifying (2.5.a), (2.5.b), and (2.5.c), we note that all of these identities still hold when the generators are replaced by their semigroups. Consider for example, $(\mathbf{a},\mathbf{b})_t$. Using (2.5.a), we have

$$\frac{d}{dt}(\mathbf{I}-\mathbf{R}_2)(\mathbf{a},\mathbf{b})_t = (\mathbf{I}-\mathbf{R}_2)(\mathbf{a},\mathbf{b})\cdot(\mathbf{a},\mathbf{b})_t = \mathbf{R}_2(\mathbf{a},\mathbf{a})\mathbf{L}_2\cdot(\mathbf{I}-\mathbf{R}_2)(\mathbf{a},\mathbf{b})_t. \tag{2.8}$$

By the uniqueness of this Cauchy problem, we get

$$(\mathbf{I}-\mathbf{R}_2)(\mathbf{a},\mathbf{b})_t = \exp(t\,\mathbf{R}_2(\mathbf{a},\mathbf{a})\mathbf{L}_2)(\mathbf{I}-\mathbf{R}_2). \tag{2.9}$$

Using $\mathbf{A}_t^{\#}$, we see that $\exp(t\,\mathbf{R}_2(\mathbf{a},\mathbf{a})\mathbf{L}_2) = \mathbf{R}_2(\mathbf{a},\mathbf{a})_t\mathbf{L}_2$. This fact coupled with the application of $(\mathbf{I}-\mathbf{R}_2)^{-1}$ to both sides of (2.9) gives us

$$(\mathbf{a},\mathbf{b})_t = (\mathbf{I}-\mathbf{R}_2)^{-1}\,\mathbf{R}_2(\mathbf{a},\mathbf{a})_t\mathbf{L}_2\,(\mathbf{I}-\mathbf{R}_2).$$

The semigroup versions of identities (2.5.b) and (2.5.c) can be proved in a similar manner. Finally, writing out in detail such matrix expressions gives us the theorem.

3. Solving for Hitting Time Distributions in Terms of Exit Time Distributions

With $(a,a)_t[\mathbf{m},\mathbf{n}]$, $(a,b)_t[\mathbf{m},\mathbf{n}]$, and $(b,a)_t[\mathbf{m},\mathbf{n}]$, we now associate the following set of hitting time distributions:

$$(a^{\dagger},a^{\dagger})_t[\mathbf{m},\mathbf{n}] = P_{\mathbf{m}}(Q(T)=\mathbf{n},\ T=t) \tag{3.1.a}$$

$$(a^{\dagger},b)_t[\mathbf{m},\mathbf{n}] = P_{\mathbf{m}}(Q(T_1)=\mathbf{n},\ T_1=t) \tag{3.1.b}$$

$$(b,a^{\dagger})_t[\mathbf{m},\mathbf{n}] = P_{\mathbf{m}}(Q(T_2)=\mathbf{n},\ T_2=t). \tag{3.1.c}$$

The right-hand side of (3.1.a) is shorthand for the density or derivative of the distribution $P_{\mathbf{m}}(Q(T)=\mathbf{n},\ T\le t)$. The others are defined similarly. A special case of Theorem 1 in [1] gives us the solution for (3.1.a),

$$(a^{\dagger},a^{\dagger})_t[\mathbf{m},\mathbf{n}] = \begin{cases} \mu_1\cdot(a,a)_t[\mathbf{m},\mathbf{n}+\mathbf{e}_1-\mathbf{e}_2] & \mathbf{n}=(0,n_2), \\ \mu_2\cdot(a,a)_t[\mathbf{m},\mathbf{n}+\mathbf{e}_2] & \mathbf{n}=(n_1,0), \\ 0 & \text{otherwise}. \end{cases} \tag{3.2}$$

Results of this type suggest a deeper relation that we will state and prove here.

Lemma 3.1. *Let X_t be a Feller-Dynkin process (see Williams [5]), with Polish state space E, generator A, and domain $D(A)$. Moreover, let $f\in D(A)$ be a continuous function on E such that both* *f and Af are bounded. If U is a Borel-measurable subset of E and disjoint from the support of f, then*

$$E_x(f(X_T);\, T \le t) = \int_0^t E_x(Af(X_s);\, T > s)ds$$

for all $x \in U$, where $T = \inf\{t \mid X_t \notin U\}$.

Proof: By Williams [5], page 129, we know that the process $M_t = f(X_t) - \int_0^t Af(X_s)ds$ is a martingale. Now T is a stopping time for our Markov process X_t so by the optional sampling theorem, $M_{T\wedge t}$ is a martingale as well. This means that $E_x(M_{T\wedge t}) = f(x) = 0$, and so

$$E_x(f(X_{T\wedge t})) = E_x\left(\int_0^{T\wedge t} Af(X_s)ds\right) = \int_0^t E_x(Af(X_s);\, T > s)ds$$

by Fubini's theorem. Finally, we note that $E_x(f(X_{T\wedge t})) = E_x(f(X_T);\, T \le t)$, since $f(X_t) = 0$ for all $t < T$ by hypothesis. ■

We can alternatively write the result of Lemma 3.1 as

$$E_x(f(X_T);\, T = t) = E_x(Af(X_t);\, T > t)\,.$$

Note that this result is *only* useful for Markov processes with pure jumps or discontinuous sample paths. The relation is vacuous for diffusions. Equation (3.2) and Theorem 1 of [1] can be viewed as simple consequences of Lemma 3.1 and we can now solve for our remaining hitting time probabilities:

$$(a^\dagger, b)_t[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_1 \cdot (a, b)_t[\mathbf{m}, \mathbf{n}+\mathbf{e}_1-\mathbf{e}_2] & \mathbf{n} = (0, n_2)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$$(b, a^\dagger)_t[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_2 \cdot (b, a)_t[\mathbf{m}, \mathbf{n}+\mathbf{e}_2] & \mathbf{n} = (n_1, 0)\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

4. Generalizing to the N-node Series Jackson Network

We now give a sketch of how these results can be extended to N dimensions, using lattice Bessel functions of rank N. Let $\mathbf{Q}(t) = (Q_1(t), \ldots, Q_N(t))$ be the vector queue length process for an N-node series Jackson network, whose state space is $\mathbb{Z}_+^N$, the non-negative orthant of the N

dimensional integer lattice. If J is an arbitrary subset of $\{1, \ldots, N\}$, we define $(a_J)_t[\mathbf{m}, \mathbf{n}]$ for all $\mathbf{m}, \mathbf{n} \in \mathbb{Z}_+^N$ to be the transition probabilities for the following stopped process

$$(a_J)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n},\ T_J > t \mid \mathbf{Q}(0) = \mathbf{m}\},$$

where $T_J = \inf\{t \mid Q_j(t) = 0 \text{ for some } j \in J\}$. We also can define the following hitting time probabilities

$$(a_J^\dagger)_t[\mathbf{m}, \mathbf{n}] = \Pr\{\mathbf{Q}(t) = \mathbf{n},\ T_J = t \mid \mathbf{Q}(0) = \mathbf{m}\}$$

where $n_j = 0$ for some $j \in J$. By Lemma 3.1, we can solve for these in terms of the $(a_J)_t[\mathbf{m}, \mathbf{n}]$

$$(a_J^\dagger)_t[\mathbf{m}, \mathbf{n}] = \begin{cases} \mu_j \cdot (a_J)_t[\mathbf{m}, \mathbf{n} - \mathbf{v}_j] & n_j = 0,\ n_{j+1} > 0, \text{ and } j \in J, \\ 0 & \text{otherwise}. \end{cases}$$

As before, the $(a_J)_t[\mathbf{m}, \mathbf{n}]$ are the entries for an operator semigroup $(\mathbf{a}_J)_t$ with generator $(\dot{\mathbf{a}_J})$. They are both bounded operators on the Banach space $l_1(\mathbb{Z}_+^N)$. Since $(a_J)_t[\mathbf{m}, \mathbf{n}] = 0$ if any $m_j = 0$ or $n_j = 0$ for some $j \in J$, this translates into operators as

$$(\mathbf{a}_J)_t \cdot \mathbf{L}_j\mathbf{R}_j = \mathbf{L}_j\mathbf{R}_j \cdot (\mathbf{a}_J)_t = (\mathbf{a}_J)_t$$

for all $j \in J$, and consequently

$$(\mathbf{a}_J) \cdot \mathbf{L}_j\mathbf{R}_j = \mathbf{L}_j\mathbf{R}_j \cdot (\mathbf{a}_J) = (\mathbf{a}_J). \tag{4.1}$$

Now let $\mathbf{R}_J = \prod_{j \in J} \mathbf{R}_j$ and $\mathbf{L}_J = \prod_{j \in J} \mathbf{L}_j$. Constructing a process similar to $\mathbf{Q}^\#(t)$ in Section 2, adding tagged customers to every node indexed by J, we can deduce that

$$\mathbf{R}_J(\mathbf{a}_J)\mathbf{L}_J = \mu_0\mathbf{R}_1 + \sum_{j=1}^{N} \mu_j\mathbf{L}_j\mathbf{R}_j + \mu_N\mathbf{L}_N - \sum_{j \in \{0\} \cup J} \mu_j\mathbf{I} - \sum_{j \notin J} \mu_j\mathbf{L}_j\mathbf{R}_j. \tag{4.2}$$

Moreover, from (4.1) we can show that

$$\exp(t\ \mathbf{R}_J(\mathbf{a}_J)\mathbf{L}_J) = \mathbf{R}_J(\mathbf{a}_J)_t\mathbf{L}_J.$$

From now on, we will only consider J to be the following sets:

1. $\{1, \ldots, N\}$

2. $(1) \equiv \{1\}^c$

3. $(N) \equiv \{N\}^c$

4. $(1,N) \equiv \{1,N\}^c$,

where c denotes the complement of the indicated set. Setting $(\mathbf{a}_\bullet) = (\mathbf{a}_{\{1,\ldots,N\}})$, $\mathbf{R}_\bullet = \prod_{j=1}^{N} \mathbf{R}_j$, and $\mathbf{L}_\bullet = \prod_{j=1}^{N} \mathbf{L}_j$, we get from (4.2),

$$\mathbf{R}_\bullet(\mathbf{a}_\bullet)\mathbf{L}_\bullet = \mu_0\mathbf{R}_1 + \sum_{j=1}^{N-1} \mu_j\mathbf{L}_j\mathbf{R}_{j+1} + \mu_N\mathbf{L}_N - \sum_{j=0}^{N} \mu_j\mathbf{I}$$

$$\mathbf{R}_{(1)}(\mathbf{a}_{(1)})\mathbf{L}_{(1)} = \mu_0\mathbf{R}_1 + \sum_{j=1}^{N-1} \mu_j\mathbf{L}_j\mathbf{R}_{j+1} + \mu_N\mathbf{L}_N - \mu_0\mathbf{I} - \mu_1\mathbf{L}_1\mathbf{R}_1 - \sum_{j=2}^{N} \mu_j\mathbf{I}$$

$$\mathbf{R}_{(N)}(\mathbf{a}_{(N)})\mathbf{L}_{(N)} = \mu_0\mathbf{R}_1 + \sum_{j=1}^{N-1} \mu_j\mathbf{L}_j\mathbf{R}_{j+1} + \mu_N\mathbf{L}_N - \sum_{j=0}^{N-1} \mu_j\mathbf{I} - \mu_N\mathbf{L}_N\mathbf{R}_N$$

$$\mathbf{R}_{(1,N)}(\mathbf{a}_{(1,N)})\mathbf{L}_{(1,N)} = \mu_0\mathbf{R}_1 + \sum_{j=1}^{N-1} \mu_j\mathbf{L}_j\mathbf{R}_{j+1} + \mu_N\mathbf{L}_N - \mu_0\mathbf{I} - \mu_1\mathbf{L}_1\mathbf{R}_1 - \sum_{j=2}^{N-1} \mu_j\mathbf{I} - \mu_N\mathbf{L}_N\mathbf{R}_N .$$

Constructing the appropriately weighted Banach space, and following arguments like those in Section 2, we get

$$(\mathbf{a}_{(1)}) = \left(\mathbf{I} - \frac{\mu_1}{\mu_0}\mathbf{L}_1\right)\mathbf{R}_1(\mathbf{a}_\bullet)\mathbf{L}_1\left(\mathbf{I} - \frac{\mu_1}{\mu_0}\mathbf{L}_1\right)^{-1} \tag{4.3.a}$$

$$(\mathbf{a}_{(N)}) = (\mathbf{I} - \mathbf{R}_N)^{-1}\,\mathbf{R}_N(\mathbf{a}_\bullet)\mathbf{L}_N(\mathbf{I} - \mathbf{R}_N) \tag{4.3.b}$$

$$(\mathbf{a}_{(1,N)}) = \left(\mathbf{I} - \frac{\mu_1}{\mu_0}\mathbf{L}_1\right)(\mathbf{I} - \mathbf{R}_N)^{-1}\mathbf{R}_1\mathbf{R}_N(\mathbf{a}_\bullet)\mathbf{L}_N\mathbf{L}_1(\mathbf{I} - \mathbf{R}_N)\left(\mathbf{I} - \frac{\mu_1}{\mu_0}\mathbf{L}_1\right)^{-1} . \tag{4.3.c}$$

Now by [illegible], we have

$$(a_\bullet)_t[\mathbf{m},\mathbf{n}] = e^{-(N+1)\alpha t}\cdot \boldsymbol{\beta}^{\mathbf{n}-\mathbf{m}} \cdot \sum_{g\in G_N} (-1)^g\, I(\mathbf{n} - g(\mathbf{m}), (N+1)\gamma t) . \tag{4.4}$$

From (4.3.a), (4.3.b), and (4.3.c), we have

$$(a_{(N)})_t[\mathbf{m},\mathbf{n}] = \sum_{j=1}^{\infty} (a_*)_t[\mathbf{m}+j\mathbf{e}_N,\mathbf{n}+\mathbf{e}_N] - (a_*)_t[\mathbf{m}+j\mathbf{e}_N,\mathbf{n}]$$

$$(a_{(1)})_t[\mathbf{m},\mathbf{n}] = \sum_{j=1}^{\infty} \left(\frac{\mu_1}{\mu_0}\right)^{j-1} \left((a_*)_t[\mathbf{m}+\mathbf{e}_1,\mathbf{n}+j\mathbf{e}_1] - \frac{\mu_1}{\mu_0}(a_*)_t[\mathbf{m},\mathbf{n}+j\mathbf{e}_1]\right)$$

$$(a_{(1,N)})_t[\mathbf{m},\mathbf{n}] = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \left(\frac{\mu_1}{\mu_0}\right)^{k-1} \Big((a_*)_t[\mathbf{m}+\mathbf{e}_1+j\mathbf{e}_N,\mathbf{n}+k\mathbf{e}_1+\mathbf{e}_N] - (a_*)_t[\mathbf{m}+\mathbf{e}_1+j\mathbf{e}_N,\mathbf{n}+k\mathbf{e}_1]$$

$$- \frac{\mu_1}{\mu_0}(a_*)_t[\mathbf{m}+j\mathbf{e}_N,\mathbf{n}+k\mathbf{e}_1+\mathbf{e}_N] + \frac{\mu_1}{\mu_0}(a_*)_t[\mathbf{m}+j\mathbf{e}_N,\mathbf{n}+k\mathbf{e}_1]\Big).$$

By (4.4) we have a solution for $(a_N)_t[\mathbf{m},\mathbf{n}]$, $(a_{(1)})_t[\mathbf{m},\mathbf{n}]$, and $(a_{(1,N)})_t[\mathbf{m},\mathbf{n}]$ in terms of lattice Bessel functions of rank N.

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