Induction of open properties

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INDUCTION ON OPEN PROPERTIES (*)

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Abstract: *We show that proving properties by induction is a characteristic of noetherian orders, but that nevertheless one can prove a property by induction as soon as the set of elements over which it is true is open for the Scott topology. We also prove that two equivalent properties of complete orders can be parameterised by an arbitrary cardinal.*

RECURRENCE OUVERTE

Résumé : *On montre qu'une propriété caractéristique des ensembles noetheriens est d'y pouvoir prouver des propriétés par récurrence, mais qu'on peut prouver une propriété par récurrence dès qu'elle est ouverte pour la topologie de Scott. On prouve aussi qu'on peut paramétrer par un cardinal arbitraire deux caractérisations équivalentes des ensembles ordonnés complets.*

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OPEN INDUCTION

I. Noetherian induction

The so-called strong induction, for natural numbers, and transfinite induction using ordinals are old and powerful proof methods. Actually, they are two avatars of noetherian induction. Recall the definition of a noetherian order.

> **Definition.** *A partially ordered set is noetherian if it satisfies the following equivalent properties:*
> *(i) every strictly increasing chain is finite;*
> *(ii) every increasing chain is eventually constant;*
> *(iii) every non empty subset contains a maximal element.*

Well-founded orders are the opposite of noetherian orders: every non empty subset contains at least one minimal element. And a set is well ordered when it is totally ordered by a well founded order: every non empty set contains exactly one minimal element. Noetherian induction has been so far the most powerful way of proving properties inductively; it is indeed the most general in a precise sense.

> **Proposition 1.** *Let* E *be a partially ordered set. Say that a property* P *is "inductive" when it satisfies the following schema:*
> (I) $\forall y(\forall x(x > y \Rightarrow Px) \Rightarrow Py)$
> *Then* E *is noetherian if and only if all inductive properties are true on* E.

Proof: The necessity is known: If Q is the subset of elements that do not satisfy P, it must be empty; otherwise it should contain a maximal element q: All elements greater than q satisfy P. But by (I), q itself should satisfy P, contradiction. The sufficiency is proved by considering the property

Px = "*every strictly increasing chain starting at x is finite*".

This property is obviously inductive: it must therefore be true everywhere on E, and this is the definition of a noetherian order, QED.

It is well-known that finite products of noetherian partial orders are again noetherian, but that infinite products need not be. For instance, define $E_n = \{0,1\}$ for all natural number n. It is a finite set, a fortiori noetherian. But the product of all the E_n ordered componentwise admits the following strictly increasing chain:

$$0000... < 1000... < 1100... < 1110... < ...$$

We shall show that some properties can nevertheless be proved by induction on the product.

II. Alpha-complete partially ordered sets

We are interested in sets having lowest upper bounds, and therefore, subsets having the same lub are interesting to pinpoint. Here is a sufficient condition.

> **Definition.** *Two subsets of an ordered set are cofinal if every element of each one is bounded upper by some element of the other.*

The interest of cofinal subsets lies in the following proposition.

> **Proposition 2.** *Two cofinal subsets of a partially ordered set have same upper bounds.*

Proof: Let X and Y be the subsets. By symmetry, it is enough to show that any upper bound m of X is also an upper bound of Y. For every element y of Y, there exists an element x of X such that $y \leq x \leq m$. Therefore, m is an upper bound of Y, QED.

This is often used when Y is a subset of X. In particular, if $x \in X$, and X is directed (see below), the set $[x,\infty[= \{y \in X;\ x \leq y\}$ is cofinal in X and thus has same upper bounds as X.

Recall that in a partially ordered set E, a *directed* subset X is such that any two points of X have a common upper bound in X. If every directed subset has an upper bound, the partial order is complete. This notion can be parameterised by the maximum cardinality α of the directed sets that must have lowest upper bounds. Here, we define (1) an ordinal as a well-ordered set, (2) an α-sequence where α is an ordinal as a monotone function $\alpha \to E$ and (3) a cardinal as an ordinal which is minimal: it admits no one-to-one mapping onto a smaller ordinal.

> **Proposition 3.** *Let* E *be a partially ordered set and* α *a cardinal. Then the following assertions are equivalent in E:*
> (i) *Every* β*-sequence has a least upper bound, for all ordinal* $0<\beta\leq\alpha$.
> (ii) *Every non empty chain of cardinality at most* α *has a least upper bound.*
> (iii) *Every non empty directed set of cardinality at most* α *has a least upper bound.*

Proof: (iii) => (ii) because a directed set is a fortiori a chain, and (ii) => (i) because a β-sequence for $\beta\leq\alpha$ is a fortiori a chain of cardinality at most α. Conversely to prove (i) => (iii) consider a directed subset X together with a mapping $f : \alpha \to X$ which is one-to-one and onto. We define a cofinal sequence $g : \alpha \to X$ by transfinite induction on the ordinal $\beta < \alpha$ of g, and such that for all ordinal $\delta \leq \beta$, the relation $g(\delta) \geq f(\delta)$ holds: If $\beta = 0$ (X cannot be empty), then $g(0) = f(0)$. If $\beta = \delta + 1$, let $g(\beta)$ be an upper bound in X of $g(\delta)$ and $f(\beta)$ (this bound exists because X is directed). If β is a limit ordinal, then we have $\beta = \vee\, \delta$ for $\delta < \beta$, and g is defined for all $u < \beta$, the set $\{g(u);\ u < \beta\}$ has a least upper bound x, by induction hypothesis. Let $g(\delta)$ be an upper bound in X of x and $f(\beta)$. The sequence thus constructed is cofinal, because if $x \in X$, then

$x = f(\beta)$ for some $\beta < \alpha$, and we have $f(\beta) \leq g(\beta)$ by construction; therefore, it has same upper bounds as X, QED.

Definition. *An ordered set satisfying the equivalent conditions of the proposition above is α-complete.*

The proposition above is also true when empty subsets are allowed: It amounts to requiring a smallest element in E, and is actually the usual definition; but the sequel is true also when the smallest element does not exist and we shall not assume its existence.

The proposition is an extension of a well-known result when $\alpha = \mathbf{N}$. Actually, in the case where $\alpha = \mathbf{N}$, a stronger result is true, viz. every countable directed set contains a cofinal sequence. It is also a parameterisation by an ordinal of a classical result proved, for instance, in P. Cohn's Universal Algebra [1981], ch. V, prop. 9.

III. Open induction

Let E be a partial order. The intuition from topology is that a subset is closed when it contains all its limits.

Proposition 4. *Let a subset be closed when it contains the lub of its directed subsets, when they exist:*

$$X \text{ directed \& included in } F \Rightarrow \mathbf{V}X \in F$$

Then closed sets are preserved by intersection and finite unions, and thus define a topology, called the lower topology.

Proof: We must prove that any intersection of closed sets is a closed set, which is clear, and that the union of two closed sets F and G is again a closed set. Suppose that X is a directed subset of the union, then either $X \cap F$ is cofinal in F:

$$\forall x \in X (\exists y \in X \cap F) \quad x \leq y$$

Then $\mathbf{V}(X \cap F) = \mathbf{V}X$ belongs to F, because F is closed. Or else there exists an element x in X such that $x \leq y$ for no element $y \in F$. Then the subset $X \cap [x, \infty[$ of elements of X that are greater than x is cofinal in X and is included in G. Its lub therefore belongs to G, QED.

A function f : A -> B is continuous for this topology if and only if, for all directed set X having a lub $\mathbf{V}X$, it satisfies

$$f(\mathbf{V}X) = \mathbf{V}\, f(X).$$

This implies that the function is increasing by taking for X the set $\{x, y\}$ with $x < y$. Now the coarsest topology for which increasing functions are continuous is classical: Its closed sets are those subsets that are closed downwards and thus the order can easily be recovered from the topology. The Scott topology is its intersection with the lower topology defined in the proposition above. The lower topology is therefore finer than the Scott topology, but the order cannot be recovered from the lower topology: For instance, in a noetherian order, a singleton set is open and closed; the topology is discrete. But this would also be the case if the order had been reduced to the equality.

Open sets are complementary to closed sets. It is easy to check that their defining property is that they cannot contain the lub of a chain if they do not contain already some element of the chain. A property P is open when the set of elements that satisfy the property is open for the lower topology:

$$\text{(O)} \qquad P(\mathbf{V}X) \Rightarrow P(x) \text{ for some } x \in X, \text{ for all chain } X$$

For instance, in discrete topological spaces, and in particular in noetherian orders, every property is open. Recall from section II:

Definition. *A partially ordered set is complete if every non empty directed subset, or equivalently every non empty chain (cf. proposition above) admits a lub.*

Note that we do not require a smallest element: noetherian orders, for instance, are complete. Complete Partial Orders will be called cpos, as tradition has it.

Theorem. *In a cpo, an inductive and open property, i.e. a property satisfying* (I) *and* (O) *above, is true everywhere.*

Proof: Let Q be the negation of the property. We shall prove that Q is true for no x. Let us restate schemas (I) and (O) for Q:

$$\begin{array}{ll} \text{(I)} & Qy \Rightarrow \exists x(x > y \ \& \ Qx) \\ \text{(O)} & \{x;\ Qx\} \text{ is closed} \end{array}$$

We suppose Qx for some x and arrive at a contradiction. Consider the set of all chains of elements satisfying Q, ordered by set inclusion. It is not empty, since it contains the chain $\{x\}$. The union of a chain of such chains is a chain the elements of which satisfy Q. We can now apply Zorn's lemma: There is a maximal such chain M. Because M is in a cpo, it has lub y. Because Q is closed, Qy is true. Because M is maximal, it must contain y. Because of (I), there must exist some $x > y$ such that Qx. But then M is not maximal, contradiction, QED.

The simplest corollary concerns the well-known notion of a property of finite character.

Definition. *A property* P *defined on the subsets of a given set is of finite character if every subset satisfying* P *contains a finite subset satisfying* P.

Let E be a set and P a property defined on the subsets of E. Assign to the set **P**(E) of subsets of E the lower topology, or the Scott topology. An element of E cannot belong to the lub of an increasing chain, and not belong to some element in the chain. Therefore a property of finite character is a fortiori an open property and by the theorem, such a property can be proved by induction.

Another corollary, less straightforward is applied for instance, in the usual proofs of Kruskal's embedding theorems.

Proposition 5. *Suppose a property defined on a projective limit of noetherian partial orders, which is true if it is true on some component of the limit. Then if the property is inductive, it is true everywhere on the limit.*

Proof: If p_i : E -> E_i is the projection of the limit into its component, a basis of the topology on the limit consists of the finite intersections of open sets of the form $p_i^{-1}U_i$. Or again, finite conjunctions of properties defined on the components (and open on these components, not a demanding requirement with a discrete topology) are open, hence the result, QED.

This applies in particular to projective limits of finite spaces, over which any order is noetherian: in the proof above, U_i can be any subset of the component. A typical instance is when one wants to prove results on well-quasi-orders. Well-quasi-orders are well-founded orders that are not total, but in which non-empty subsets can only have a finite number of minimal elements. An equivalent statement is that any infinite sequence (t_n) of elements satisfies the following property:

$$\text{(W)} \qquad t_i \leq t_j \text{ for some } i < j$$

Infinite sequences are considered as projective limits of finite sequences with index set **N**: a finite sequence of length n yields a finite sequence of length n-1 by erasing its last element. If an infinite sequence satisfies the characteristic property (W) of well-quasi-orders, then a (finite) left factor of the sequence already satisfies (W). Therefore to prove (W), it suffices to prove that it is inductive for some quasi-order. For instance, in Kruskal's result on tree embeddings [1960], the finite sequences of trees are strictly ordered by lexicographic order on the cardinality of their trees, and the usual proof is actually a proof that (W) is inductive for the order on the limit.

For the sake of completeness, we include a definition of functions by open induction, although its usefulness is not clear at present.

Proposition 6. *If a function f* : E -> F *where* E *is a cpo has an inductive and open domain the function is defined everywhere.*

Proof: This is a direct corollary of the theorem.

A function has an inductive domain if it is defined by induction. Most functions have an open domain: this means that a function cannot be defined at the lub of a chain if it is not defined already somewhere along the chain. It is the case, for instance, of increasing functions into a noetherian order.

IV. Conclusion

We have stated few really new results; all stand on the border between ordered set theory and topology. One extreme is to forget about topology. Logicians, for instance, given a theorem concerning infinite objects, will test the expressive power of a finite version of it. See a discussion on the subject by C. Smorynski [1982] where it is noticed, in the case of Kruskal's embedding theorem, that "[...] the proof cited is higly nonconstructive. A direct constructive proof is a bit more difficult.... Quite a bit more difficult." To avoid these difficulties, people proving such results usually take the nonconstructive approach and add up some quantity of topology or ordered set theory, according to their inclination: They are led to prove the theorem above or its main corollary: proposition 5. But then, the proof would either be mixed up with the inductive proof of property P - yielding a deterrent piece of mathematics - or omitted, leaving doubts on the validity of the remaining induction. Further, there is little hope to mechanise a nonconstructive proof by contradiction, while existing theorem-provers happen to use induction.

Proposition 3 is the second useful result. It states precisely what everyone expects when the existence of limits is restricted. Many results in ordered set theory can be stated and proved in the language of topology, as witnessed by a recent book on continuous lattices by Gierz *et al.* [1980]. Accordingly, many authors will use general directed subsets (probably thinking of the associated filters), even when chains, or even α-chains, are easier to handle. Proposition 3 ensures that no generality is lost when subsets are restricted to the comfortable situation of total, or even well orders.

« Le défaut unique de tous les ouvrages
est d'être trop longs. »
(Vauvenargues)

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