Analysis of a stack algorithm for random length packet communication

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ANALYSE D'UN PROTOCOLE EN ARBRE POUR DES MESSAGES DE LONGUEUR ALEATOIRE

Résumé—Nous analysons de manière exacte les performances du protocole de Capetanakis-Tsybakov-Mikhailov, sous l'hypothèse des réseaux locaux, où les messages sont de longueurs différentes. En particulier, nous donnons une évaluation précise de la distribution (moyenne et écart type) des délais des messages et nous déterminons les valeurs exactes des débits maximaux accessibles, quelque soit la distribution des longueurs de message en donnée.

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1. INTRODUCTION

This paper analyses the performance of a protocol for managing the use of a *single-channel packet switching communication network* like the one used in the Ethernet [10].

We consider the following model, commonly taken as the basis of mathematical studies of the multiple-access channel [3]. The time is slotted and stations can start transmitting only at the beginning of slots. A packet may have a length of several slots. Each transmission is within the reception range of every user. When more than one user transmit simultaneously, packets are said to *collide*, none is correctly transmitted and the colliding users abort their transmission at the end of the slot. We assume that for a given packet, collision can occur only on the first slot to transmit. For the next slots, the other users are aware that a packet is in current transmission and wait for its final slot (carrier sense).

Thus the status of a slot is a *blank* if no user transmit on it, a *success* if there is only one user transmitting and a *collision* when two or more users transmit. This ternary feedback is the only source of information available for monitoring the communication process and resolving eventual collision. The collision resolution algorithm is clearly a major determinant of the behaviour of such a transmission process. In this paper we focus on the Capetanakis-Tsybakov-Mikhailov tree protocol [7], [8], [9] coupled with free, or continuous access of newly arriving packets into the contention. This protocol enjoys nice properties such as simplicity and robustness, stability under a large population of users, and last but not least, according to our point of view, tractability to analysis.

In 1985, Fayolle, Flajolet, Hofri and Jacquet [1] published a complete analysis of this protocol with an exhaustive determination of the channel utilization and of the distribution of packet delay. At this time it was the first complete and exact evaluation of the performance of a collision resolution algorithm. But the analysis was done under the simple hypothesis that all packets be of the same length, namely one slot. Our purpose is to extend this former analysis under the more realistic hypothesis of variable length packets (*e.g.* traffic mixture of *voice* and *data* in local area network).

This extension is not trivial. In a recent paper, Tsybakov and Fedortsov [4] showed some of the intricacies of the problem of variable packet length for this communication process. Given a distribution of packet length, they finally extracted fair upper and lower bounds of the maximum throughput of the protocol (*i.e.* the maximum admissible arrival rate before the channel destabilizes). This led us to slightly *modify* the tree protocol in order to get the analysis tractable. The modification does not affect the basic properties of the protocol and, surprisingly, even slightly *improves* the performance when the mean packet length is above about 10 slots. This is a reasonable hypothesis considering that the slot is a fair estimation of the delay of reaction of an ethernet chipment monitoring the channel.

A. *Specifications of the protocol*

Each station with a pending packet uses a variable C_s to schedule its transmissions. The variable C_s is interpreted as the stack level at which the packet is located at slot s. Each station updates the value of C_s at the end of each slot, in accordance with the instructions of the protocol. A station has to monitor the stack variable C_s only during the time it has a waiting packet (idle station can switch off). Below, we give a set of instructions for our slightly modified protocol:

I1 If a packet appeared at a station at slot s, then $C_s = -1$.
I2 If $C_s = -1$ and the slot s is a success, then the station waits for the end of the current message, and then set C_s to 0. For any other state of slot s, we have $C_{s+1} = 0$.
I3 While $C_s = 0$, then the station initiates packet transmission.
I4 If $C_s = 0$ and the slot s is a success, then the station transmits the following next slots of the packet and becomes idle.
I5 If $C_s = 0$ and the slot s is a collision, then C_{s+1} assumes one of the values 0 and 1 with respective probabilities p and q (of course, $p + q = 1$). If it turns out that $C_{s+1} = 0$, then the station initiates packet transmission in slot $s + 1$.
I6 If $C_s \geq 1$ and the slot s is a collision, then $C_{s+1} = C_s + 1$.
I7 If $C_s \geq 1$ and the slot s is a success, then $C_{s+1} = C_s$.
I8 If $C_s \geq 1$ and the slot s is a blank, then $C_{s+1} = C_s - 1$.

From a global point of view this protocol consists in monitoring a virtual stack containing the stations which are waiting for retransmission. The stack is a sequence of cells, numbered from 0 to ∞ and during the slot s, the elements of the cell number i ($i \geq 0$) are the stations such that $C_s = i$. The stack behaves as following. Suppose that at slot s, n stations collide ($n \geq 2$). Thus there were n stations in cell number 0 during the slot s. Because of the collision, stations which were waiting at level greater than zero, will increment their stack level. Meanwhile, the colliding stations use their random generator. Those which get a 0, suppose that their number be I ($I \geq n$), remain at level 0 and transmit again during the slot $s + 1$. The other colliding stations $(n - I)$ reach the level 1, for the slot $s + 1$, and wait for the resolution of the eventual conflict of the I first stations (plus some eventual new incomers) before retransmitting (when their counters C_s will read zero again).

The process underlies a recursive structure, since any conflict is resolved via a partition and the resolution of conflicts of smaller size.

B. *Characteristic Parameters*

We are interested here in several characteristic parameters of this protocol :

a) An n-session is the minimal set of slots separating the two following events:
  (i) There are exactly n stations inserted in the global stack, and each of them is stored at level 0.
  (ii) There is no station in the global stack.

The length of such an n-session, which is usually termed the *collision resolution interval* (CRI) will be here denoted by l_n.

b) Capetanakis-Tsybakov-Mikhailov tree protocols are stable random-access systems when the packet arrival rate is not too high. Thus, the *maximum traffic rate* that our protocol will allow before destabilizing is an important characteristic of the system. It will be denoted by $\lambda_{\max}$.

c) The *delay* experienced by a packet is the time from its generation at a station to the end of its successful transmission and is denoted by W.

Many other parameters are of interest, but our aim is not to give an exhaustive analysis of our protocol. However, these could be evaluated by the same following methods.

With the basic protocol (without our modification), the final set of slots of an n-session (which is a blank or a successful transmission) has a variable length. This is the reason why Tsybakov and Fedortsov failed in providing a complete analysis of it. However, with our protocol, the final set of slots of an n-session is reduced to a single blank, and this feature makes it tractable to analysis.

B. The probabilistic Model

First of all, it is assumed that the number N of stations is large, so that the assumption $N = \infty$ is valid. Moreover, the number of newly created active users in each slot is supposed to be independent of the state of the stack and its history, and to be a Poisson arrival process with a fixed rate λ. Thus, if this number is denoted by X,

$$Pr(X = n) = e^{-\lambda}\frac{\lambda^n}{n!}.$$

At last, we assume that the length T of packets is independent of X, of the stack and its history. $Pr(T = n)$ will be denoted by T_n, and the mean of T, by M. Thus,

$$M = \sum_{n=0}^{\infty} nT_n.$$

2. The basic equation and CRI duration

A. Preliminaries

Notations: if ϕ is a complex variable fonction, let

$$\mathbf{R}\phi(z) = \phi(z) - \phi(\lambda + pz) - \phi(\lambda + qz) .$$

Moreover, let $\sigma_1(z) = \lambda + pz$ and $\sigma_2(z) = \lambda + qz$. We denoted by $\mathcal{H}$ the noncommutative semigroup generated by composition of σ_1 and σ_2 plus the identity. At last, for each σ in $\mathcal{H}$, let

$$(p,q)^{\sigma} = p^{|\sigma|_1} q^{|\sigma|_2}$$

where $|\sigma|_i$ is the number of occurences of σ_i in σ (when σ is the identity, by convention $|\sigma|_i = 0$).

Thanks to these notations, we now can state the following lemma which will be used in the next sections in order to solve some functional equations. The reader is referred to [2] for a detailed proof of it.

Lemma 0.

a) *Let f be an entire function such that $f(\frac{\lambda}{p}) = f(\frac{\lambda}{q})$. The equation $\mathbf{R}g(z) = f(z)$, with specified values for $g(0)$ and $g'(0)$, has the unique regular solution:*

$$g(z) = g(0) + zg'(0) + \mathbf{S}(f;z)$$

where

$$\mathbf{S}(f;z) = \sum_{\sigma \in \mathcal{H}} [f(\sigma(z)) - f(\sigma(0)) - (p,q)^{\sigma} z f'(\sigma(0))] .$$

b) *The equation $g(z) - p\,g(\sigma_1(z)) - q\,g(\sigma_2(z)) = f(z)$, where f is an entire function such that $\int_{\frac{\lambda}{p}}^{\frac{\lambda}{q}} f(u)du = 0$, has, for each specified value of $g(0)$, the unique following regular solution:*

$$g(z) = g(0) + \mathbf{T}(f; z)$$

where

$$\mathbf{T}(f; z) = \sum_{\sigma \in \mathcal{H}} (p,q)^{\sigma} [f(\sigma(z)) - f(\sigma(0))] \,.$$

B. Mean value analysis and determination of $\lambda_{\max}$

The recursive structure of the algorithm leads to the following identity, when there is an initial collision ($n \geq 2$):

$$l_n = 1 + l_{I+X} + l_{n-I+Y}. \tag{2.1a}$$

The integer I is the number of stations which remain at level 0 after the collision, and X and Y the numbers of arrivals generated on single slots. X and Y are independent and $Pr(X = k) = Pr(Y = k) = e^{-\lambda}\frac{\lambda^k}{k!}$, and I has the binomial distribution $B_P(n)$, namely $Pr(I = i) = \binom{n}{i} p^i q^{n-i}$. To complete the identity, we have the obvious relations

$$\begin{aligned} l_0 &= 1 \\ l_1 &= T + l_m, \end{aligned} \tag{2.1b}$$

where T is the length of the packet and m the number of arrivals generated during its transmission. The second equality, which is less obvious than the first one, means that packets, which are waiting at level 1, cannot immediately transmit after detecting the end of a successful transmission like in the basic protocol : now they have to wait for the end of the conflict resolution of the m packets which have been generated during that successful transmission. The distribution of m is easy to obtain from the distribution of T. Thus, if the mean of l_n is denoted by L_n, we have

$$\begin{cases} L_0 = 1 \\ L_1 = M + \sum_{n=0}^{+\infty} T_n \sum_{k=0}^{+\infty} e^{-\lambda n} \frac{(\lambda n)^k}{k!} L_k \\ L_n = 1 + \sum_{x,y \geq 0} e^{-\lambda} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^y}{y!} \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} (L_{j+x} + L_{n-j+y}) \quad \text{for } n \geq 2. \end{cases} \tag{2.2}$$

Let us define the function ϕ by $\phi(z) = e^{-z} \sum_{n=0}^{+\infty} L_n \frac{z^n}{n!}$. We introduce the quantity K defined by

$$K = \begin{cases} \dfrac{e^{-\frac{\lambda}{q}} - e^{-\frac{\lambda}{p}}}{\frac{\lambda}{p} e^{-\frac{\lambda}{p}} - \frac{\lambda}{q} e^{-\frac{\lambda}{q}}} & \text{if } p \neq q, \\ \dfrac{1}{1 - 2\lambda} & \text{if } p = q = \frac{1}{2} \end{cases},$$

the function $t(z)$, by

$$t(z) = (1 + K\,z)e^{-z}$$

and function $\chi(z)$, by

$$\chi(z) = \sum_{n=0}^{\infty} T_n \mathbf{S}(t; n\, z) \; .$$

THEOREM 1. *The maximum throughput, $\lambda_{\max}$ is the smallest positive root of the equation*

$$2\lambda\chi(\lambda) + (1 - \lambda M)(1 + 2\mathbf{S}(t; \lambda) = 0 \; :$$

and if $\lambda < \lambda_{\max}$, then the mean value of CRI duration is $\phi(\lambda)$, with the expression

$$\phi(\lambda) = \frac{1}{2\lambda\chi(\lambda) + (1 - \lambda M)(1 + 2\mathbf{S}(t; \lambda))} \; .$$

Moreover, we have

$$L_1 = 1 + \frac{M\,(1 + 2\mathbf{S}(t; \lambda)) - 2\chi(\lambda)}{2\lambda\chi(\lambda) + (1 - \lambda M)(1 + 2\mathbf{S}(t; \lambda))} \; ,$$

and

$$\phi(z) = 1 + z(L_1 - 1) - 2\phi(\lambda)\mathbf{S}(t; z) \; .$$

Proof. These results are very similar to those obtained in [1], and by considering L_1 as a parameter, computations can be led in the same way. Thus, we just give a sketch of our computations, and the reader is referred to [1] for a more detailed discussion of them.

From (2.2), we obtain the following functional equation for ϕ:

$$\mathbf{R}\phi(z) = 1 - e^{-z}\left(2\phi(\lambda)(1 + z) + z\,(\phi'(\lambda) - L_1 + 1)\right), \tag{2.3a}$$

and,

$$L_1 = M + \sum_{n=0}^{+\infty} T_n \phi(\lambda n). \tag{2.3b}$$

A convenient simplification of (2.3a) is possible, by noting that, if $p \neq q$, then $\mathbf{R}\phi(\frac{\lambda}{p}) = \mathbf{R}\phi(\frac{\lambda}{q}) = -\phi(2\lambda)$, and thus

$$\phi'(\lambda) - L_1 + 1 = 2\phi(\lambda)(K - 1) \; ,$$

where

$$K = \frac{e^{-\frac{\lambda}{q}} - e^{-\frac{\lambda}{p}}}{\frac{\lambda}{p}e^{-\frac{\lambda}{p}} - \frac{\lambda}{q}e^{-\frac{\lambda}{q}}} \; ,$$

and if $p = q = \frac{1}{2}$, then substituting $z = 2\lambda$ in the derivative of (2.3a) also yields

$$\phi'(\lambda) - L_1 + 1 = 2\phi(\lambda)(K - 1) \; ,$$

where

$$K = \frac{1}{1 - 2\lambda} \; .$$

This simplification produces, in both cases:

$$\mathbf{R}\phi(z) = 1 - 2\phi(\lambda)t(z) \; .$$

Moreover, $\phi(0) = 1$ and $\phi'(0) = L_1 - 1$. Thus, by using theorem 1 we get

$$\phi(z) = 1 + z(L_1 - 1) - 2\phi(\lambda)\mathbf{S}(t; z) . \tag{2.4}$$

We still have to compute L_1 and $\phi(\lambda)$. Now, from (2.3),

$$\phi(\lambda) = 1 + \lambda(L_1 - 1) - 2\phi(\lambda)\mathbf{S}(t; \lambda),$$

and from (3a.4),

$$L_1 = M + \sum_{n=0}^{+\infty} T_n \left(1 + \lambda n(L_1 - 1) - 2\phi(\lambda)\mathbf{S}(t; \lambda n)\right).$$

These two equations form a linear system whose determinant is:

$$\det(\lambda) = 2\lambda\chi(\lambda) + (1 - \lambda M)(1 + 2\mathbf{S}(t; \lambda) .$$

If $\det(\lambda) = 0$, then this system has no solution, and so the protocol is not stable. Thus, we get

$$\lambda_{\max} = \min\{\lambda > 0 / \det(\lambda) = 0\}. \tag{2.5}$$

However, if $\det(\lambda) \neq 0$, this system has a unique solution which is the one given in the statement of theorem 2.

From $\phi(z)$, we can compute an explicit form of L_n. Moreover, the renewing properties of the protocol (see [1]) yields that $\phi(\lambda) = \sum_{n=0}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} L_n$ is the unconditional length of a session. ∎

C. Moments of the CRI duration.

Let $P_n(u)$ be the characteristic function of the length of a session when the initial collision is of multiplicity n :

$$P_n(u) = \sum_{k=0}^{\infty} Pr(l_n = k)u^k .$$

Let us introduce the bivariate generating function, $P(z, u)$, defined by

$$P(z, u) = e^{-z} \left(\sum_{n=0}^{\infty} P_n(u) \frac{z^n}{n!} \right) ,$$

THEOREM 2. *We have the functionnal equation*

$$\begin{aligned} \frac{1}{u} P(z, u) =& P(\lambda + pz, u)P(\lambda + qz, u) \\ &- e^{-z} P(\lambda, u) \left(P(\lambda, u)(1 + z) + zP_z(\lambda, u)\right) + e^{-z} \left(1 + z\frac{P_1(u)}{u}\right) , \end{aligned} \tag{2.6a}$$

where

$$P_1(u) = \sum_{n=0}^{\infty} T_n u^n P(u, \lambda n) \tag{2.6b}$$

(P_z means partial derivation of function $P(z, u)$ with respect to variable z).

Proof. We get from (2.1a) and (2.1b) the following recursion

$$\begin{cases} P_0(u) = u \\ P_1(u) = \sum_{n=0}^{\infty} T_n\, u^n \sum_{k=0}^{\infty} e^{-\lambda n} \frac{(\lambda n)^k}{k!} P_k(u) \\ P_n(u) = u \Big(\sum_{x,y \geq 0} e^{-\lambda} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^y}{y!} \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} (P_{j+x}(u) P_{n-j+y}(u)) \Big) & \text{for } n \geq 2. \end{cases}$$

some elementary algebric manipulations yield the functionnal equation of $P(z,u)$ (For further explanations, see [1]). ∎

Let $\Psi(z)$ defined by

$$\Psi(z) = \frac{\partial^2}{\partial u^2} (P(z,u))_{u=1}.$$

Introducing the quantity $\overline{K}$, defined by

$$\begin{cases} \overline{K} = 4\phi(\lambda) e^{\frac{-\lambda}{pq}} \dfrac{\left(\phi(\frac{\lambda}{q}) - \phi(\frac{\lambda}{p})\right)\left(\frac{\lambda}{p} - \frac{\lambda}{q}\right)}{\cdot \left[\frac{\lambda}{q} e^{-\frac{\lambda}{q}} - \frac{\lambda}{p} e^{-\frac{\lambda}{p}}\right]^2} & \text{if } p \neq q, \\ \overline{K} = 4\phi(\lambda) \dfrac{\phi'(2\lambda)}{(1-2\lambda)^2} & \text{if } p = q = \frac{1}{2}, \end{cases}$$

the function $G(z)$, defined by

$$G(z) = 2\phi(z) - 2 + 2\phi(\lambda + pz)\phi(\lambda + qz) - e^{-z}\{2\phi^2(\lambda)(1 + Kz) + z\overline{K}\}\ .$$

the function $R(z)$, defined by

$$R(z) = \sum_{n=0}^{\infty} n(n-1) T_n + 2 \sum_{n=0}^{\infty} T_n n \phi(z\, n)\ ,$$

and $Y(z)$, defined by

$$Y(z) = \sum_{n=0}^{\infty} T_n \mathrm{S}(G; n\, z)\ ,$$

we obtain the following theorem:

THEOREM 3. *If $\lambda < \lambda_{\max}$, the inconditionnal variance $Var(l)$, of the length of session is*

$$Var(l) = \Psi(\lambda) + \phi(\lambda) - \phi^2(\lambda)\ ,$$

with

$$\Psi(\lambda) = \frac{(1 - \lambda M)\mathrm{S}(G; \lambda) + \lambda(R(\lambda) + Y(\lambda))}{2\lambda\chi(\lambda) + (1 - \lambda M)(2S(t; \lambda) + 1)}\ .$$

Moreover we have

$$\Psi'(0) = \frac{-2\chi(\lambda)\mathrm{S}(G; \lambda) + (R(\lambda) + Y(\lambda))(2\mathrm{S}(t; \lambda) + 1)}{2\lambda\chi(\lambda) + (1 - \lambda M)(2S(t; \lambda) + 1)}$$

and the expression

$$\Psi(z) = \mathbf{S}(G; z) - 2\Psi(\lambda)\mathbf{S}(t; z) + \Psi'(0)z$$

Proof. Of course $e^{-z}\sum_{n\geq 0} E(l_n^2)\frac{z^n}{n!} = \Psi(z) + \phi(z)$, which leads to the expression of the unconditionnal variance.

Derivating twice (2.6a) and substituting $u = 1$, we obtain

$$\begin{aligned} \mathbf{R}\Psi(z) =& 2\phi(z) - 2 + 2\phi(\lambda + pz)\phi(\lambda + qz) - e^{-z}(2\Psi(\lambda)(1 + z) + z\Psi'(\lambda)) \\ & - 2e^{-z}\phi(\lambda)(\phi(\lambda)(1 + z) + z\phi'(\lambda)) - ze^{-z}\Psi_1 , \end{aligned} \tag{2.7}$$

where

$$\Psi_1 = \frac{\partial^2}{\partial u^2}\left(\frac{P_1(u)}{u}\right)_{u=1} ,$$

since $\phi(z) = P_u(z, 1)$. The same methods of simplification as those developped in a) lead to

$$\mathbf{R}\Psi(z) = G(z) - 2\Psi(\lambda)e^{-z}(1 + Kz) ,$$

Moreover, $\Psi(0) = 0$ and

$$\Psi'(0) = L_1''(1) = R(\lambda) + \sum_{n=0}^{\infty} T_n\Psi(\lambda n) .$$

From theorem 1, $\Psi(z) = \mathbf{S}(G; z) - 2\Psi(\lambda)\mathbf{S}(t; z) + \Psi'(0)z$. We now just have to evaluate $\Psi(\lambda)$ and $\Psi'(0)$ by solving the following linear system:

$$\begin{cases} \Psi(\lambda) = \mathbf{S}(G; \lambda) - 2\Psi(\lambda)\mathbf{S}(t; \lambda) + \Psi'(0)\lambda \\ \Psi'(0) = R(\lambda) + \sum_{n=0}^{\infty} T_n(\mathbf{S}(G; \lambda n) - 2\Psi(\lambda)\mathbf{S}(t; \lambda n) + \Psi'(0)\lambda n), \end{cases}$$

whose determinant is the same as the one we got in a), so that for $\lambda < \lambda_{\max}$, it has a unique solution which has an explicit expression. ■

Remark We could get, by using the third derivative of (2.6), an explicit form of the third moment of l_n, and so on, it is possible to compute all moments of the distribution of l_n.

3. Distribution of the packet delay W.

A. *Direct evaluation of the mean packet delay*

LEMMA 4. *The mean number of users that are active per session is* $\lambda\phi(\lambda)$

Proof. This is a direct application of the renewing properties of the protocol. ■

Let us define c_n as the total sojourn time experienced by all users that are active during an n-session, and denote by C_n the mean of c_n. It is convenient to introduce the function $C(z)$, defined by $C(z) = e^{-z}(\sum_{n=0}^{+\infty} C_n\frac{z^n}{n!})$. Introducing the quantity $A(\lambda)$, defined by

$$\begin{cases} A(\lambda) = \dfrac{\frac{1}{q} - \frac{1}{p} + \phi(\frac{\lambda}{q}) - \frac{q}{p}\phi(2\lambda)}{\frac{1}{q}e^{-\frac{\lambda}{q}} - \frac{1}{p}e^{-\frac{\lambda}{p}}} & \text{if } p \neq q \\ A(\lambda) = e^{2\lambda}\dfrac{2 + \phi(2\lambda) + \lambda\phi'(2\lambda)}{2(1 - 2\lambda)} & \text{if } p = q = \frac{1}{2}, \end{cases}$$

the function $F(z)$, defined by

$$F(z) = qz\phi(\lambda + pz) - zA(\lambda)e^{-z} ,$$

the function $X(z)$, defined by

$$X(z) = \sum_{n=0}^{\infty} T_n \mathbf{S}(F; n\, z) ,$$

and the quantity D,

$$D = M + \lambda \sum_{n=0}^{\infty} T_n \frac{n(n-1)}{2} ,$$

we obtain the following theorem:

THEOREM 5. *When $\lambda < \lambda_{\max}$, the mean delay $E[W]$ has the expression :*

$$E(W) = \frac{1}{\lambda}[(1 - \lambda M)\mathbf{S}(F; \lambda) + \lambda(D + X)] .$$

Proof. The renewing properties and lemma 4 tell that $E[W] = \dfrac{C(\lambda)}{\lambda\phi(\lambda)}$. The specifications of the protocol produce

$$\begin{cases} C_0 = 0 \\ C_1 = M + \displaystyle\sum_{n=0}^{\infty} T_n \sum_{k=0}^{\infty} e^{-\lambda n} \frac{(\lambda n)^k}{k!} \left(\frac{n-1}{2} k + C_k \right) \\ C_n = n + \displaystyle\sum_{x,y \geq 0} e^{-\lambda} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^y}{y!} \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} (C_{j+x} + C_{n-j+y} + (n-j) L_{j+x}) \quad \text{for } n \geq 2. \end{cases}$$

From this recursion, we get an equation relative to C whose simplified form is

$$\mathbf{R}C(z) = z + qz\phi(\lambda + pz) - 2C(\lambda)(1 + Kz)e^{-z} - zA(\lambda)e^{-z} . \tag{3.1a}$$

Moreover, $C(0) = 0$ and,

$$C'(0) = C_1 = D + \sum_{n=0}^{\infty} T_n C(\lambda n), \tag{3.1b}$$

Using lemma 0 yields

$$C(z) = C'(0)z + \mathbf{S}(F; z) - 2C(\lambda)\mathbf{S}(t; z) . \tag{3.2}$$

From (3.1b) and (3.2), we obtain

$$C(\lambda) = \frac{(1 - \lambda M)\mathbf{S}(t; \lambda) + \lambda(D + X)}{2\lambda\chi(\lambda) + (1 - \lambda M)(2S(t; \lambda) + 1)} , \tag{3.3}$$

■

Once more the reader is referred to [1] for a more detailed explanation of these computations. The above procedure yields $E(W)$, but it cannot lead to the values of higher moments, due to the linearity it utilizes. That is why we have to resort to a more involved analysis.

B. Moments of the packet delay

Let Ω_n be the delay experienced by a packet that had its first transmission at the beginning of an n-session. Introducing the notations $\omega_n(u) = \sum_{i\geq 1} Pr(\Omega_n = i)u^i$, and

$$h(z,u) = e^{-z}\sum_{n\geq 1}\omega_n(u)\frac{z^{n-1}}{(n-1)!} ,$$

and defining ω by

$$\omega(z) = e^{-z}\sum_{n\geq 1}E(\Omega_n)\frac{z^{n-1}}{(n-1)!} = \left(\frac{\partial}{\partial u}h(z,u)\right)_{u=1} ,$$

produces:

THEOREM 6. *The bivariate generating function $h(z,u)$ satisfies the functional equation*

$$\begin{aligned}\frac{1}{u}h(z,u) =& p\,h(\lambda+pz,u) + q\,h(\lambda+qz,u)P(\lambda+pz,u)\\ &+ e^{-z}\left(\sum_{n=1}^{+\infty}T_n u^{n-1} - h(\lambda,u)(p+qP(\lambda,u))\right) ,\end{aligned} \tag{3.3}$$

and we have the following expression for the generating function of the first moment of Ω:

$$\omega(z) = M + T(J;z) - \omega(\lambda)T(e^{-u};z) ,$$

where

$$J(z) = q\phi(\lambda+pz) - e^{-z}(q\phi(\lambda) - M + 1) ,$$

with

$$\omega(\lambda) = \frac{M + T(J;\lambda)}{1 + T(e^{-u};\lambda)} .$$

Proof. Using the line of reasonning introduced in [1] we have

$$\begin{cases}\omega_1(u) = \sum_{n=0}^{\infty} T_n u^n & \\ n\,\omega_n(u) = u\Big(\sum_{x,y\geq 0} e^{-\lambda}\frac{\lambda^x}{x!}e^{-\lambda}\frac{\lambda^y}{y!}\times & \\ \qquad\times\sum_{j=0}^{n}\binom{n}{j}p^j q^{n-j}(j\,\omega_{j+x}(u) + (n-j)P_{j+x}(u)\omega_{n-j+y}(u))\Big) & \text{for } n\geq 2 ,\end{cases}$$

which, translated in terms of generating function, leads to the functionnal equation. Derivating this functional equation with respect to u and substituting $u=1$ yields the functional equation for $\omega(z)$

$$\begin{aligned}\omega(z) - p\omega(\lambda+pz) - q\omega(\lambda+qz) =& 1 + q\phi(\lambda+pz)\\ &- e^{-z}(\omega(\lambda) + q\phi(\lambda) - M + 1).\end{aligned} \tag{3.4}$$

As $\omega(0) = M$, using theorem 1 proves the theorem 5. ∎

Let W_n^j be the mean number of packets that experienced a delay of j slots during an n-session. Let $W_n(u)$ be defined by

$$W_n(u) = \sum_{j=0}^{\infty} W_n^j u^j ,$$

and the bivariate generating function $W(z,u)$, by

$$W(z,u) = \sum_{n=0}^{\infty} W_n(u) \frac{z^n}{n!} e^{-z} .$$

THEOREM 7. *The characteristic function of the distribution of the delay of a random packet satisfies the identity*

$$\sum_{k=0}^{\infty} Pr(W = k) u^k = \frac{W(\lambda, u)}{\lambda \phi(\lambda)} ,$$

and the bivariate function $W(z,u)$ satisfies the functional equation

$$\begin{aligned} W(z,u) =& W(\lambda + pz, u) + W(\lambda + qz, u) \\ & + z(h(z,u) - p\,h(\lambda + pz, u) - q\,h(\lambda + qz, u) + h(\lambda, u) e^{-z}) \\ & - \left(2W(\lambda,u)(1+z) + z \left(W_z(\lambda,u) - W_1(u) + \sum_{n=0}^{\infty} T_n u^n \right) \right) e^{-z} , \end{aligned} \tag{3.5a}$$

with,

$$\begin{aligned} W_1(u) = & \sum_{n=0}^{\infty} T_n u^n - \lambda \sum_{n=0}^{\infty} T_n n\, h(\lambda n, u) \\ & + \lambda \sum_{n=0}^{\infty} T_n \Big(\sum_{\alpha=0}^{n-1} u^\alpha \Big) h(\lambda n, u) \\ & + \sum_{n=0}^{\infty} T_n W(\lambda n, u). \end{aligned} \tag{3.5b}$$

Proof . The Law of Large Numbers and the renewing properties of the communication process provides that $\sum_{n=0}^{\infty} W_n^j \frac{\lambda^n}{n!} e^{-\lambda}$ divided by the mean number of users per session is the stationary probability for a random packet to experience a delay of j slots. Since the mean number of users per session is exactly $\lambda\phi(\lambda)$, we prove the first part of the theorem.

We have

$$\begin{cases} W_0(u) = 0 \\ W_1(u) = \displaystyle\sum_{n=0}^{\infty} T_n T_n u^n + \sum_{n=0}^{\infty} T_n \sum_{k=0}^{\infty} e^{-\lambda n} \frac{(\lambda n)^k}{k!} \times \\ \qquad \times \left(\dfrac{1-u^n}{n(1-u)} k\omega_k(u) + W_k(u) - k\omega_k(u) \right) \\ W_n(u) = n\,\omega_n(u) + \displaystyle\sum_{x,y \geq 0} e^{-\lambda} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^y}{y!} \sum_{j=0}^{n} \binom{n}{j} p^j q^{n-j} \times & \text{when } n \geq 2 , \\ \qquad \times \Big(W_{j+x}(u) - j\,\omega_{j+x}(u) + W_{n-j+y}(u) - (n-j)\omega_{n-j+y}(u) \Big) , \end{cases}$$

which, translated in terms of bivariate generating function, yields the functional equation. ∎

Introducing the function $\Xi(z)$, defined by

$$\Xi(z) = z\left[-1 + 2\omega(z) + 2q\phi(\lambda + pz)\omega(\lambda + qz) + q(\Psi(\lambda + pz) + \phi(\lambda + pz)) \right. \\ \left. -e^{-z}\left(2q\phi(\lambda)\omega(\lambda) + q(\Psi(\lambda) + \phi(\lambda))\right)\right] ,$$

and the quantity B, defined by

$$\begin{cases} B = \dfrac{\Xi(\frac{\lambda}{p}) - \Xi(\frac{\lambda}{q})}{\frac{\lambda}{p}e^{-\frac{\lambda}{p}} - \frac{\lambda}{q}e^{-\frac{\lambda}{q}}} & \text{if } p \neq p \\ B = e^{2\lambda}\dfrac{\Xi'(2\lambda)}{1-2\lambda} & \text{if } p = q = \frac{1}{2} , \end{cases}$$

and the function $v(z)$, by

$$v(z) = -B\, ze^{-z} + \Xi(z) ,$$

we can state the following theorem :

THEOREM 8. *When $\lambda < \lambda_{\max}$, the second moment $E(W^2)$ of the packet delay has the expression*

$$E(W^2) = \frac{1}{\lambda}[(1 - \lambda M)\mathbf{S}(v;\lambda) + \lambda(\eta + \Delta(\lambda))] ,$$

where

$$\Delta(z) = \sum_{n=0}^{\infty} T_n \mathbf{S}(v; z\,n) ,$$

and

$$\eta = M + \sum_{n=0}^{\infty} T_n n(n-1) + \lambda \sum_{n=0}^{\infty} T_n \frac{n(n-1)(2n-1)}{6} + 2\lambda \sum_{n=0}^{\infty} T_n \frac{n(n-1)}{2}\omega(\lambda n) .$$

Proof. Of course, we have

$$C(z) = \left(\frac{\partial}{\partial u}W(z,u)\right)_{u=1} .$$

Let $W^{(2)}(z) = \left(\dfrac{\partial^2}{\partial u^2}W(z,u)\right)_{u=1} + C(z)$. Then, substituting $u = 1$ in the second derivative of (3.5a) and simplifying the result, produces the following equation relative to $W^{(2)}$:

$$\mathbf{R}W^{(2)}(z) = -2W^{(2)}(\lambda)e^{-z}(1+z) - Hze^{-z} + \Xi(z) , \tag{3.6}$$

where H is a known real which will be eliminated afterwards. After simplification this equation becomes

$$\mathbf{R}W^{(2)}(z) = -2W^{(2)}(\lambda)e^{-z}(1 + Kz) - Bze^{-z} + \Xi(z),$$

Then, using Lemma 0 yields

$$W^{(2)}(z) = -2W^{(2)}(\lambda)\mathbf{S}(t;z) + \mathbf{S}(v;z) + W^{(2)\prime}(0)z .$$

Moreover, $W^{(2)\prime}(0) = W_1''(1) + C'(0)$, hence,

$$W^{(2)\prime}(0) = \eta + \sum_{n=0}^{\infty} T_n W^{(2)}(\lambda n). \tag{3.7}$$

We are now able to compute $W^{(2)}(\lambda)$, and the fact that $E(W^2) = \dfrac{W^{(2)}(\lambda)}{\lambda\phi(\lambda)}$ completes the proof of the theorem. ■

Remark The variance is obtained from theorems 4 and 5 by using the formula: $Var(W) = E(W^2) - E(W)^2$. The other moments could be evaluated in the same way.

4. Numerical Results and Discussion

A. Analysis of $\lambda_{\max}$

Figure 1 shows $\lambda_{\max}$ as a function of M; quantity M is the mean packet length and varies there between 1 and 50. The two curves correspond to two types of packet length distribution. The curve C1 is obtained with the case where all packets are of the same length M. The curve C2 is obtained with the case where the packets are either of length 1, either of length 100, the probabilities are being tuned in order to have the mean M. These two cases illustrate the differences between packet length distributions with zero variance and distributions with large variances.

The curve C2 is below C1, thus the variance of the packet length slightly affects the protocol performance. But computations and analysis show that the degradation remains confined between strict bounds: the protocol does not collapse when the variance tends to infinity, for a fixed mean M. The curve C2 is of interest, because, in real life, local area networks are generally submitted to two types of traffic: short packets corresponding to user oriented communications, and large packets corresponding to data exchanges between computers.

It is possible to fairly compare the basic protocol to our modified version. When the packets are all of length one slot, the modified protocol entails $\lambda_{\max} = 0.328226$ which is below the 0.360177 reached by the basic protocol [2]. The explanation of this difference is clear: the basic protocol destabilizes when the proportion of collision slots equals the proportion of non collision slots (empty and success slots); meanwhile the modified protocol naturally destabilizes when the proportion of collision slots only equals the proportion of empty slots. Reference [4] presents an upper bound of the equivalent of curve C1 for the basic tree protocol, which clearly shows that the modified protocol outperforms the basic one as $M > 10$. This assertion is rather surprising, but it can be explained as follow.

Let us suppose that all packets are exactly of length M, the quantity M being large compared to 1. Let us assume that λ is very close to $\lambda_{\max}$. The proportion of empty or collision slots is negligible compared to the proportion of success slots (which occur in sequences of M units in a row — see below for a rigorous proof). The proportion of success slots is exactly λM and we have then $\lambda M \approx 1$. The number of users which become active during the emission of a packet follows a Poisson distribution of rate λM, which is approximately 1. The probability that no user becomes active during a packet transmission is $e^{-\lambda M} \approx \frac{1}{e}$. In the basic protocol, a user whose stack level is 1 at the beginning of a success will transmit just after this packet and experience a collision, with probability greater than $1 - \frac{1}{e} \approx 0.632$. With the modified protocol, this collision is avoided, because no user in stack is allowed to transmit just after a success.

The following proposition give a precise illustration of the behaviour of $\lambda_{\max}$ when $M \to \infty$.

Theorem 9. *When the mean length of packet tends to infinity, the quantity* $\lambda_{\max} M$ *tends to* 1

Proof: the quantity λM is the proportion of success slots in steady state. Thus, we necessarily have $\lambda M < 1$, and $\lambda_{\max} M \leq 1$ consequently. Thus $\lambda_{\max} \to 0$ when $M \to \infty$.

Let $\varphi_\lambda(z)$ be the generating function of the mean session length for the basic protocol, when all packets are of length one slot with a Poisson arrival of rate λ packet per slot. We know that ([2])

$$\varphi_\lambda(z) = 1 - \frac{2S(t;z)}{1+2S(t;\lambda)} .$$

The $\lambda_{\max}$ of our protocol satisfyes, according to theorem 2,

$$(1 - \lambda_{\max} M) = \lambda_{\max} \sum_{n=0}^{\infty} T_n \Big(\varphi_{\lambda_{\max}}(n\lambda_{\max}) - 1\Big) .$$

The analysis done in [2] yields that, for any given $\lambda < 0.360177$, $\varphi_\lambda(z) = O(z)$, when $z \to \infty$, and $\varphi_\lambda(z) = 1 + O(z^2)$, when $z \to 0$. Let A_λ be defined by

$$A_\lambda = \sup_{x \geq 0} \frac{\varphi_\lambda(x) - 1}{x} ;$$

We have $A_\lambda < \infty$ and A_λ is a continuous increasing function of λ. Thus

$$\lambda_{\max} \sum_{n=0}^{\infty} T_n \Big(\varphi_{\lambda_{\max}}(n\lambda_{\max}) - 1\Big) \leq \lambda_{\max} \sum_{n=0}^{\infty} T_n A_{\lambda_{\max}} \lambda_{\max} = M \lambda_{\max}^2 A_{\lambda_{\max}} .$$

Since $A_{\lambda_{\max}}$ is bounded ($\lambda_{\max} = O(\frac{1}{M})$), we get

$$1 - \lambda_{\max} M = O(\frac{1}{M}) .$$

■

It is interesting to analyse how $M\lambda_{\max}$ tends to 1. A good way for that is to determine the *marginal throughput*, μ, of the channel. The marginal throughput is the throughput we obtain when we follow the events on the channel in the natural way, except that each packet duration is collapsed into a single slot (we take a movie of the channel, but we stop the camera after every first slot of a successful transmission and we resume at the end of the successful transmission).

The following lemma is obvious

LEMMA 10. *The marginal throughput, μ, of the channel, when the input load is λ and the mean packet duration is M, is a monotonous function of λ and is defined by*

$$\lambda = \frac{1}{M - 1 + \frac{1}{\mu}} .$$

Like maximum throughput $\lambda_{\max}$ we can define *marginal maximum throughput*, $\mu_{\max}$.

THEOREM 11. *When $M \to \infty$, we have*

$$\liminf \mu_{\max} > 0.25730 .$$

Proof. According to the previous proposition and theorem 2, we have $\lambda_{\max} \geq \lambda^*$, where λ^* satisfyes

$$1 - \lambda^* M = M\lambda^{*2} A_{\lambda^*} .$$

Since $\lambda^* M \leq 1$, we have

$$1 - \lambda^* M \leq A_{\lambda^*}\lambda^* ,$$

and finally

$$\lambda^* \geq \frac{1}{M + A_{\lambda^*}} .$$

Since $\lambda^* \to 0$ and $A_{\lambda^*} \to A_0$, and $A_0 \approx \frac{2}{\log 2} = 2.88539$, the proof is complete. ∎

Thus, we proved that there are lower bounds of the stability of our protocol that are only dependent of the mean packet length, M. When all packet, are of length M, we can precise the previous proposition.

THEOREM 12. *When all packets are of length M, then*

$$\mu_{\max} = 0.4277 + O(\frac{1}{M}) .$$

Proof. We have

$$1 - \lambda_{\max} M = \lambda_{\max}\Big(\varphi_{\lambda_{\max}}(\lambda_{\max} M) - 1\Big) .$$

We know that

$$\lambda_{\max} M = 1 + O(\frac{1}{M}) .$$

The precise analysis in [2] leads to the fact that $\varphi_\lambda(z)$ is analytic with respect to its two arguments λ and z; thus

$$\varphi_{\lambda_{\max}}(\lambda_{\max} M) = \varphi_0(1) + O(\frac{1}{M}) .$$

The estimate

$$\varphi_0(1) \approx \frac{1}{0.4277}$$

completes the proof. ∎

B. Numerical evaluation of means and variances of L and W.

Means and variances of L and W are given in figures 2 and 3 as functions of λ and p. These two tables correspond to two types of packet length distribution. The results of figure 2 have been computed under the assumption that all packets have a length of 10 slots, and those of figure 3 under the assumption that packets are either of length 2 slots, either of length 18 slots, with a probability of 0.5. In each box of those tables are given respectively (from top to bottom) $E(L)$, $Var(L)$, $E(W)$ and $Var(W)$.

First of all, we have to note the symmetry with respect to the value $p = \frac{1}{2}$, for the values of $E(L)$ and $V(L)$. This is not a surprising fact since the functional equation (3b.4) is also symetric with respect to the value $p = \frac{1}{2}$. However, this remark is no longer true for the values of $E(W)$ and $V(W)$. Indeed, we can prove ([1]) that $E(W)$ is minimized at $p = 2 - \sqrt{2} \approx 0.586$. A little reflection will show that this can be expected, as $p > \frac{1}{2}$ somewhat decreases the probability of wasting the first slots. As λ increases, the importance of efficient splitting predominates to reduce the value of p that minimizes the mean delay to very nearly one-half when $\lambda \to \lambda_{\max}$. At last, by comparing the two tables, we can notice once more that the protocol performance is slightly affected by the variance of the packet length.

5. CONCLUSION

We have analyzed the throughput and delay characteristics of a simple Stack Algorithm in the realistic conditions of Local Area Network communication, when the distribution of the packet duration is given. The salient features of this algorithm are its simplicity (in terms of implementation) and, as opposed to the well-known ALOHA or *Binary Exponential Back-off* algorithms (©ETHERNET) systems [11], [12], its inherent stability (under the assumption of a Poisson arrival process). The algorithm is very robust and can be used with any type of traffic mixture (voice mixed with data, for example). We restrict ourselves on the evaluation of some user-oriented parameters such that throughput and delay. Some other parameters are of interest, for example, we did not extend ourselves on the high robustness to channel errors, which is one of the typical properties of most versions of the stack algorithm [3].

There are several versions of the stack algorithms that can be implemented (and slightly modified) for our purpose. For example the stack algorithm with Q-ary, instead of binary, splitting ([5]) can be easily analyzed with exactly the same techniques. We know that $Q = 3$ optimizes the throughput when the packets are one slot long. But this property does not persist when the length of packet increases. For example, when all packets are of length M we can show expansion of the marginal maximum throughput, following the techniques of proposition 3 :

$$\mu_{\max} = 0.4114 + O(\frac{1}{M})$$

when $Q = 3$, and

$$\mu_{\max} = 0.3785 + O(\frac{1}{M})$$

when $Q = 4$. Thus the binary stack algorithm is better than the other generalized Q-ary versions. It is also posssible to improve the tree algorithms by an adjustment that "saves doomed slots" ([3]). The throughput is higher, within few percent, but the robustness to channel errors slightly decreases (risk of deadlocks).

In conclusion, we think that the basic binary stack algorithm, which we analyzed in this paper, enjoys nice properties such simplicity, robustness and stability, and certainly is the best alternative to the deficient binary exponential back-off protocol for Local Area Network communication.

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Figure 1

Curves $\lambda_{max} M = f(M)$ for two examples of packet length distribution.

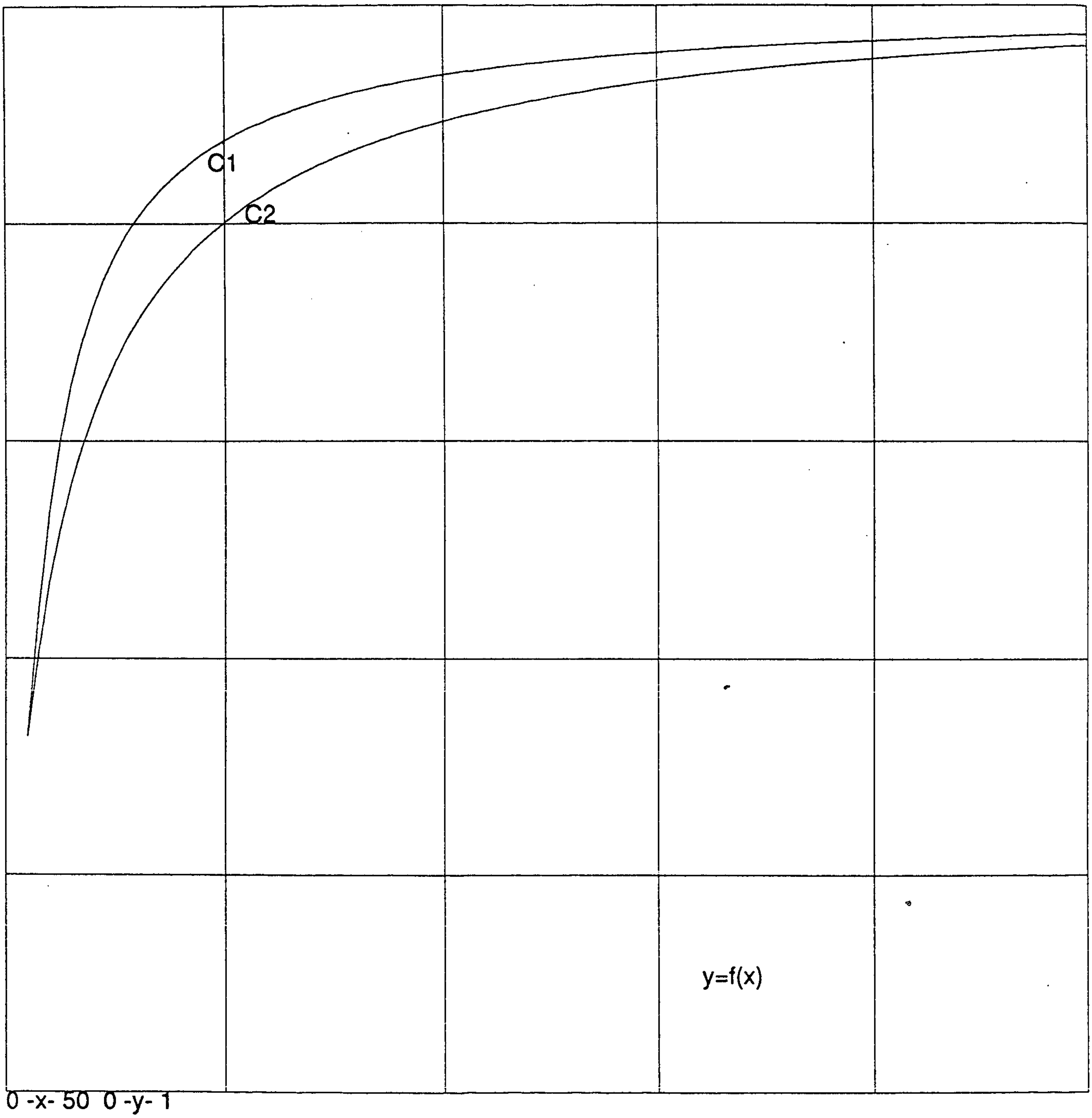

Figure 2

$E(L)$, $V(L)$, $E(W)$, and $V(W)$ while all packets have a length of ten slots.

$M\lambda$, p	0.25	0.35	0.4	0.48	0.5	0.52	0.56	0.58	0.6	0.75
0.01	1.010 .1037 1.005 e1 .4487	1.010 .1038 1.005 e1 .4257	1.010 .1038 1.005 e1 .4190	1.010 .1038 1.005 e1 .4112	1.010 .1031 1.005 e1 .4541	1.010 .1038 1.005 e1 .4081	1.010 .1038 1.005 e1 .4054	1.010 .1038 1.005 e1 .4041	1.010 .1038 1.005 e1 .4029	1.010 .1037 1.005 e1 .3961
0.10	1.111 1.467 1.065 e1 7.138	1.111 1.467 1.063 e1 6.537	1.111 1.467 1.062 e1 6.364	1.111 1.467 1.062 e1 6.167	1.111 1.383 1.062 e1 7.063	1.111 1.467 1.062 e1 6.093	1.111 1.467 1.061 e1 6.029	1.111 1.467 1.061 e1 6.001	1.111 1.467 1.061 e1 5.975	1.111 1.467 1.062 e1 5.854
0.20	1.254 4.449 1.159 e1 2.350 e1	1.253 4.440 1.152 e1 2.101 e1	1.253 4.439 1.149 e1 2.029 e1	1.253 4.439 1.147 e1 1.950 e1	1.253 3.975 1.147 e1 2.278 e1	1.253 4.439 1.147 e1 1.920 e1	1.253 4.439 1.147 e1 1.895 e1	1.253 4.439 1.147 e1 1.884 e1	1.253 4.439 1.147 e1 1.874 e1	1.254 4.449 1.151 e1 1.831 e1
0.30	1.446 1.065 e1 1.297 e1 5.936 e1	1.443 1.058 e1 1.278 e1 5.219 e1	1.442 1.056 e1 1.273 e1 5.018 e1	1.441 1.055 e1 1.268 e1 4.796 e1	1.441 9.018 1.267 e1 5.617 e1	1.441 1.055 e1 1.267 e1 4.715 e1	1.442 1.056 e1 1.266 e1 4.648 e1	1.442 1.056 e1 1.266 e1 4.619 e1	1.442 1.056 e1 1.267 e1 4.593 e1	1.446 1.065 e1 1.278 e1 4.495 e1
0.40	1.721 2.458 e1 1.506 e1 1.419 e2	1.711 2.415 e1 1.465 e1 1.230 e2	1.708 2.406 e1 1.455 e1 1.178 e2	1.707 2.401 e1 1.445 e1 1.123 e2	1.706 1.957 e1 1.444 e1 1.295 e2	1.707 2.401 e1 1.443 e1 1.103 e2	1.707 2.403 e1 1.443 e1 1.088 e2	1.708 2.404 e1 1.443 e1 1.082 e2	1.708 2.406 e1 1.444 e1 1.076 e2	1.721 2.458 e1 1.470 e1 1.065 e2
0.50	2.153 6.063 e1 1.847 e1 3.587 e2	2.122 5.827 e1 1.764 e1 3.048 e2	2.115 5.780 e1 1.743 e1 2.908 e2	2.110 5.750 e1 1.724 e1 2.767 e2	2.110 4.459 e1 1.722 e1 3.076 e2	2.110 5.750 e1 1.720 e1 2.721 e2	2.111 5.759 e1 1.720 e1 2.689 e2	2.113 5.768 e1 1.721 e1 2.677 e2	2.115 5.780 e1 1.723 e1 2.668 e2	2.153 6.063 e1 1.784 e1 2.725 e2
0.60	2.933 1.816 e2 2.481 e1 1.089 e3	2.839 1.663 e2 2.296 e1 8.900 e2	2.817 1.632 e2 2.251 e1 8.427 e2	2.802 1.612 e2 2.213 e1 7.996 e2	2.801 1.180 e2 2.209 e1 8.309 e2	2.802 1.612 e2 2.207 e1 7.882 e2	2.807 1.618 e2 2.208 e1 7.824 e2	2.811 1.624 e2 2.211 e1 7.815 e2	2.817 1.632 e2 2.217 e1 7.821 e2	2.933 1.816 e2 2.369 e1 8.589 e2
0.70	4.789 8.836 e2 4.015 e1 5.314 e3	4.412 7.076 e2 3.484 e1 3.860 e3	4.331 6.750 e2 3.365 e1 3.572 e3	4.277 6.541 e2 3.272 e1 3.350 e3	4.275 4.447 e2 3.263 e1 3.096 e3	4.277 6.541 e2 3.259 e1 3.315 e3	4.294 6.607 e2 3.267 e1 3.324 e3	4.310 6.668 e2 3.280 e1 3.346 e3	4.331 6.750 e2 3.299 e1 3.381 e3	4.789 8.836 e2 3.788 e1 4.488 e3
0.80	1.493 e1 2.814 e4 1.247 e2 1.664 e5	1.070 e1 1.089 e4 8.276 e1 6.019 e4	1.003 e1 9.107 e3 7.596 e1 4.927 e4	9.617 8.110 e3 7.138 e1 4.299 e4	9.601 4.960 e3 7.107 e1 3.201 e4	9.617 8.110 e3 7.104 e1 4.278 e4	9.748 8.415 e3 7.189 e1 4.433 e4	9.869 8.703 e3 7.281 e1 4.589 e4	1.003 e1 9.107 e3 7.411 e1 4.810 e4	1.493 e1 2.814 e4 1.163 e2 1.583 e5
0.85	... ...	4.227 e1 6.827 e5 3.238 e2 3.782 e6	3.215 e1 3.063 e5 2.405 e2 1.672 e6	2.758 e1 1.963 e5 2.018 e2 1.059 e6	2.743 e1 1.105 e5 2.000 e2 6.511 e5	2.758 e1 1.963 e5 2.007 e2 1.058 e6	2.891 e1 2.250 e5 2.100 e2 1.216 e6	3.022 e1 2.558 e5 2.197 e2 1.387 e6	3.215 e1 3.063 e5 2.341 e2 1.668 e6	... ...

Figure 3

$E(L)$, $V(L)$, $E(W)$, and $V(W)$ while all packets have a length of 2 or 18 slots.

$M\lambda, p$	0.25	0.35	0.4	0.48	0.5	0.52	0.56	0.58	0.6	0.75
0.01	1.010 .1697 1.008 e1 6.511 e1	1.010 .1697 1.008 e1 6.509 e1	1.010 .1697 1.008 e1 6.508 e1	1.010 .1697 1.008 e1 6.507 e1	1.010 .1691 1.008 e1 6.512 e1	1.010 .1697 1.008 e1 6.507 e1	1.010 .1697 1.008 e1 6.506 e1	1.010 .1697 1.008 e1 6.506 e1	1.010 .1697 1.008 e1 6.506 e1	1.010 .1697 1.008 e1 6.505 e1
0.10	1.111 2.351 1.104 e1 8.080 e1	1.111 2.349 1.101 e1 7.985 e1	1.111 2.349 1.100 e1 7.958 e1	1.111 2.349 1.099 e1 7.928 e1	1.111 2.271 1.099 e1 8.047 e1	1.111 2.349 1.099 e1 7.917 e1	1.111 2.349 1.099 e1 7.908 e1	1.111 2.349 1.099 e1 7.904 e1	1.111 2.349 1.099 e1 7.901 e1	1.111 2.351 1.100 e1 7.893 e1
0.20	1.256 7.029 1.254 e1 1.161 e2	1.255 6.998 1.243 e1 1.119 e2	1.254 6.993 1.240 e1 1.108 e2	1.254 6.989 1.237 e1 1.095 e2	1.254 6.589 1.237 e1 1.143 e2	1.254 6.989 1.236 e1 1.090 e2	1.254 6.990 1.236 e1 1.087 e2	1.254 6.991 1.236 e1 1.085 e2	1.254 6.993 1.236 e1 1.084 e2	1.256 7.029 1.242 e1 1.082 e2
0.30	1.453 1.673 e1 1.473 e1 1.891 e2	1.449 1.653 e1 1.446 e1 1.770 e2	1.448 1.649 e1 1.438 e1 1.736 e2	1.447 1.647 e1 1.432 e1 1.700 e2	1.447 1.522 e1 1.431 e1 1.833 e2	1.447 1.647 e1 1.430 e1 1.688 e2	1.447 1.648 e1 1.430 e1 1.678 e2	1.447 1.648 e1 1.430 e1 1.674 e2	1.448 1.649 e1 1.430 e1 1.671 e2	1.453 1.673 e1 1.448 e1 1.671 e2
0.40	1.743 3.876 e1 1.808 e1 3.526 e2	1.728 3.779 e1 1.749 e1 3.198 e2	1.725 3.759 e1 1.733 e1 3.111 e2	1.722 3.746 e1 1.719 e1 3.020 e2	1.722 3.400 e1 1.718 e1 3.334 e2	1.722 3.746 e1 1.717 e1 2.991 e2	1.723 3.750 e1 1.716 e1 2.969 e2	1.724 3.754 e1 1.717 e1 2.962 e2	1.725 3.759 e1 1.719 e1 2.956 e2	1.743 3.876 e1 1.762 e1 2.987 e2
0.50	2.215 9.774 e1 2.367 e1 7.863 e2	2.171 9.270 e1 2.238 e1 6.874 e2	2.161 9.168 e1 2.206 e1 6.626 e2	2.153 9.101 e1 2.179 e1 6.388 e2	2.153 8.111 e1 2.176 e1 7.077 e2	2.153 9.101 e1 2.174 e1 6.319 e2	2.156 9.122 e1 2.175 e1 6.277 e2	2.158 9.141 e1 2.178 e1 6.266 e2	2.161 9.168 e1 2.182 e1 6.262 e2	2.215 9.774 e1 2.289 e1 6.546 e2
0.60	3.121 3.123 e2 3.457 e1 2.364 e3	2.979 2.781 e2 3.146 e1 1.938 e3	2.948 2.712 e2 3.073 e1 1.843 e3	2.926 2.668 e2 3.014 e1 1.762 e3	2.925 2.327 e2 3.009 e1 1.903 e3	2.926 2.668 e2 3.006 e1 1.745 e3	2.933 2.682 e2 3.011 e1 1.740 e3	2.939 2.695 e2 3.019 e1 1.743 e3	2.948 2.712 e2 3.031 e1 1.750 e3	3.121 3.123 e2 3.319 e1 1.998 e3
0.70	5.580 1.890 e3 6.437 e1 1.406 e4	4.936 1.380 e3 5.361 e1 9.536 e3	4.805 1.292 e3 5.134 e1 8.728 e3	4.718 1.236 e3 4.968 e1 8.162 e3	4.714 1.047 e3 4.955 e1 8.228 e3	4.718 1.236 e3 4.952 e1 8.105 e3	4.745 1.254 e3 4.979 e1 8.184 e3	4.771 1.270 e3 5.009 e1 8.278 e3	4.805 1.292 e3 5.052 e1 8.416 e3	5.580 1.890 e3 6.152 e1 1.259 e4
0.80	3.755 e1 5.527 e5 4.532 e2 4.183 e6	1.648 e1 5.245 e4 1.848 e2 3.748 e5	1.456 e1 3.718 e4 1.601 e2 2.613 e5	1.347 e1 3.005 e4 1.455 e2 2.082 e5	1.343 e1 2.422 e4 1.448 e2 1.822 e5	1.347 e1 3.005 e4 1.450 e2 2.078 e5	1.380 e1 3.213 e4 1.486 e2 2.226 e5	1.412 e1 3.418 e4 1.522 e2 2.373 e5	1.456 e1 3.718 e4 1.572 e2 2.590 e5	3.755 e1 5.527 e5 4.320 e2 4.200 e6
0.85	... ...	... ...	... ...	4.709 e2 1.277 e9 5.160 e3 9.233 e9	4.193 e2 7.110 e8 4.585 e3 5.155 e9	4.709 e2 1.277 e9 5.142 e3 9.248 e9	... ...	... ...	... ...	... ...

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