Parallel manipulators. Part 2 : Theory. Singular configurations and Grassmann geometry

Jean-Pierre Merlet

PARALLEL MANIPULATORS
Part 2 : Theory
Singular configurations and Grassmann Geometry

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Résumé: Les manipulateurs parallèles sont des robots où les différents segments ne sont pas placés successivement à partir de la base vers la "main", mais au contraire tous connectés directement à la fois à la base et à la main. Ils présentent un grand intérêt pour la réalisation d'opérations mécaniques ainsi qu'une alternative aux articulations classiques utilisées dans les manipulateurs actuels. Nous avons exposé dans une première partie les propriétés cinématiques et cinétiques des manipulateurs parallèles. Nous étudions ici la possibilité éventuelle d'avoir plusieurs positions de la main pour des longueurs fixées des segments. Dans ce cas le manipulateur devient non rigide: on est dans une configuration singulière. Le manipulateur n'est alors plus commandable. On sait que trouver ces positions revient à la recherche des racines du déterminant de la jacobienne. Mais cette méthode n'est pas utilisable ici vu la complexité de cette matrice. Nous proposons une autre approche basée sur la géométrie des lignes de Grassmann (parfois appelée aussi géométrie des lignes de Plücker). Si l'on considére que l'ensemble des lignes de P^3 présente une structure de variété de rang au plus 6 on montre qu'une structure telle qu'un robot parallèle sera non rigide si la variété engendrée par les lignes associées aux segments est dégénérée. L'intérêt de cette approche est que l'on sait caractériser **géométriquement** toutes les cas de dégénerescence. On cherchera donc à trouver les configurations du manipulateur qui satisfont à ces conditions.

Dans cette étude partielle on montre des conditions de singularité inédites. Les cas restant à étudier sont exposés.

Summary: Parallel manipulators have a specific mechanical architecture where all the links are connected both at the basis and at the gripper of the robot. This kind of manipulator has a better positionning ability than the classical robot. We have addressed in the first part of this paper the direct kinematics problems . We have shown that it may be suspected that in some case for a given set of links lengths more than one solution may be given to this problem : we have thus a singular configuration for the manipulator. To determine these singular configurations the classical method is to find the roots of the determinant of the jacobian matrix. However in our case the jacobian matrix is too complicated to find these roots. We propose here a new method based on Grassmann line-geometry. If we consider the set of lines of P^3, it constitutes a linear variety of rank 6. It can be easily shown that a singular configuration is obtained when the variety spanned by the lines associated to the robot links has a rank less than 6. An important feature of this geometry is that each degeneracy case can be described by simple **geometric** features.

Thus the difficult problem is partitionned in a small number of geometric problems.

Although all the cases has not been treated yet we propose new singular configurations found with this method.

1 Parallel manipulator

1.1 Introduction

We deal here with the study of fully parallel manipulator like the model presented in Figure 1.

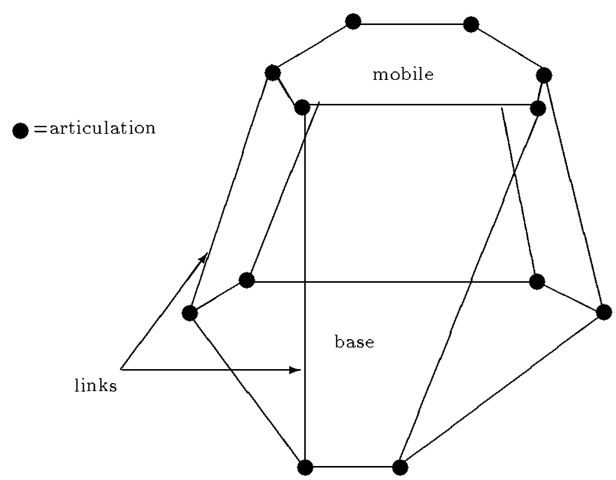


Figure 1: fully parallel manipulator: the links have a variable length

Basically it consists in two plates connected by 6 articulated links. In the following chapters the smaller plate will be called the *mobile* and the greater (which is in general fixed) will be called the *base*. In each link there is at least one actuator. In the case of one actuator by link we have a so-called *fully parallel* manipulator.

Some manipulator of this type have been designed or studied since a long time. The first one, to the author's knowledge, was designed for testing tyres (see Mc Gough in Stewart paper [16]). But the main use of this mechanical architecture consists in the flight simulator (see for example Stewart [16], Watson [17], Baret [15]). The first design as a manipulator system has been done by Mac Callion in 1979 for an assembly workstation [5] but Minsky [9] has presented in the early 70's some design related to various mechanical architectures. Some other researcher have also addressed this problem: Reboulet [11], Inoue [4], Tanaka (as a platform for movies camera!) [12], Fichter [1],Yang [13], Mohamed [8], Zamanov [14]. Even a commercial manipulator was sold by Marconi under the name "Gadfly" for the assembly of integrated circuits. This kind of manipulator have a great positionning ability and are very convenient for force-feedback command (see Merlet [6]). A prototype of parallel manipulator is currently under development at INRIA. The architecture of this manipulator is based on the prototype developed at the CERT-DERA laboratory in Toulouse with a design which has been modified by the author. In the first part of this paper we have presented various features and problem involved by the parallel architecture. We will deal here with the special problem of singular configurations.[1]

1.2 Notation

We introduce the absolute frame R with origin C and a relative frame R_b fixed to the mobile with origin O (see Figure 2). The rotation matrix relating a

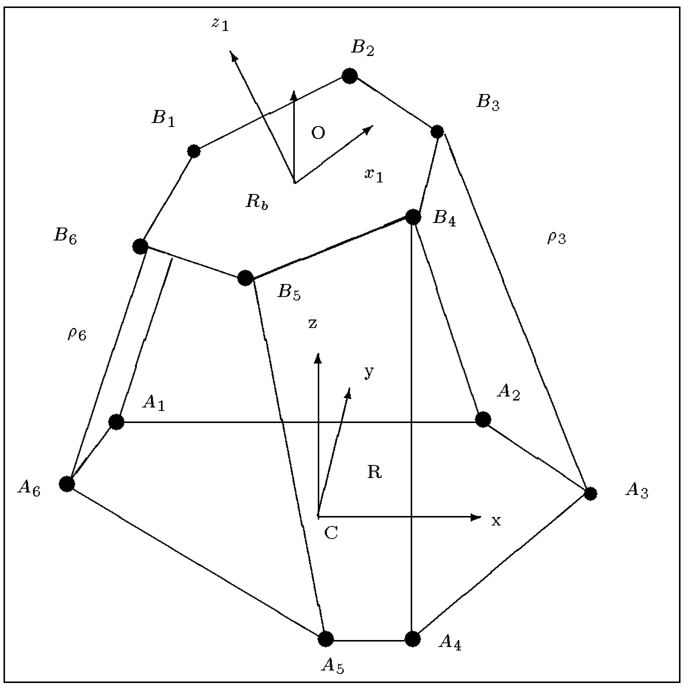


Figure 2: notation

vector in R_b to the same vector in R will be denoted by M with:

$$M = \begin{pmatrix} v1 & v2 & v3 \\ v4 & v5 & v6 \\ v7 & v8 & v9 \end{pmatrix}$$

[1]A lot of calculation (in particular the results presented in the Appendices) has been obtained with the aid of MACSYMA [26], a large symbolic manipulation program developed at MIT or with REDUCE.

The Eulers's angles can be used with the rotation matrix Me :

$$Me = \begin{pmatrix} cos\psi cos\phi - sin\psi cos\theta sin\phi & -cos\psi sin\phi + sin\psi cos\theta cos\phi & sin\psi sin\theta \\ sin\psi cos\phi + cos\psi cos\theta sin\phi & cos\psi cos\theta cos\phi - sin\psi sin\phi & -cos\psi sin\theta \\ sin\theta sin\phi & sin\theta cos\phi & cos\theta \end{pmatrix}$$

with ψ, θ, ϕ the Euler's angles.

The centers of the articulations on the base for link i will be denoted A_i and those on the mobile B_i. The length of link i will be noted ρ_i , and the unit vector of this link $\mathbf{n_i}$. The coordinates of A_i in frame R are (xa_i, ya_i, za_i) , the coordinates of B_i in frame R_b are (x_i, y_i, z_i) and the coordinates of O, the origin of the relative frame, (x_o, y_o, z_o).

For the sake of simplicity the subscript i is omitted whenever it is possible and vectors will be noted in **bolt** character. A vector which coordinates are expressed in the relative frame will be denoted by the subscript $_r$.

We will restrict our study in three particular cases of fully parallel manipulators which are the most currently used.

The first one is the case where all the articulation points of both the base and the mobile lie in a plane and are symmetric along one axe (see Figure 3). The articulation points are supposed to be on circles with a base radius greater than the mobile radius. The mobile is homotetic to the base and is rotated at 180 degrees for the connection of the links. In this case, without loss of generality, we will define R such that $za_i = 0$ and R_b such that $z_i = 0$. The symmetry axes will be used as an axe of each frame R, R_b. We exclude the case where three or more articulation points are collinear. We will call this architecture the simplified symmetric manipulator (SSM). The second

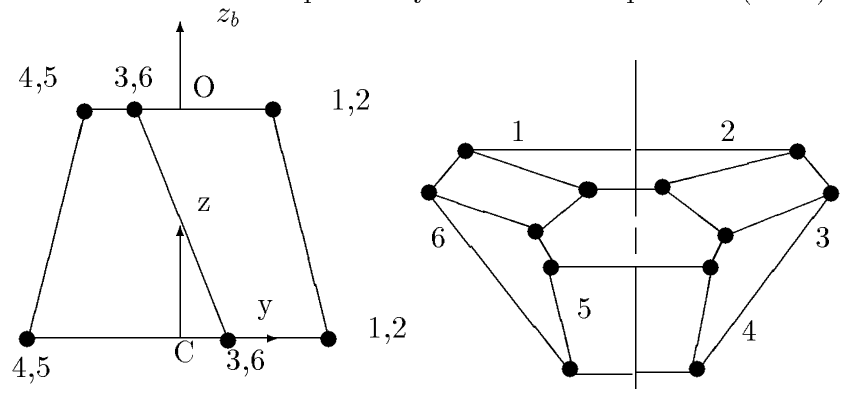


Figure 3: simplified symmetric manipulator SSM (side view, top view)

interesting design, the triangular simplified symmetric manipulator (TSSM),

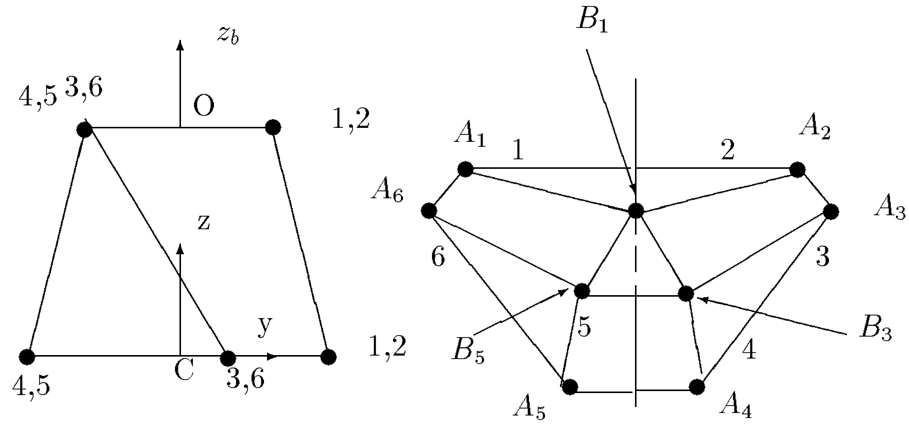


Figure 4: triangular simplified symmetric manipulator TSSM

is presented on Figure 4. In this case the articulation points on the mobile are located only in three different positions. The last interesting design is presented in Figure 5 : basically it is a further simplification of the TSSM where the couples of articulations on the base have the same center of rotation: we will call this manipulator the minimal simplified symmetric manipulator (MSSM). It must be noted that a practical realisation of a MSSM is very difficult.

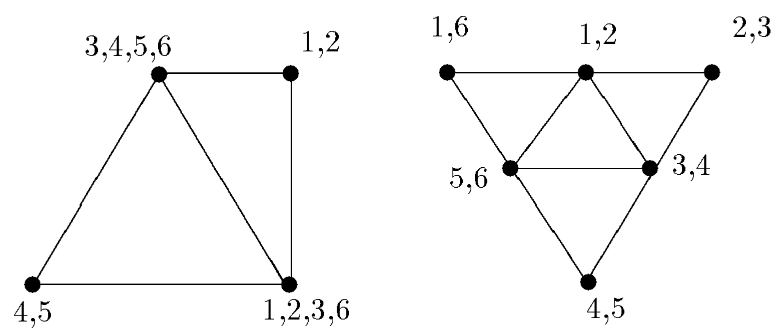


Figure 5: the minimal simplified symmetric manipulator (MSSM)

1.3 Singular configurations and the jacobian matrix

The fundamental relations relating the links lengths to the position and orientation of the mobile is:

$$\begin{aligned}\rho^2 &= (x_o - x_a + x.v1 + y.v2 + z.v3)^2 + (y_o - y_a + x.v4 + y.v5 + z.v6)^2 \\ &\quad +(z_o - z_a + x.v7 + y.v8 + z.v9)^2 \qquad (1)\end{aligned}$$

Thus we have to solve a system of 6 non-linear equations of type (1) to get the position of the mobile x_o, y_o, z_o and its orientations from the links lengths. The solution is unique if the rank of the jacobian matrix J of this system is equal to 6 with:

$$J = ((\frac{\partial \rho}{\partial \mathbf{q}})) \qquad (2)$$

with $\mathbf{q}$ the position and orientation parameters vector. Note that this matrix is in fact the inverse jacobian (in a robotics sense) of the manipulator. The symbolic computation of the determinant of J is rather tedious (see [7] for the formulation of this determinant). Mac Callion [5] used a numerical deflation method to find all the roots of the determinant. Mac Callion have found up to nine roots to this determinant all outside the range of the links lengths and Hunt has shown that there can be up to 16 roots [3]. Thus we need another method to find the singular configurations of the parallel manipulator.

2 Plücker coordinates of lines,rigidity and geometry

It is well known that a line can be described by its Plücker coordinates. Let us introduce briefly these coordinates. We consider two points on a line, say M_1 and M_2, and a reference frame R_0 which origin is O (see Figure 6). Let us consider now the two three dimensional vectors $\mathbf{S}$ and $\mathbf{M}$ defined by :

$$\mathbf{S} = \mathbf{M_1M_2}$$

$$\mathbf{M} = \mathbf{OM_1} \wedge \mathbf{OM_2} = \mathbf{OM_2} \wedge \mathbf{S} = \mathbf{OM_1} \wedge \mathbf{S}$$

If we assemble these vectors to form a 6-dimensional vector we get the vector $\mathbf{U}$ of the Plücker-coordinates of this line.

$$\mathbf{U} = [S_x, S_y, S_z, M_x, M_y, M_z]$$

It is useful to introduce the normalized vector $\mathbf{U}'$ defined by :

$$\mathbf{U}' = \frac{\mathbf{U}}{||\mathbf{S}||} = \left[S'_x, S'_y, S'_z, M'_x, M'_y, M'_z\right]$$

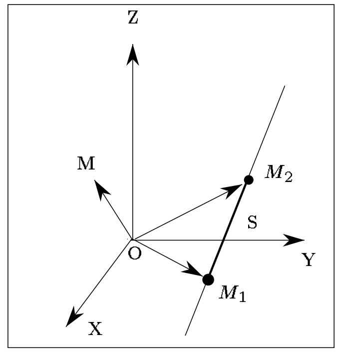


Figure 6: Plücker coordinates

It may be seen that the first three components of this vector are the components of the unit vector $\mathbf{n_i}$ of the line. The last three components are given by :

$$\mathbf{OM} \wedge \mathbf{n_i}$$

M being any point of the line. We will show now that the matrix P constituted by

$$P = ((U'_1, U'_2, ...U'_6))$$

,where U'_i is the coordinate vector of line i, is identical to the transpose of the matrix J. We introduce the articular force vector $\mathbf{f}$ i.e. the axial forces which are exerted by the links on the mobile plate. Let us consider now the external force vector $\mathbf{F}$ and torque vector $\mathbf{M}$ about the origin C acting on the mobile and let $\mathcal{T} = [\mathbf{F}, \mathbf{M}]$ be the generalized force vector. It is well known that :

$$\mathcal{T} = J^T \mathbf{f}$$

The system being in equilibrium we may write :

$$\sum_{i=1}^{i=6} f_i \mathbf{n_i} = \sum_{i=1}^{i=6} f_i \mathbf{S'_i} = \mathbf{F}$$

$$\sum_{i=1}^{i=6} \mathbf{CB_i} \wedge f_i \mathbf{n_i} = \mathbf{M}$$

where $\wedge$ denotes the cross product. If we choose C as origin for the reference point of the Plücker frame we are able to write from the two previous equations:

$$P\mathbf{f} = \mathcal{T} \tag{3}$$

and therefore :

$$J^T = P$$

Equation 3 is a linear system of equations in the unknown f_i. If the system is rigid this mean that whatever are the generalized forces it exists one solution to this system (and in our case the solution will be unique). It will be true if the matrix P is of full rank which is equivalent to say that the Plücker vectors are linearly independent.

Let us assume now that these vectors belong to a vector space V_6 and we consider the one-dimensional subspaces of V_6 as points of a projective P_5. Then every line g in P_3 corresponds to exactly one point $\mathbf{G}$ in P_5.

It is well known that point $\mathbf{G}$ belongs to a quadric Q_p (see [21], [25], [18]). Indeed we have for every line of P_3 :

$$\mathrm{S_x M_x + S_y M_y + S_z M_z} = 0$$

This equation defines the quadric Q_p which is called the *Grassmannian* or the *Plücker quadric.* At this point we have defined a one-to-one relation between the set of lines in the real P_3 and the quadric Q_p in P_5. The rank of this mapping is 6 (there is at most 6 independent Plücker vectors).

Let us consider now the various sub-spaces of P_5 (or more precisely their intersection with Q_p). We get various varieties which rank ranges from 0 to 6. As a matter of example a point in P_5 (rank=1) corresponds to a line in P_3. As for Q_p (which represents the set of line of P_3) it is defined through 6 linearly independent Plücker vectors and is therefore of rank 6.

Let us come back to the rigidity of a parallel manipulator. We have seen that this manipulator is rigid (and therefore not in a singular configuration) if and only if the 6 lines are linearly independent. Therefore any subset spanned by n lines must have a rank equal to n. At this point the problem is far to be solved because we are not able to find the generalized coordinates of the mobile for which there is a linear dependency between the n Plücker vectors. But we will see that these dependencies can be described by geometric considerations.

3 Grassmann Geometry

The varieties of lines has been studied by H. Grassmann (1809-1877). The purpose of this study was to find geometric characterization of each varieties. We will introduce now the various results which can be found in [22] or with more mathematical justifications in [25].

Let us begin with the varieties of rank 0 through 3 (Figure 7) We have first the empty set of rank 0. Then the *point* (rank=1) which is a line in the 3D

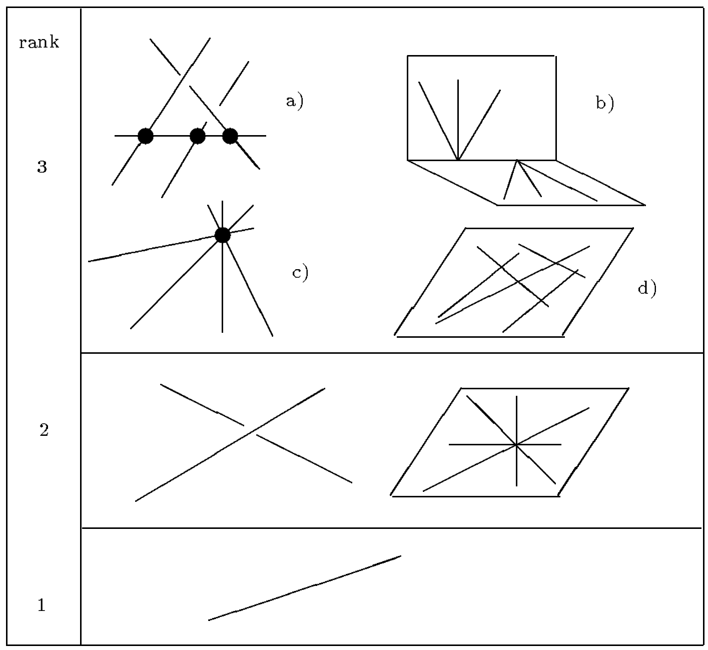


Figure 7: Grassmann varieties of rank 1,2,3

space. The *lines*(rank=2) are either a pair of skew lines in R^3 or a flat pencil of lines: those lying in a plane and passing through some point on that plane.

The *planes* (rank=3) are of four types:

-all lines in a plane (3d)

- all lines through a point (3c)

- the union of two flat pencils having a line in common but lying in distincts planes and with distinct centers (3b)

-a regulus (3a)

Let us define the regulus. Let three skew lines in space and consider the set of lines which pass through the three lines : this set of line build a surface which is an hyperboloïd of one sheet (a quadric surface, Figure 8) and is called a **regulus**. Each line belonging to the regulus is called a *generator* of the regulus.

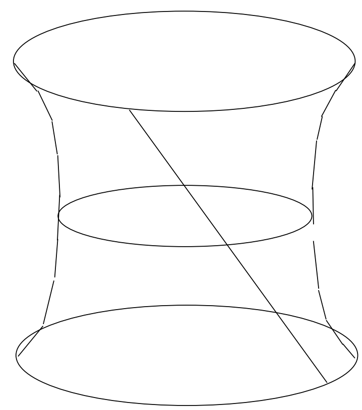


Figure 8: hyperboloid of one sheet

It is shown in [23],[25] that this surface is *doubly ruled*. This mean that it exists two reguli (a regulus and its "complementary" regulus) which generate the same surface or that each point on the surface is on more than one line. The only double ruled surface are the plane, the hyperboloïd of one sheet and the hyperbolic paraboloïd.

Let us come back to the regulus: we have seen that there are two families of straight lines on the hyperboloïd and each family covers the surface completely. A line on this surface is dependent of the lines of either the regulus or the complementary regulus. An interesting property is that a line of one family intersects all the lines of the other family and that any two lines of the same family are mutually skew (see [24] for the hairy details).

Another interesting property of the generators is that their equations are easily written if we use homogeneous coordinates (x_0, x_1, y_0, y_1) based on a pair of skew lines X,Y which contain the reference points X_0, X_1, Y_0, Y_1. As a matter of example let us consider two points A_1, B_1 belonging to X and A_2, B_2 belonging to Y. The homogeneous coordinates of a point M are x_0, x_1, y_0, y_1 such that:

$$\mathbf{OM} = x_0\mathbf{OA_1} + x_1\mathbf{OB_1} + y_0\mathbf{OA_2} + y_1\mathbf{OB_2} \tag{4}$$

We shall write x (resp. y) for the column vector of the x_i (resp. y_i). We note that any line which is skew to Y has a matrix equation of the form:

$$y = Ax \tag{5}$$

where A is a uniquely defined 2x2 matrix of constants. Let us consider a regulus R with generators X, Y, and a third line L. The matrix equation of L with respect to the reference frame X, Y is

$$y = A_L x \tag{6}$$

An important result is that any generator of R has a matrix equation of the form:

$$y = \lambda A_L x \tag{7}$$

where λ is a real constant

Let us describe now the varieties of higher rank of the Grassmann geometry (Figure 9). Varieties of dimension 4 are called *congruences* and are of four types:

- a linear spread generated by 4 skew lines i.e.no lines meet the regulus generated by the three others lines in a proper point (*elliptic congruence*, 4a)
- all the lines concurrent with two skew lines (*hyperbolic congruence*, 4b)
- a one-parameter family of flat pencil, having one line in common and forming a variety (*parabolic congruence*, 4c)
- all the lines in a plane or passing through one point in that plane (*degenerate congruence*, 4d)

Varieties of dimension 5 are called *complexes* and are of two types:

- *non singular*: generated by 5 independent skew lines (5a)
- *singular* (or *special*): all the lines meeting one given line (5b)

The complexes may be defined to be the set of lines which are dependent of a skew pentagon. An interesting property of the special complex is that four of the vertices of the skew pentagon are coplanar. As for a general complex its geometric characterization is that through any point of the space there is one and only one flat pencil of line such that all the lines which belong to the pencil belong also to the complex. In other words all the lines of a complex which are coplanar intersect one point.

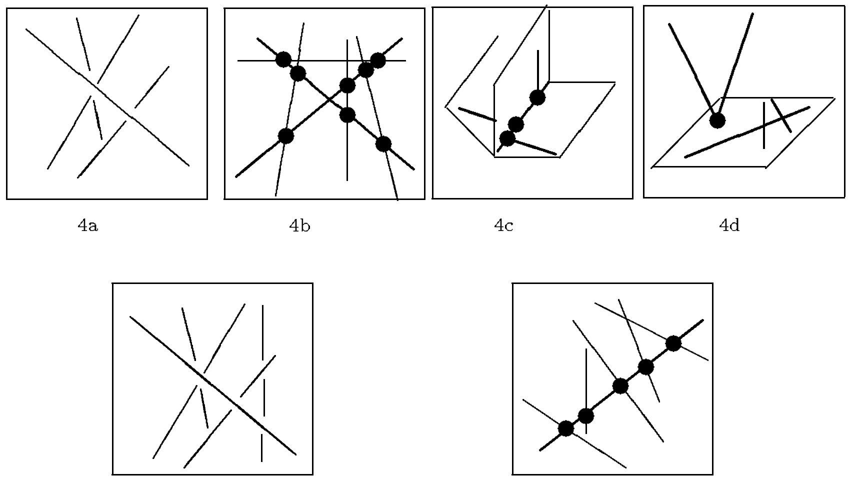


Figure 9: Grassmann varieties of rank 4,5

4 An example: the 2D parallel manipulator

Let us consider a basic example: a 2D parallel manipulator (Figure 10) In this

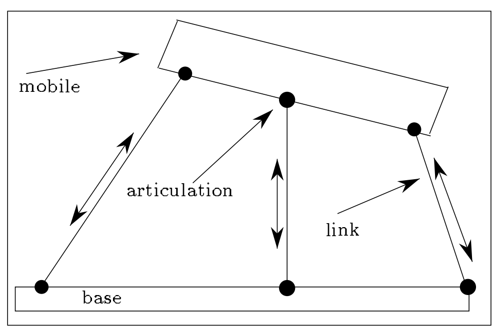


Figure 10: 2D parallel manipulator

case we have three segments and these three lines must constitute a variety of Grassmann of rank 3. Thus we will consider any subset of 1,2,3 segments and determine the condition for which any such subset has a rank 1,2,3.

The case of 1 and 2 segments are rather trivial: for one line we have only to verify that this line exists and for two lines that the lines are distincts. This is clearly the case if we except the configuration where the base and the mobile are collinear.

We will consider now the whole system of three bars. By reference to Figure 7 we can see that the only possibility for a system of three bars to be a 2-rank Grassmann variety is obtained when the three lines cross the same point (Figure 11). In particular if the mobile and the base are homotetic we

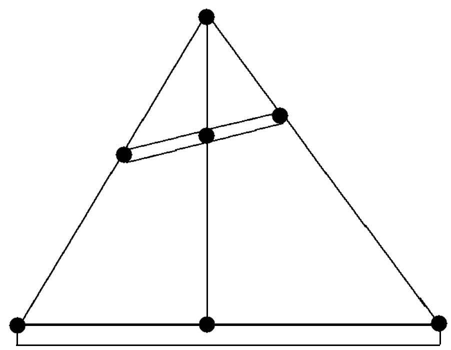


Figure 11: singular configuration for the 2D parallel manipulator

get a rather disturbing singular configuration when the base and the mobile are parallel, whatever is their relative position.

Another design is straightforward to avoid the above singular configuration (Figure 12).

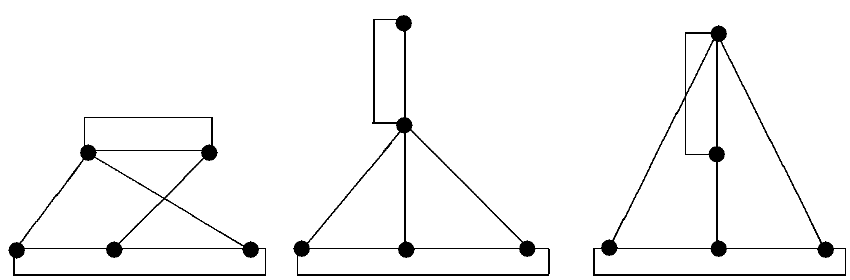


Figure 12: a 2D parallel manipulator and its two singular configurations

We can see here that practically, whatever is the position and orientation of the mobile the three segments cannot cross the same point. We see on the figure the two possible singular configurations which cannot be reached by a real manipulator. Furthermore this design is also interesting because we can solve the difficult direct kinematics problem i.e. find the position and orientation of the mobile for a given set of links lengths.

5 Study of the MSSM

5.1 Introduction

For the general case we will make two assumptions :

- a segment cannot lie in the plane of the base . A consequence of this assumption is that the mobile and the base cannot be coplanar.
- the links lengths cannot be equal to zero (i.e. an articulation point on the mobile and the base cannot be in the same location).

We will begin the study of he rigidity of the parallel manipulator by the simplest case: the MSSM (Figure 13).

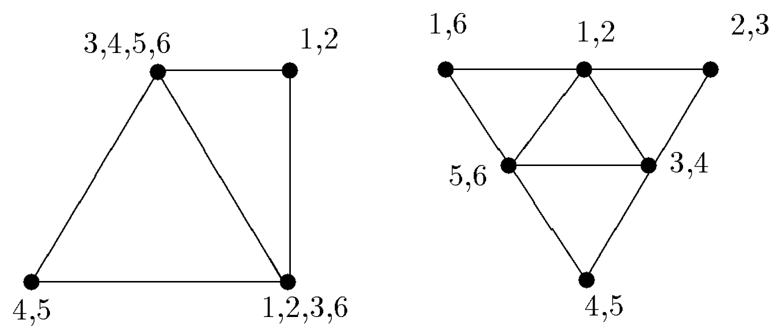


Figure 13: the minimal simplified symmetric manipulator (MSSM)

5.2 Case by case study

5.2.1 Subsets of 2,3 bars

Subset of 2 bars In the case of 2 bars we have only to count if there is 2 distincts bars in our mechanism. Thus this verification is trivial.

Subset of 3 bars In the case of 3 bars the rank of the system is 2 if the lines belong to a flat pencil of lines i.e. are in a plane and pass through some point on that line.

Let us notice first that for any set of 3 lines which may be coplanar we have two lines which have a common articulation point on the mobile (say B_i), and the third line has a common articulation point on the base (say A_j) with one of the preceding lines. For example consider the set 1,2,3: lines 1,2 have a common point on the mobile (B_1) and lines 2,3 a common point on the base (A_2).

All the lines being distincts it is thus impossible that three lines have a common intersection point except if the base and the mobile are coplanar or if the articulation point on the base and the mobile have the same location (but this configuration cannot be reached by a real manipulator). In fact the coplanarity of the base and the mobile is a general case of singularity which is excluded in practical applications.

Thus 3 lines of a MSSM cannot constitute a flat pencil of line.

5.2.2 Subsets of 4 bars

Degeneracy of type 3d In this case four lines are coplanar.

The first possible case is obtained when the base and the mobile are coplanar. But other cases can be found. Let us suppose that lines 2,3,4 are coplanar. We may rotate then the mobile around the line B2B4 until the point B5 lie in the plane defined by 2,3,4 (Figure 14). In this case line 2,3,4,5 are coplanar. The appendix 1 in section 7, page 27, gives the conditions which must be

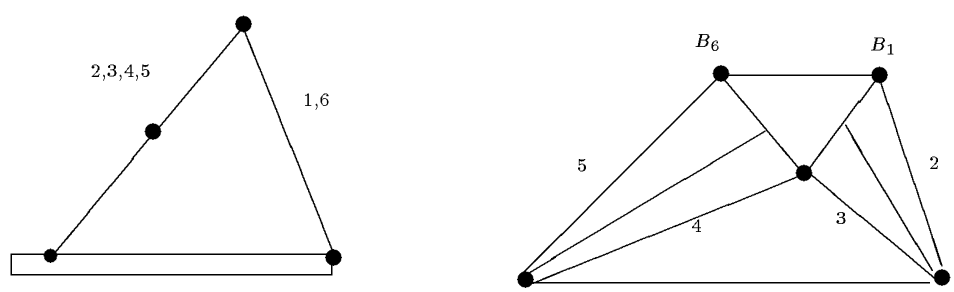


Figure 14: singular configuration of type 3d

satisfied by the parameters if the system is in such singular configurations. We get the following equation:

$$(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0 \qquad (8)$$

which express the coplanarity of 2-3-4. The coplanarity of 5 with 2-3-4 is more difficult to express.

This configuration is known as Hunt singular configuration. We will see later that this configuration yields also to a special complex for the 6 lines.

An open problem is to determine if , for a given design of a MSSM (i.e. the position of the articulation points being given), and for a given range of the links lengths such a singular configuration lie in the working area of the MSSM.

Let us make a useful remark for the following part : in this case we may notice that the 6 lines cross one line (ligne B_4B_5 in the above example).

Degeneracy of type 3c In this case we will have four lines passing through the same point.

Let us remark first that among a set of four lines we have two lines (say T,U) with a common intersection point A_i on the base and two (say V,W) on the mobile (point B_j). The lines being distinct they cannot have any other intersection point. We have assumed that every A_i are different from the B_j and thus any set of four lines of a MSSM cannot have a common intersection point.

Degeneracy of type 3b In this case 4 lines constitute two flat pencils having a line in common but lying in distinct planes (Figure 15).

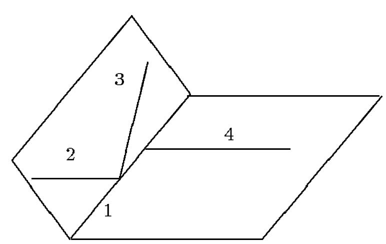


Figure 15: 3-dimensional Grassmann variety of type 3b

In this case we must have three coplanar lines which constitute a flat pencil. But we have seen in the part devoted to the subset of three lines that it does not exist three lines such that they constitute a flat pencil. Thus this configuration is not to be considered.

Degeneracy of type 3a The problem is to find 4 lines which are on the same regulus.

The hyperboloïd of one sheet has two regulus $\Re_1$ and $\Re_2$ and we denote by (1) the family of lines which are spanned by $\Re_1$ and (2) the family of lines spanned by $\Re_2$. Remember that each lines of (1) has an intersection point with every line of (2) and none with the other lines of (1).

Let us suppose that lines 1 belongs to (1). Whatever is the configuration of the MSSM line 2 intersects the line 1 and thus must belong to (2). Lines 3 intersects line 2 and therefore belongs to (1). Applying the same reasoning for the other lines it is clear that lines 1,3,5 belong to $\Re_1$ and line 2,4,6 belong to $\Re_2$ whatever is the configuration of the MSSM. Thus it is impossible that 4 lines of the MSSM belongs to the same regulus.

5.2.3 Subsets of 5 bars

configuration 4d All the five lines are in a plane or pass through one point of this plane.

Let us notice first that we cannot have five coplanar lines : indeed we have at most 4 collinear articulation points on the base and thus at most 4 coplanar lines.

Consider now the case of 4 coplanar lines. We have investigated this case for the singular configuration of type 3d. It may be seen on Figure 14 that in this case the fifth line cross the plane at its articulation point on the mobile. Thus the singular configuration in this case is identical to the singular configuration of type 3d.

Let us suppose now that three lines are coplanar.

We notice that the three articulation points on the base must be collinear. We can now distinguish two cases. The first one is obtained when two of these lines have a common articulation on the mobile (2,3,4 for example) and the second one none of the three lines have a common articulation point (2,3,5 for example). In this later case 4 will be coplanar to 2,3,5 and therefore 4 lines are coplanar. The above result can be applied.

In the first case line 5 crosses the plane at point A_5 and line 1 at point B_2. Thus lines 1,5 can intersect the plane at the same point if B_2=A_5. For the lines 6,5 we get by the same manner $B_6 = A_5$ or 5 in the plane 2,3,4 (but this is configuration 3d). As for the lines 1,6 they cannot have a common point with the plane 2,3,4. We assume that these cases cannot be realized by real manipulators.

The last case to be considered is when 3 lines cross at a same point a plane spanned by two others lines. Without loss of generality consider the plane spanned by 1,2. Lines 3,6 cross the plane in two different points A_1 and A_2 and thus cannot belong both to the set of lines to be considered. Its remains thus two sets (1,2,3,4,5) or (1,2,4,5,6). In this case the intersection point with the plane is either A_3 or A_6. But among the sets of three non coplanar lines there is two line which have a common point (B_3 or B_5). Thus these lines cannot cross in the same time an other point in plane 1,2.

configuration 4c In this case three lines must constitute a flat pencil of lines and therefore this configuration is not to be considered.

configuration 4b Five lines must pass through two skew lines (Figure 16).

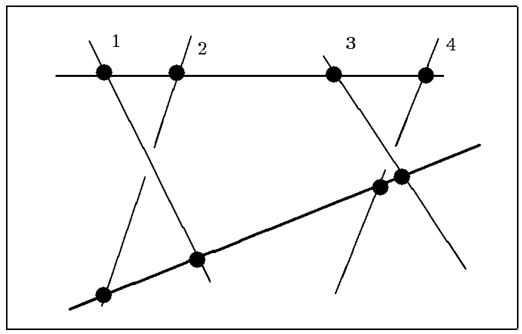


Figure 16:

We will consider without loss of generality lines 1,2,3,4,5. Let us notice first that there is no two skew lines which can intersect three coplanar lines.

Let us consider lines 1,2,3,4 : if 1,2,3 are coplanar we cannot find skew lines intersecting 1,2,3. If 1,2,3 are not coplanar and if a line intersect lines 1-2-3 then this line lie in the plane 2-3 and pass through the articulation point of 1,2 on the mobile or lie in the plane 1-2 and intersect 3 at its common point with 2 on the base (Figure 17) or is the line crossing both B_1 and B_3.

But a line which has a common point with lines 4,5 either lie in the plane 4,5 or crosses the articulation point common to 4,5 on the base. In the former case a line crossing line 1,2,3 and 4,5 must be coplanar to 4,5 and 1,2 or to 4,5 and 2,3: this is clearly impossible except if line 2,3,4 are coplanar. Indeed a line crossing B_2 and A_4 will intersect 5 segments (Figure 18) but in this case we have shown that is is the impossible to find 2 skew lines crossing 2,3,4.

In the latter case let us notice that there is no line in the planes 1,2 or 2,3 crossing the articulation point common to 4,5. Thus no line in the above plane can cross at the same time 1,2,3,4,5.

Thus there is only one line (one edge of the mobile) which can cross a set of 5 lines and therefore it is impossible to find 2 skew lines crossing a set of 5 lines.

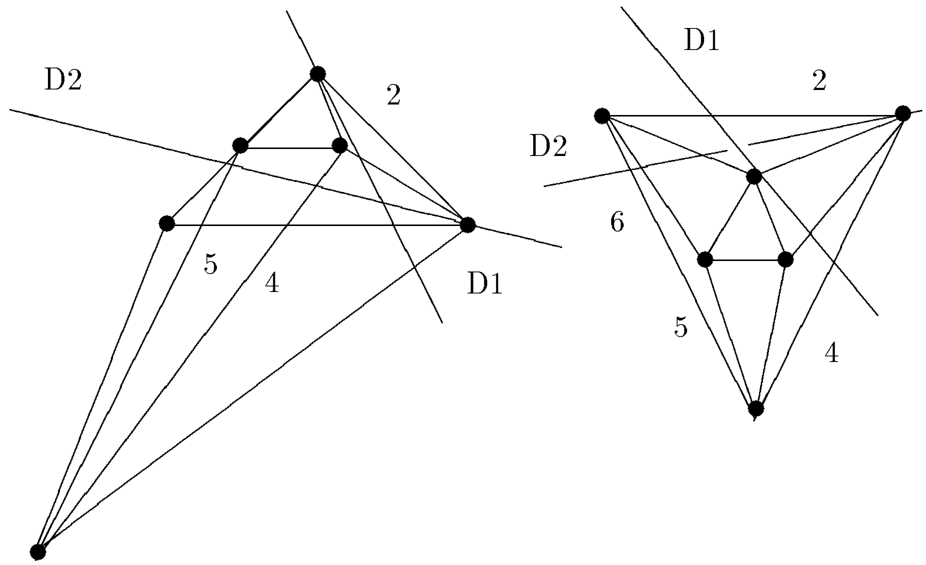


Figure 17: 2 skew lines intersecting line 1,2,3

configuration 4a In this case one line is dependent from four lines. Among this four lines none of them intersect the regulus spanned by the three others in a proper point.

Let us consider lines 1,2,3,4,5. From this set we consider 3 skew lines which spanned a regulus. There is only one set (1,3,5) where the lines may be skew. But the lines 2 or 4 intersect one line of the above set and therefore we cannot find 4 lines which meet the required condition.

5.2.4 Subsets of 6 bars

Configuration 5b 6 lines cross the same line D in space.

We may justify one more time Hunt's singular configuration described for 3d type(Figure 20). In this case we see that all the segments pass through the line B_4B_5.

We have seen for the 4b case that one line can have a common intersection point with five links if and only if :

-this line is an edge of the mobile and 4 segments are coplanar (these segments are successive).

- 3 segments are coplanar and the intersection line crosses the articulation points on the base which are not common to the segments (see Figure 18).

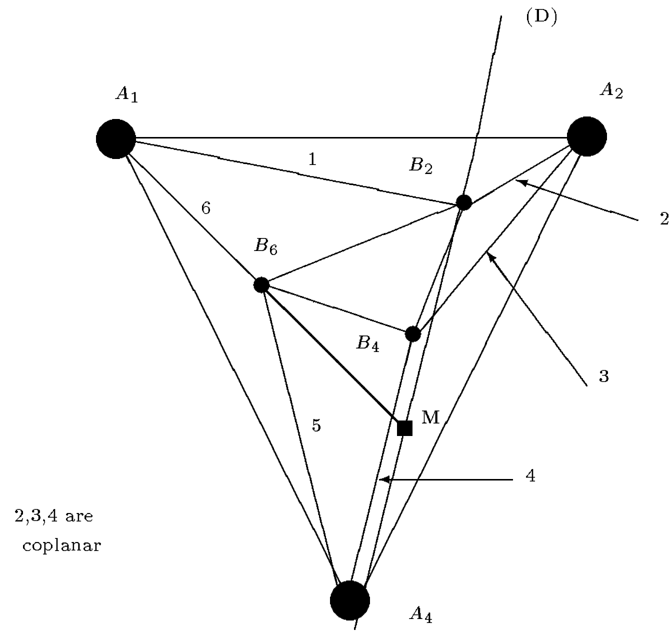


Figure 18: line (D) cross 1,2,3,4,5 if 2,3,4 are coplanar

Let us remark me may have found directly that at least 3 segments must be coplanar. Indeed let us remember that for a special complex 4 vertices of the skew pentagon must be coplanar. The geometry of the MSSM yields then directly that at least there must be 3 coplanar segments.

Let us consider the first case. If we rotate the mobile around the edge B_4B_5 until line 1,2 are coplanar with 6,3 the line based on this edge intersects the 6 links. We find again Hunt's singular configuration. Now we will consider edge B_2B_3 as being the intersection line of 5 segments (1,2,3,4,5). Let us notice that in that case the line based on edge B_2B_5 intersects 1,2,4,5,6 and may intersect 3 if the edge is not parallel to 3. In summary this case yields no more singular configuration than case 4d.

We will consider now the case of three coplanar segments (say 2,3,4) and consider the line (D) intersecting B_2 and A_4. (D) intersects segments 1,2,3,4,5. Let us rotate now the mobile around its edge B_2B_4. Let M be the intersection point with the plane spanned by 2,3,4. It may be possible that M belongs to (D) and therefore (D) intersects the 6 segments(Figure 21). The appendix 2 in section 8, page 29, shows that to obtain this configuration we must have

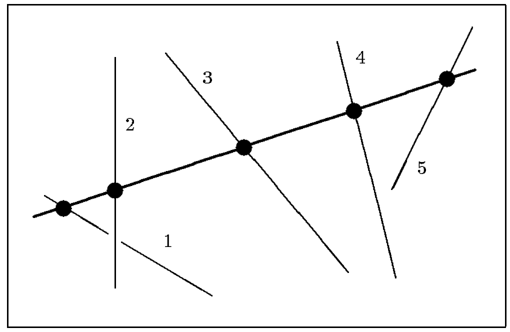


Figure 19: 5-dimensional Grassmann variety of type 5b

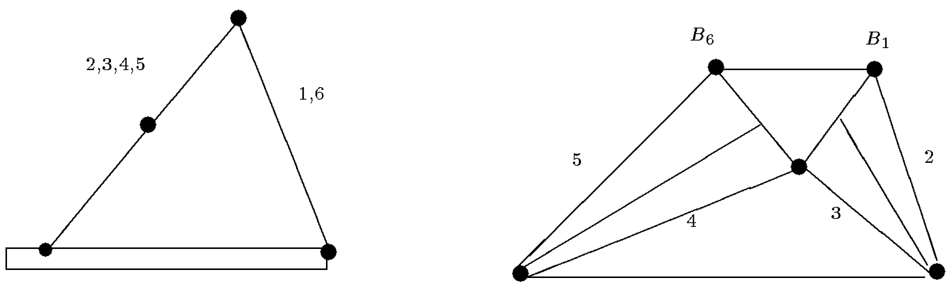


Figure 20: Hunt's singular configuration

:

$$(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0 \tag{9}$$
$$Az_0^2 + Bz_0 + C = 0 \tag{10}$$

configuration 5a In this case 6 lines spanned a general complex. FICHTER [1] has shown by using an intuitive method that this will be the case if we rotate the mobile around the vertical axis with an angle $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ whatever is the position of its center.

First we will show that if we consider only rotation around a vertical axis the above singular configurations are the only cases where we have a general complex.

To demonstrate this results we consider the pencils of lines spanned by 1-6,2-3 and 4-5. Then we consider in each of these pencils the line D_i which is

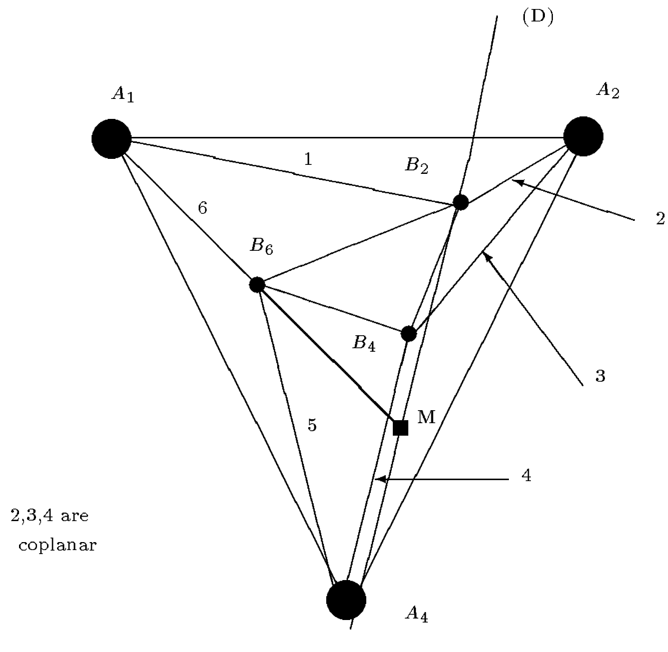


Figure 21: A new singular configuration

coplanar to the base (see Figure 22).

Condition C1

Lines 1-2-3-4-5-6 spanned a general complex if and only if the lines D_i have a common point.

Proof

From the classical property of a general complex we know that all the lines of a complex which are coplanar have a common point.

Let us consider now the complex spanned by lines 1-2-3-4-6 and let M the common intersection point of D_1, D_2. Then lines A_4M belongs to the complex together with the lines of the flat pencil which center is A_4 and lines are 4 and A_4M. By construction line 5 belongs to this pencil and therefore 5 belong to the complex.

If we apply condition C1 in the case where we have only a rotation around a vertical axis we may show (see the appendix 3 in section 9, page 31) that

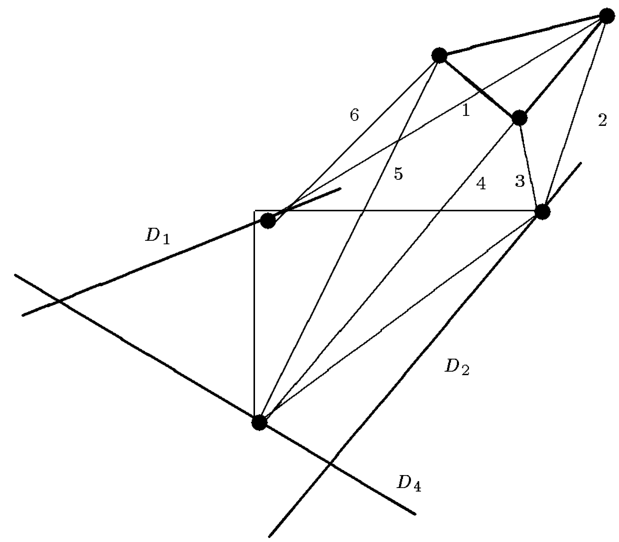


Figure 22:

C1 is equivalent to:

$$2cos(\psi)z_0(-x_3ya_1 + x_3ya_4 + xa_1y_1 - xa_1y_3) = 0$$

whatever is x_0 and y_0. The above condition is true only for $\psi = \frac{\pi}{2}$ or $-\frac{\pi}{2}$ Indeed the last term of the above equation will be zero if the edge of the base and the mobile were parallel.

Using the same method we will show however that this not the only case where we get such a singular configuration. In the general case condition C1 yields to a third degree polynomial in the variable z_0 (see the appendix 4 in section 10, page 32). Thus for a given orientation of the mobile and given value of x_0 and y_0 we have at least one singular configuration where the MSSM is a general complex.

Let us consider an example. We choose $x_0 = 0$, $y_0 = 0$, $\psi = 40°$, $\theta = 40°$, $\phi = 40°$. In this case the polynomial has three real roots. Figures 23, 24, 25 show the three singular configurations. The first singular configurations can be easily interpreted. We see that lines 4 and 5 are identical. Thus the variety spanned by the lines of the MSSM is in fact spanned by only 5 lines.

6 Results : the singular configurations of a MSSM

The table below summarizes the results.

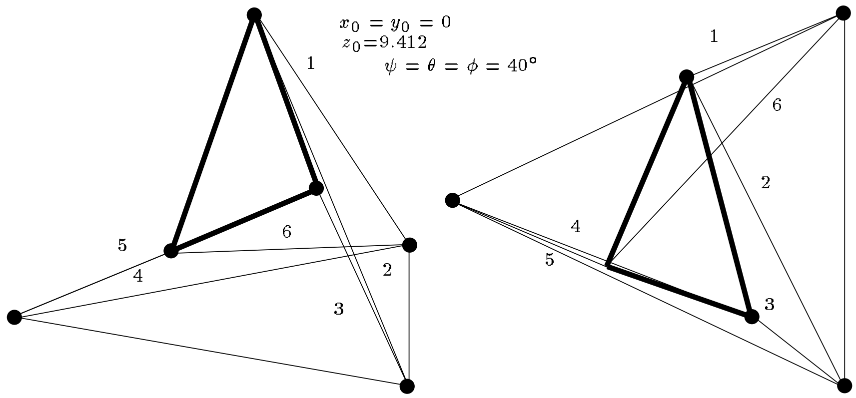


Figure 23: Perspective and top view of the first 5a singular configuration

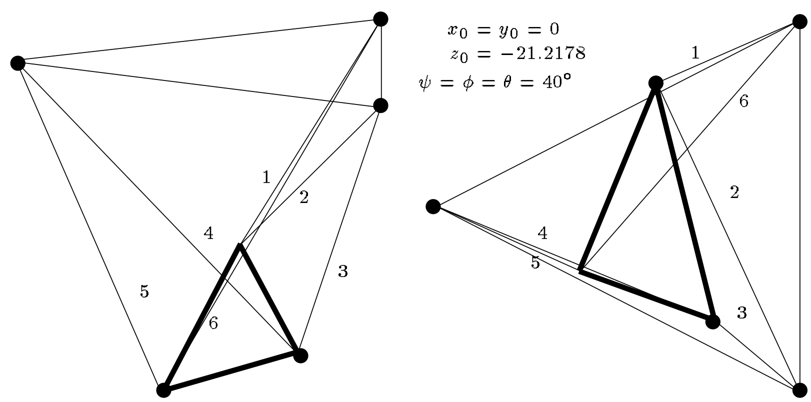


Figure 24: Perspective and top view of the second 5a singular configuration

4 lines coplanar (Hunt configuration)	$A(y_0, z, \psi, \theta, \phi)x_0 + B(y_0, z_0, \psi, \theta, \phi) = 0$ $(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0$
special complex	$U_1x_0^2 + V_1x_0 + W_1 = 0$ $(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0$
general complex	$\psi = \pm\frac{\pi}{2},\ \theta = 0\ ,\ \phi = 0$
general complex (second case)	$a(x_0, y_0, \psi, \theta, \phi)z_0^3 + b(x_0, y_0, \psi, \theta, \phi)z_0^2$ $+d(x_0, y_0, \psi, \theta, \phi)z_0 + e(x_0, y_0, \psi, \theta, \phi) = 0$

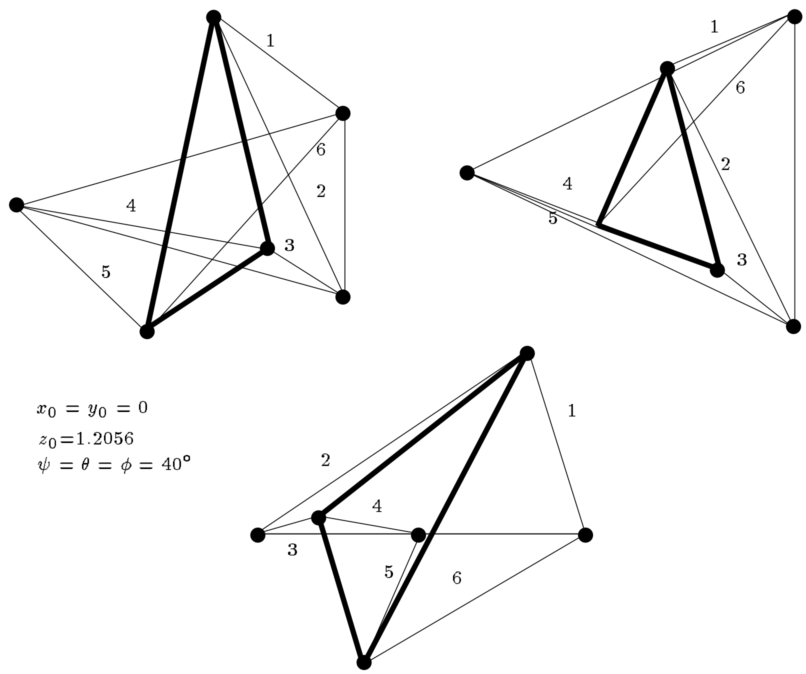


Figure 25: Perspective,top and side view of the third 5a singular configuration

7 Appendix

1)Condition for a MSSM singularity of type 3d

We will consider here that line 2,3,4 are coplanar and that the mobile is rotated around B_2B_4 until point 5 lie in the plane. The others possibilities can be easily deduced from this case by simple rotation.

The condition of 2,3,4 being coplanar is that the projection $(B_2B_4)^p$ of B_2B_4 on the base plane is parallel to A_2A_4 which yields :

$$(B_2B_4)^p \wedge A_2A_4 = 0 \tag{11}$$

If 5 lie in the plane 2-3-4 then :

$$(A_2B_2 \wedge A_3B_3).A_5B_5 = 0$$

This two equations define the two constraints on the parameters of the system. We get:

$$(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0 \tag{12}$$

```
(c3) /* solve the problem of singularities of type 3d
for the MSSM. Line 2,3,4 are coplanar and point 5
is in the plane of 2,3,4 and thus 2,3,4,5 are coplanar
*/

/* rotation matrix of the mobile */

rot: matrix([b1,b2,b3],[b4,b5,b6],[b7,b8,b9])$

(c4) /* position of the articulation points on the mobile */

b1r: matrix([0],[y1],[0])$
(c5) b2r: matrix([0],[y1],[0])$
(c6) b3r: matrix([x3],[y3],[0])$
(c7) b4r: matrix([x3],[y3],[0])$
(c8) b5r: matrix([-x3],[y3],[0])$
(c9) b6r: matrix([-x3],[y3],[0])$

(c10) /* position of the articulation points on the base */

a1: matrix([-xa2],[ya1],[0])$
(c11) a2: matrix([xa2],[ya1],[0])$
(c12) a3: matrix([xa2],[ya1],[0])$
(c13) a4: matrix([0],[ya4],[0])$
(c14) a5: matrix([0],[ya4],[0])$
(c15) a6: matrix([-xa2],[ya1],[0])$

(c16) /* position of the center of the mobile */

cen: matrix([x0],[y0],[z0])$

(c17) /* articulation point of the mobile in absolute coordinates */

bb1:cen+rot.b1r$
(c18) bb2:cen+rot.b2r$
(c19) bb3:cen+rot.b3r$
(c20) bb4:cen+rot.b4r$
(c21) bb5:cen+rot.b5r$
(c22) bb6:cen+rot.b6r$

(c23) a1b1:ev(bb1-a1)$
(c24) a2b2:ev(bb2-a2)$
(c25) a3b3:ev(bb3-a3)$
(c26) a4b4:ev(bb4-a4)$
```

```
(c27) a5b5:ev(bb5-a5)$
(c28) a6b6:ev(bb6-a6)$

(c29) cross(u1,u2,u3):=block(
u3[1]:u1[2]*u2[3]-u1[3]*u2[2],
u3[2]:u1[3]*u2[1]-u1[1]*u2[3],
u3[3]:u1[1]*u2[2]-u1[2]*u2[1]
)$

(c30) dot(u1,u2):=block([p],
p:sum(u1[i]*u2[i],i,1,3),
return(p)
)$
(c31) a2a4:a4-a2$

(c32) /* the projection of B2 on the base plane*/

b2proj:b2$

(c33) b2proj[3,1]:0$

(c34) /* the projection of B4 on the base plane*/

b4proj:b4$

(c35) b4proj[3,1]:0$

(c36) /* the projection of B2B4 on the base plane*/

b2b4proj:b4proj-b2proj$
(c37) /* 2,3,4 coplanar*/

eq1:b2b4proj[1,1]*a2a4[2,1]-b2b4proj[2,1]*a2a4[1,1];

(d37)  (v2 y3 - v2 y1 + v1 x3) (ya4 - ya1) + xa2 (v5 y3 - v5 y1 + v4 x3)

(c38) nn1:matrix([0],[0],[0])$

 (c39) cross(a2b2,a3b3,nn1)$

 (c40) eq2:dot(nn1,a5b5)$

 (c41) eq2:eq2[1]$

 (c42) ratsimp(eq2,z0);
(d42)((- v2 y3 + v2 y1 - v1 x3) ya4 + (v2 y3 - v2 y1 + v1 x3) ya1
+((2 v1 v5-2 v2 v4) x3-v5 xa2) y3+(v5 xa2+(2 v2 v4 - 2 v1 v5) x3)
y1 - v4 x3 xa2) z0 + ((v8 x0 - v8 xa2) y3
+(v8 xa2 + (v2 v7-v1 v8) x3-v8 x0) y1-v7 x3 xa2 + v7 x0 x3) ya4
+(((2 v1 v8-2 v2 v7) x3-v8 x0) y3+((v2 v7-v1 v8) x3 + v8 x0) y1
- v7 x0 x3) ya1 + ((v8 xa2 + (2 v2 v7 - 2 v1 v8) x3) y0
+ (2 v5 v7 - 2 v4 v8) x3 xa2 + (2 v4 v8 - 2 v5 v7) x0 x3) y3
+(((2 v1 v8-2 v2 v7) x3- v8 xa2) y0 + (v4 v8 - v5 v7) x3 xa2
+ (2 v5 v7 - 2 v4 v8) x0 x3) y1 + v7 x3 xa2 y0
```

8 Appendix

2) The MSSM as a special complex

In this case 2-3,4 are coplanar and the line which cross A_4 and B_2 has a common point with line 6. The coplanarity of 2-3-4 is given by:

$$(v2y_3 - v2y_1 + v1x_3)(ya_4 - ya_1) + xa_2(v5y_3 - v5y_1 + v4x_3) = 0$$

The other constraint yields to a second degree equation in z_0.

```
(c3) /* singularite 3d pour leMSSM: 2,3,4,5 coplanaires*/

/* les operations elementaires*/

dot(v1,v2):=block([p],
p:sum(v1[i]*v2[i],i,1,3),
return(p))$

(c4) cross(u1,u2,u3):=block(
u3[1]:u1[2]*u2[3]-u1[3]*u2[2],
u3[2]:u1[3]*u2[1]-u1[1]*u2[3],
u3[3]:u1[1]*u2[2]-u1[2]*u2[1]
)$
(c5) rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$
(c6) b1r:matrix([0],[y1],[0])$(c12)a1:matrix([-xa2],[ya1],[0])$
(c7) b2r:matrix([0],[y1],[0])$(c13) a2:matrix([xa2],[ya1],[0])$
(c8) b3r:matrix([x3],[y3],[0])$(c14) a3:matrix([xa2],[ya1],[0])$
(c9) b4r:matrix([x3],[y3],[0])$(c15) a4:matrix([0],[ya4],[0])$
(c10) b5r:matrix([-x3],[y3],[0])$(c16) a5:matrix([0],[ya4],[0])$
(c11) b6r:matrix([-x3],[y3],[0])$(c17) a6:matrix([-xa2],[ya1],[0])$

(c18) /* position of the center of the mobile */

cen:matrix([x0],[y0],[z0])$
(c19) /* articulation point of the mobile in absolute coordinates */

b1:cen+rot.b1r$(c25) a1b1:b1-a1$
(c20) b2:cen+rot.b2r$(c26) a2b2:b2-a2$
(c21) b3:cen+rot.b3r$(c27) a3b3:b3-a3$
(c22) b4:cen+rot.b4r$(c28) a4b4:b4-a4$
(c23) b5:cen+rot.b5r$(c29) a5b5:b5-a5$
(c24) b6:cen+rot.b6r$(c30) a6b6:b6-a6$

(c31) a2a4:a4-a2$
(c32) /* the projection of B2 on the base plane*/
b2proj:matrix([b2[1,1]],[b2[2,1]],[0])$

(c33) /* the projection of B4 on the base plane*/
b4proj:matrix([b4[1,1]],[b4[2,1]],[0])$

(c34) /* the projection of B2B4 on the base plane*/
b2b4proj:b4proj-b2proj$

(c35) /* 2,3,4 coplanar*/
eq1:b2b4proj[1,1]*a2a4[2,1]-b2b4proj[2,1]*a2a4[1,1];
(d35)  (v2 y3 - v2 y1 + v1 x3) (ya4 - ya1) + xa2 (v5 y3 - v5 y1 + v4 x3)

 (c36) /* a line D crossing A4 and B2*/
m:matrix([x],[y],[z])$
```

```
(c37) a4m:m-a4$

(c38) a4b2:b2-a4$

(c40) cross(a4m,a4b2,nn)$
(c41) /* line 6 */
a1m:m-a1$
(c42) a1b5:b5-a1$

(c44) cross(a1m,a1b5,nn1)$

(c45) eq2:nn[1,1]$
(c46) eq3:nn[2,1]$

(c47) eq4:nn1[1,1]$

(c48) linsolve([eq4,eq2,eq3],[x,y,z]),globalsolve:true$

(c49) eq5:nn1[2,1]$

(c52) eq5:ratsimp(num(%),z0)$
(d53)
((v2 y3-v2 y1- v1 x3) ya4 + (- v2 y3 + v2 y1 + v1 x3) ya1 - v5 xa2 y3
                           2             2
+ v5 xa2 y1 + v4 x3 xa2) z0  + ((v2 v8 y3
+ (- v2 v8 y1 - v8 xa2 + (- v1 v8 - v2 v7) x3 - v8 x0) y3
                                                                   2
+ (v8 xa2 + (2 v2 v7 - v1 v8) x3 + v8 x0) y1 + v7 x3 xa2 + v1 v7 x3
                             2
+ v7 x0 x3) ya4 + (- v2 v8 y3  + (v2 v8 y1 + (v1 v8 + v2 v7) x3 + v8 x0) y3
                                              2
+ ((v1 v8 - 2 v2 v7) x3 - v8 x0) y1 - v1 v7 x3  - v7 x0 x3) ya1
              2
- v5 v8 xa2 y3  + (v5 v8 xa2 y1 + v8 xa2 y0 + (v4 v8 + v5 v7) x3 xa2) y3
                                                                     2
+ ((v4 v8 - 2 v5 v7) x3 xa2 - v8 xa2 y0) y1 - v7 x3 xa2 y0 - v4 v7 x3  xa2) z0
        2         2       2       2                        2         2
+ ((- v8  xa2 - v8  x0) y3  + ((v8  xa2 + (v2 v7 v8 - v1 v8 ) x3 + v8  x0) y1
+ 2 v7 v8 x3 xa2 + 2 v7 v8 x0 x3) y3 + (- v7 v8 x3 xa2
                   2    2                       2   2         2      2
+ (v1 v7 v8 - v2 v7 ) x3  - v7 v8 x0 x3) y1 - v7  x3  xa2 - v7  x0 x3 ) ya4
     2      2           2                    2
+ (v8  x0 y3  + (((v1 v8  - v2 v7 v8) x3 - v8  x0) y1 - 2 v7 v8 x0 x3) y3
         2               2                       2      2
+ ((v2 v7  - v1 v7 v8) x3  + v7 v8 x0 x3) y1 + v7  x0 x3 ) ya1
    2          2           2                        2
+ v8  xa2 y0 y3  + (((v4 v8  - v5 v7 v8) x3 xa2 - v8  xa2 y0) y1
                                                   2               2
- 2 v7 v8 x3 xa2 y0) y3 + (v7 v8 x3 xa2 y0 + (v5 v7  - v4 v7 v8) x3  xa2) y1
    2   2
+ v7  x3  xa2 y0
```

9 Appendix

3) The MSSM as a general complex: first case

We consider here a MSSM which is able to translate and rotate around the vertical axis we may get a general complex if the rotation angle is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ whatever is the position of its center.

We consider the pencils of lines spanned by 1-6,2-3 and 4-5. Then we consider in each of these pencils the line D_i which is coplanar to the base. If lines 1-2-3-4-5-6 spanned a general complex then the lines D_i must have a common point. The following REDUCE program computes the condition for which this is true. We get the following condition :

$$2cos(\theta)z_0(-x_3ya_1 + x_3ya_4 + xa_1y_1 - xa_1y_3) = 0$$

which is true only for $\theta = \frac{\pi}{2}$ or $-\frac{\pi}{2}$

```
comment coordinates of the articulation points on the base$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((-xa1),(ya1),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((0),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((0),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((xa1),(ya1),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$
comment : rotation matrix$
rot:=mat((v1,v2,0),(-v2,v1,0),(0,0,1))$

comment: the axis vector of the link$
a1b1:=cen+rot*b1r-a1$a2b2:=cen+rot*b2r-a2$
a3b3:=cen+rot*b3r-a3$a4b4:=cen+rot*b4r-a4$
a5b5:=cen+rot*b5r-a5$a6b6:=cen+rot*b6r-a6$

comment: the normal to the flat pencils 1-6,2-3,4-5$
n16:=mat((a1b1(2,1)*a6b6(3,1)-a6b6(2,1)*a1b1(3,1)),
(a1b1(3,1)*a6b6(1,1)-a6b6(3,1)*a1b1(1,1)),
(a1b1(1,1)*a6b6(2,1)-a6b6(1,1)*a1b1(2,1)))$
n23:=mat((a2b2(2,1)*a3b3(3,1)-a3b3(2,1)*a2b2(3,1)),
(a2b2(3,1)*a3b3(1,1)-a3b3(3,1)*a2b2(1,1)),
(a2b2(1,1)*a3b3(2,1)-a3b3(1,1)*a2b2(2,1)))$
n45:=mat((a4b4(2,1)*a5b5(3,1)-a5b5(2,1)*a4b4(3,1)),
(a4b4(3,1)*a5b5(1,1)-a5b5(3,1)*a4b4(1,1)),
(a4b4(1,1)*a5b5(2,1)-a5b5(1,1)*a4b4(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(0))$a1m:=m-a1$a2m:=m-a2$a4m:=m-a4$

comment intersection condition $
eq1:=a1m(1,1)*n16(1,1)+a1m(2,1)*n16(2,1)$
eq2:=a2m(1,1)*n23(1,1)+a2m(2,1)*n23(2,1)$
eq3:=a4m(1,1)*n45(1,1)+a4m(2,1)*n45(2,1)$

comment the two first equations yield x and y$
solve(lst(eq1,eq2),x,y)$

comment the third equation yields the condition$
eq4:=num(eq3);
2*v1*z0*( - x3*ya1 + x3*ya4 + xa1*y1 - xa1*y3)
```

10 Appendix

$\boxed{\text{4) The MSSM as a general complex: general case}}$

This appendix find the condition on z_0 such that the MSSM will be a general complex.

To get this condition we write that the line of the flat pencils spanned by 1-6,2-3,3-4 which lie on the plane of the base must intersect at the same point. This yields to a third degree equation in z_0.

```
comment coordinates of the articulation points on the base$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((-xa1),(ya1),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((0),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((0),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((xa1),(ya1),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$
comment : rotation matrix$
rot:=mat((v1,v2,3),(v4,v5,v6),(v7,v8,v9))$

comment: the axis vector of the link$
b1:=cen+rot*b1r$a1b1:=b1-a1;b2:=cen+rot*b2r$a2b2:=b2-a2;
b3:=cen+rot*b3r$a3b3:=b3-a3;b4:=cen+rot*b4r$a4b4:=b4-a4;
b5:=cen+rot*b5r$a5b5:=b5-a5;b6:=cen+rot*b6r$a6b6:=b6-a6;

comment: the normal to the flat pencils 1-6,2-3,4-5$
n16:=mat((a1b1(2,1)*a6b6(3,1)-a6b6(2,1)*a1b1(3,1)),
(a1b1(3,1)*a6b6(1,1)-a6b6(3,1)*a1b1(1,1)),
(a1b1(1,1)*a6b6(2,1)-a6b6(1,1)*a1b1(2,1)))$
n23:=mat((a2b2(2,1)*a3b3(3,1)-a3b3(2,1)*a2b2(3,1)),
(a2b2(3,1)*a3b3(1,1)-a3b3(3,1)*a2b2(1,1)),
(a2b2(1,1)*a3b3(2,1)-a3b3(1,1)*a2b2(2,1)))$
n45:=mat((a4b4(2,1)*a5b5(3,1)-a5b5(2,1)*a4b4(3,1)),
(a4b4(3,1)*a5b5(1,1)-a5b5(3,1)*a4b4(1,1)),
(a4b4(1,1)*a5b5(2,1)-a5b5(1,1)*a4b4(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(0))$a1m:=m-a1$a2m:=m-a2$a4m:=m-a4$

comment intersection condition $
eq1:=a1m(1,1)*n16(1,1)+a1m(2,1)*n16(2,1)$
eq2:=a2m(1,1)*n23(1,1)+a2m(2,1)*n23(2,1)$
eq3:=a4m(1,1)*n45(1,1)+a4m(2,1)*n45(2,1)$

comment the two first equations yield x and y$
solve(lst(eq1,eq2),x,y)$
eqfin:=num(eq3)$
    3
2 z0  v9 x3 (v1 x3 y1 ya1 - v1 x3 y1 ya4 - v1 x3 y3 ya1 + v1 x3 y3 ya4
                         2                              2        2
              - v5 xa1 y1  + 2 v5 xa1 y1 y3 - v5 xa1 y3 ) + 2 z0  x3 (
                     2                                     2
    - v1 v6 x3 y1 ya1  + v1 v6 x3 y1 ya1 ya4 + v1 v6 x3 y3 ya1  - v1
                                     2                          2
   v6 x3 y3 ya1 ya4 + v2 v3 v5 xa1 y1  y3 - 2 v2 v3 v5 xa1 y1 y3  + v2
               3                  2                      2
    v3 v5 xa1 y3  + v2 v3 v9 x3 y1  ya1 - v2 v3 v9 x3 y1  ya4 - v2 v3
                                                         2
```

```
v9 x3 y1 y3 ya1 + v2 v3 v9 x3 y1 y3 ya4 - v2 v3 xa1 y1  ya4 + 2 v2
                               2                  2
v3 xa1 y1 y3 ya4 - v2 v3 xa1 y3  ya4 - v3 v4 v9 x3  xa1 y1 + v3 v4
     2                      2                  2        2
v9 x3  xa1 y3 - v3 v4 x3 xa1  y1 + v3 v4 x3 xa1  y3 + v5  v6 xa1
  2          2             2     2          2              2
y1  y3 - 2 v5  v6 xa1 y1 y3  + v5  v6 xa1 y3  - v6 v9 x3 y1  ya1 +
           2                  2                  2
v6 v9 x3 y1  ya4 + v6 v9 x3 y3  ya1 - v6 v9 x3 y3  ya4 - v8 v9 xa1
  2                   2
y1  ya1 - v8 v9 xa1 y1  ya4 + 2 v8 v9 xa1 y1 y3 ya1 + 2 v8 v9 xa1
                        2                   2
y1 y3 ya4 - v8 v9 xa1 y3  ya1 - v8 v9 xa1 y3  ya4) + 2 z0 x3 (v2 v3
         2                        2                           2
 v6 x3 y1  y3 ya1 - v2 v3 v6 x3 y1  y3 ya4 - v2 v3 v6 x3 y1 y3  ya1
                    2                      2
 + v2 v3 v6 x3 y1 y3  ya4 + v2 v3 v8 xa1 y1  y3 ya1 - 2 v2 v3 v8
         2                      3                     2    2
xa1 y1 y3  ya1 + v2 v3 v8 xa1 y3  ya1 - v2 v6 v7 x3 y1  ya1  + v2
           2                                2
v6 v7 x3 y1  ya1 ya4 + v2 v6 v7 x3 y1 y3 ya1  - v2 v6 v7 x3 y1 y3
            2      2       2     2      2               2       2
ya1 ya4 - v3  v9 x3  xa1 y1  + v3  v9 x3  xa1 y1 y3 - v3  x3 xa1
  2     2       2   2                  2   2                  2
y1  + v3  x3 xa1  y3  - v3 v5 v7 x3 xa1  y1  + v3 v5 v7 x3 xa1  y1
                2                         2
y3 - v3 v7 v9 x3  xa1 y1 ya1 + v3 v7 v9 x3  xa1 y3 ya1 - v3 v7 x3
   2                      2                     2
xa1  y1 ya4 + v3 v7 x3 xa1  y3 ya4 + v4 v6 v7 x3  xa1 y1 ya4 - v4
        2                   2      2               2      2
v6 v7 x3  xa1 y3 ya4 + v5 v6  x3 y1  y3 ya1 - v5 v6  x3 y1  y3 ya4
        2         2            2         2                      2
 - v5 v6  x3 y1 y3  ya1 + v5 v6  x3 y1 y3  ya4 + v5 v6 v8 xa1 y1
                        2                              2
y3 ya1 + v5 v6 v8 xa1 y1  y3 ya4 - 2 v5 v6 v8 xa1 y1 y3  ya1 - 2 v5
                2                      3                      3
 v6 v8 xa1 y1 y3  ya4 + v5 v6 v8 xa1 y3  ya1 + v5 v6 v8 xa1 y3  ya4
     2      2    2     2      2             2      2    2     2
 + v6  x3 y1  ya1  - v6  x3 y1  ya1 ya4 - v6  x3 y3  ya1  + v6  x3
  2                       2
y3  ya1 ya4 + v6 v8 xa1 y1  ya1 ya4 - 2 v6 v8 xa1 y1 y3 ya1 ya4 +
            2                       2          2   2        2
v6 v8 xa1 y3  ya1 ya4) + 2 x3 ( - v3  v8 x3 xa1  y1  y3 + v3  v8 x3
    2      2              2       2                  2
 xa1  y1 y3  + v3 v6 v7 x3  xa1 y1  ya4 - v3 v6 v7 x3  xa1 y1 y3
                     2   2                      2               2
ya4 - v3 v7 v8 x3 xa1  y1  ya4 + v3 v7 v8 x3 xa1  y1 y3 ya4 + v6
        2       2     2          2                2            2
v8 x3 y1  y3 ya1  - v6  v8 x3 y1  y3 ya1 ya4 - v6  v8 x3 y1 y3
   2     2            2                2   2
ya1  + v6  v8 x3 y1 y3  ya1 ya4 + v6 v7  x3  xa1 y1 ya1 ya4 - v6
  2   2                       2       2                     2
v7  x3  xa1 y3 ya1 ya4 + v6 v8  xa1 y1  y3 ya1 ya4 - 2 v6 v8  xa1
     2                2       2
y1 y3  ya1 ya4 + v6 v8  xa1 y3  ya1 ya4)
```

11 Study of the TSSM

We will deal now with the case of the TSSM (Figure 26). Let us make a

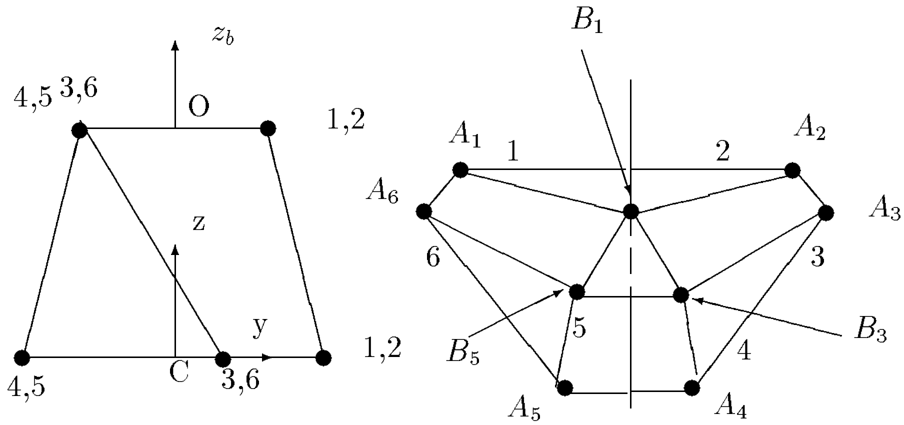


Figure 26: triangular simplified symmetric manipulator TSSM

preliminary remark: for the TSSM we may have at most 2 coplanar lines. Indeed we notice that there is at most two segments with collinear articulation points on the base.

11.1 Subset of 3 bars

Three segments must be coplanar : this is impossible.

11.2 Subset of 4 bars

11.2.1 Type 3d

Four segments must be coplanar : this is impossible.

11.2.2 Type 3c

(Four lines cross the same point).

Among a set of four bars two have a common articulation point on the mobile. Thus this common point must be the common point to the four lines. We will assume that the two lines with a common point are 1,2. Lines 3,4 and 5,6 have a common point different from B_1 and thus cannot have another one. Thus the only sets to be considered are (1,2,3,5),(1,2,3,6), (1,2,4,5) and (1,2,4,6). The demonstration of the following result will be the same in each case and we will study only the case of the set 1,2,3,5.

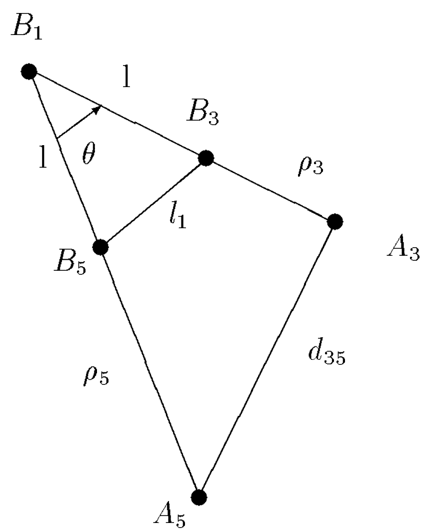


Figure 27: Four lines intersecting B_1

If 3 crosses B_1 then 3 is collinear to the edge B_1B_3. In the same manner if 5 crosses B_1 then 5 is collinear to the edge B_1B_5 and thus 3 and 5 are coplanar. Let us consider now the problem in this plane. For a given position of A_3, A_5 and a geometry of the mobile is it possible to find the links lengths ρ_3 and ρ_5 such that 3,5 intersect B_1 (Figure 27). Let d_{35} denotes the distance between A_3 and A_5 , l and l_1 the dimension of the mobile. We may write :

$$l_1^2 = 2l^2(1 - cos\theta) \tag{13}$$

$$d_{35}^2 = (l+\rho_3)^2 + (l+\rho_5)^2 - 2(l+\rho_3)(l+\rho_5)cos\theta \tag{14}$$

which yields :

$$l_1^2 = l^2\frac{d_{35}^2 - (\rho_3-\rho_5)^2}{(l+\rho_3)(l+\rho_5)} \tag{15}$$

This equation enable to test quickly if such a configuration is reachable for a given manipulator. We describe in the appendix 5 in section 13, page 44, the relations which describe such a singular configuration. Basically we must have :

$$\tan\psi = \frac{ya_3 - ya_4}{xa_3 + xa_4}$$

and we can compute x_0, y_0, z_0 for a given θ. Figure 28 shows such a case.

11.2.3 Type 3b

In this case three lines must be coplanar : this is impossible.

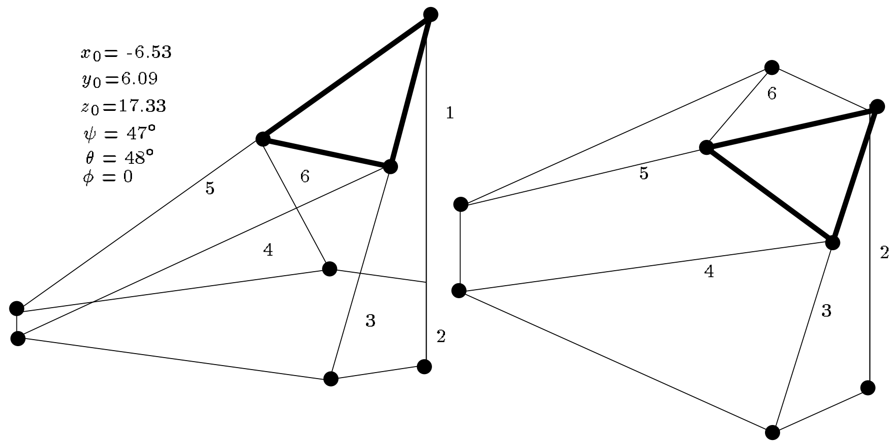


Figure 28: Perspective and top view of a singular configuration of type 3c

11.2.4 Type 3a

We will use the same method as for the MSSM. Let us suppose that line 1 belongs to the family (1) spanned by the regulus. Line 2 intersects line 1 and thus belongs to the family (2) spanned by the complementary regulus. For the same reason lines 3,4 and 5,6 cannot belong to the same family. Therefore 4 lines cannot belong to the same regulus.

11.3 Set of five bars

11.3.1 Configuration 4d

(all lines in a plane or crossing a point of this plane)

Let us remember that we have at most 2 coplanar lines. Among a set of 5 lines two pairs are coplanar and have a common point which is their articulation points on the mobile. These two points being different this imply that the plane to be considered is spanned by a pair of lines and that the common point to the three other lines is the articulation point of the second pair. For example if we consider lines 1,2,3,4,5 we will consider the plane spanned by 1,2 , put the articulation point B_3 common to 3,4 in this plane and look if 5 can intersect B_3 (Figure 29).

In appendix 6 in section 14, page 45, we show how we may calculate the value of x_0, y_0, z_0 acoording to the value of ψ, θ, ϕ to get such a singular configuration. An example is given in figure 30.

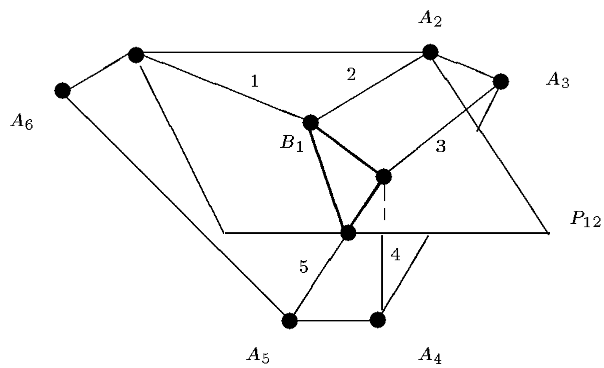
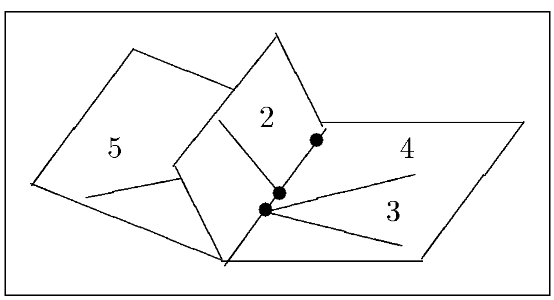


Figure 29: Singular configuration of type 4d

11.3.2 Configuration 4c



In this case we must have three coplanar lines: this is impossible.

11.3.3 Configuration 4b

Let us consider first lines 1,2,3,4. We have to find two skew lines D_1, D_2 which intersect this 4 lines. We have 4 possibilities for a line D which intersects lines 1,2,3,4:

-$D \in P_{12}$ and D intersects B_3

-$D \in P_{34}$ and D intersects B_1

-$D = P_{12} \cap P_{34}$

-D intersects both B_1 and B_3

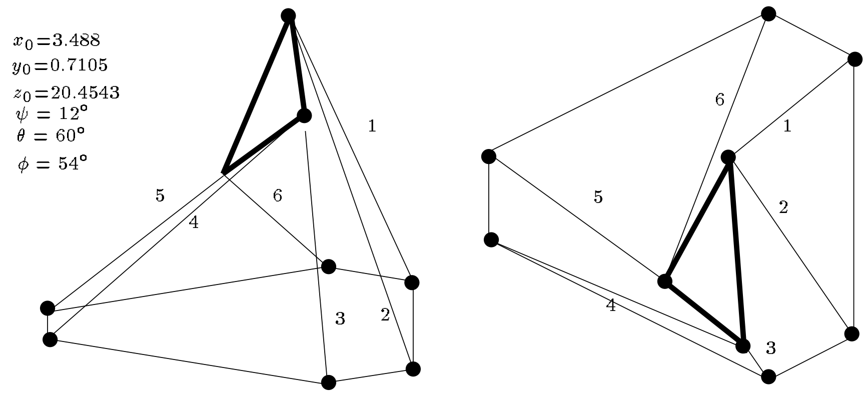


Figure 30: Perspective and top view of a singular configuration of type 4d

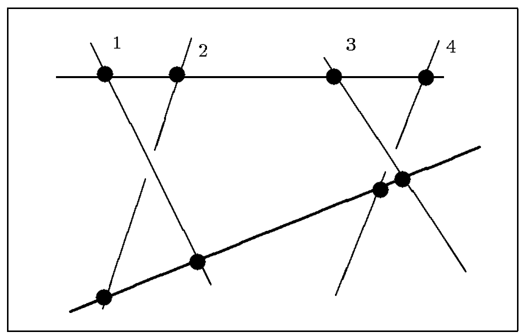


Figure 31: 3-dimensional Grassmann variety of type 4b

11.3.4 $D_1 \in P_{12}$ and $D1$ intersects B_3

If D_2 is skew to D_1 then $D_2 \notin P_{12}$ and D_2 does not intersect B_3. Therefore D_2 cannot be neither $P_{12} \cap P_{34}$ nor B_1B_3. Thus the only remaining case is $D_2 \in P_{34}$ and D_2 intersects B_1. In this case we have :

$$B_1 \in P_{34} \quad B_1 \in P_{12} \Rightarrow B_1 \in P_{12} \cap P_{34}$$

$$B_3 \in P_{34} \quad B_3 \in P_{12} \Rightarrow B_3 \in P_{12} \cap P_{34}$$

Thus C_{23}, B_1, B_3 belong to the same line. Let M_{12} be the intersection point of line 5 with P_{12} and M_{34} the intersection point of line 5 with P_{34}. If M_{12} is different from M_{34} then the lines B_3M_{12} and B_1M_{45} are skew and intersect the lines 1,2,3,4,5. This is then a singular configuration (Figure 32). In appendix 7 in section 15, page 47, we show how to get the value of x_0, y_0 according to z_0, ψ, θ, ϕ.

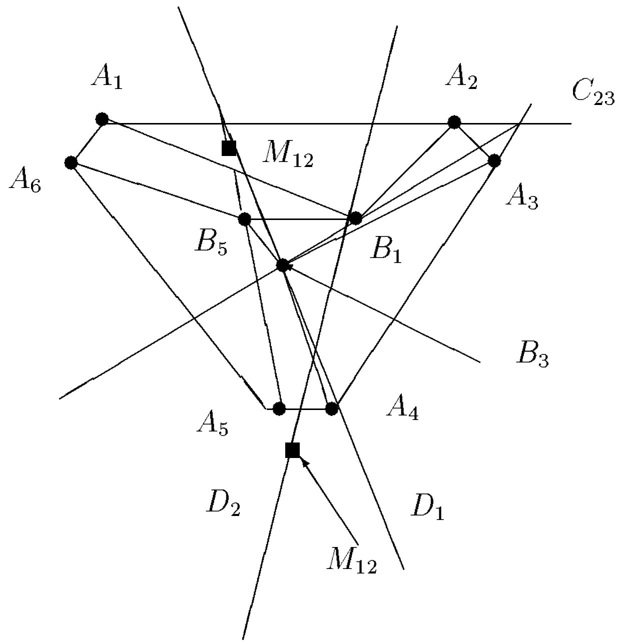


Figure 32: 2 skew lines intersecting 1,2,3,4,5

11.3.5 $D_1 \in P_{34}$ and $B_1 \in D_1$

As in the previous part we get:

$$D_2 \in P_{12} \quad B_3 \in D_2$$

which is the case we investigated above.

11.3.6 $D_1 = P_{12} \cap P_{34}$

If D_2 is not to be coplanar with D_1 then we must have:

$$D_2 = B_1 B_3$$

But if line 5 intersects D_2 then line 5 and B_1B_3 are coplanar and thus the mobile and line 5 are coplanar. We must then investigate if line 5 intersects D_1.

Appendix 8 in section 16, page 48, shows how we can verify if we are in such a configuration. Basically the coplanarity of line 5 and the mobile yields:

$$y_0 = -\frac{v9z0 - v6ya4 + v3xa4 + v3x0}{v6}$$

If we write that line 5 intersects D_1 we get four linear equations in term of the intersection point. A numerical resolution of the three first equations give the position of the intersection point and then we have to verify if the fourth equation is satisfied.

11.4 Set of six bars

11.4.1 Configuration 5a

In this case the variety spanned by the 6 lines is a general complex. We make the same analysis as for the MSSM. We consider the lines D_i belonging to the flat pencils spanned by lines 1-2, 3-4, and 5-6 and lying in the mobile plane. We get a general complex if and only if these 3 lines intersects the same point.

We will consider first the case where we have only rotation around the vertical axis. The appendix 9 in section 17, page 50 shows that we get then a singular configuration if we have $\psi = \pm\frac{\pi}{2}$ $\theta = \phi = 0$ whatever is the position of the center of the mobile.

In a second part we consider the general case. The appendix 10 in section 18, page 52, shows that we must have then either $\psi = \phi$ or $\theta = \pm\frac{\pi}{2}$. In both case we found that z_0 is then solution of a third degree equation. Figure 33 shows a configuration in the first case and Figure 34 shows a configuration in the later case

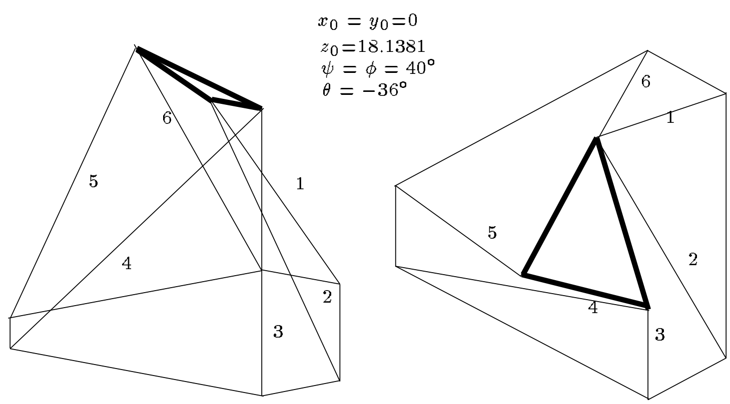


Figure 33: Perspective and top view of a singular configuration of type 5a ($\psi = \phi$)

11.4.2 Configuration 5b

We have to consider the case where the 6 segments cross the same line. Let us consider lines 1,2,3,4.

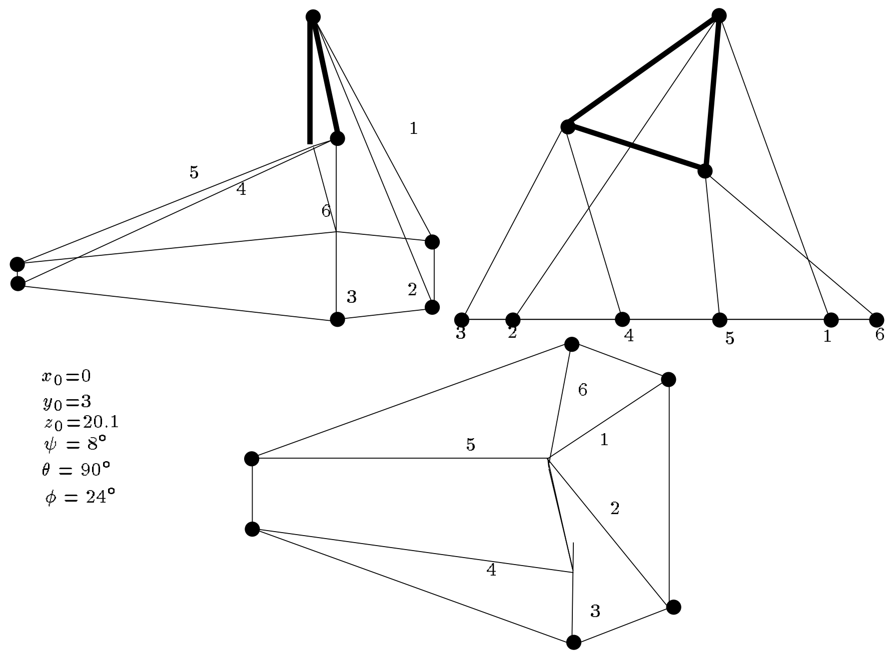


Figure 34: Perspective,side and top view of a singular configuration of type 5a ($\theta = \pm\frac{\pi}{2}$)

We have 4 possibilities for line D to intersect 1,2,3,4:
-$D = P_{12} \cap P_{34}$
-D intersects both B_1 and B_3
-$D \in P_{12}$ and D intersects B_3
-$D \in P_{34}$ and D intersects B_1
Let us now consider lines 5,6 in each of these cases.

11.5 $D = P_{12} \cap P_{34}$

We may have:
-$B_5 \in P_{12} \cap P_{34}$
-$D = P_{56} \cap P_{12} \cap P_{34}$
In the first case we may deduce from the preliminary remark that line D intersects both B_5 and C_{23}. Appendix 11 in section 19, page 54, gives the condition on x_0, y_0 according to z_0, ψ, θ, ϕ to realize the first case. Figure 36 shows such a singular configuration.

Let us consider now the second case. The three planes must have a line in common. Let us consider the intersection line of plane P_{34}, P_{56}. We know that C_{45} belongs to this line. If the intersection line lie also in the plane P_{12} then

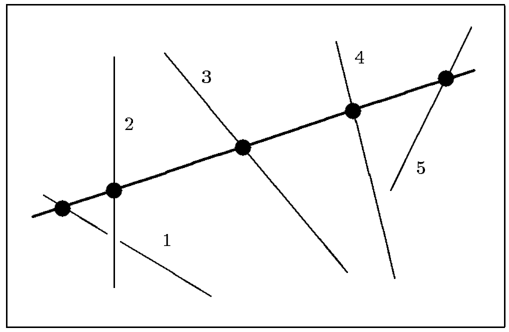


Figure 35: 5-dimensional Grassmann variety of type 5b

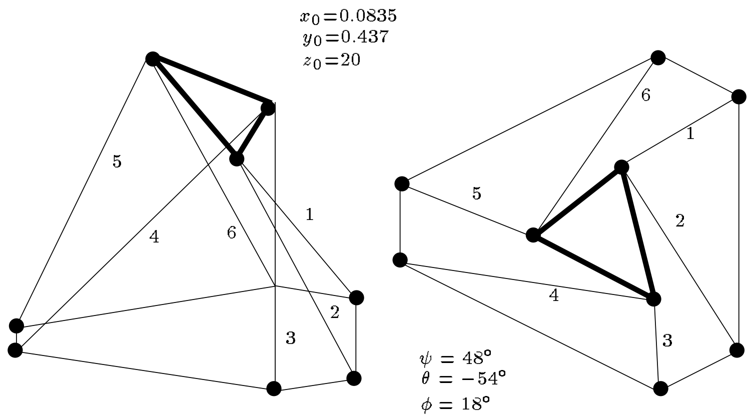


Figure 36: Perspective and top view of a singular configuration of type 5b

C_{45} must also lie in this plane. This is impossible under our assumption and therefore the three planes cannot have a line in common.

11.6 D intersects both B_1 and B_3

Thus the line common to the 6 segments is the edge B_1B_3 of the mobile. If lines 5,6 intersect both this edge this mean that the edge is coplanar to P_{56}. This is the classical Hunt's singular configuration (Figure 37).

11.7 $D \in P_{12}$ and D intersects B_3

Thus B_3 belongs to P_{12}. If D intersects also 5,6 we may have two possibilities.

-$B_5 \in P_{12}$

-the intersection line $P_{12} \cap P_{56}$ intersects B_3

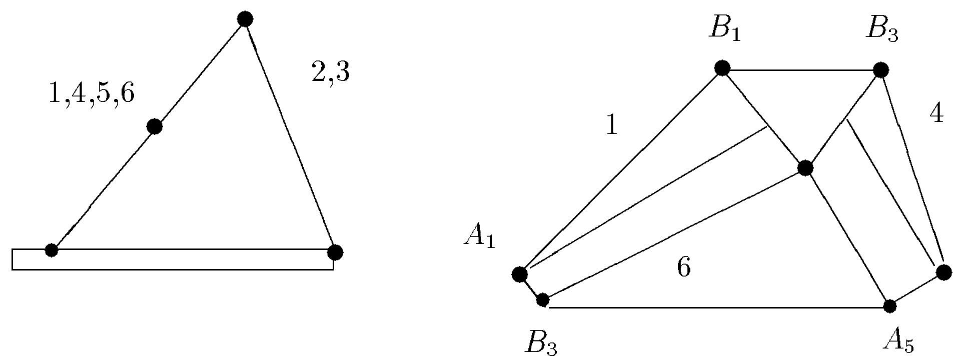


Figure 37: Hunt's singular configuration for the TSSM

In the first case that D is the edge B_3B_5 of the mobile and two of the segments are coplanar to the mobile. This is again Hunt's singular configuration.

In the second case we may deduce that the intersection line must be the line joining C_{16} and B_3. We have dealt with a similar problem in a previous part ($D = P_{12} \cap P_{34}$ and $B_5 \in D$).

11.8 $D \in P_{34}$ and D intersects B_1

This case is similar to the previous one.

12 Summary of the results

The table below summarizes the results.

3c	$\tan\psi = (ya_3 - ya_4)/(xa_3 + xa_4)$ $x_0 = A(\psi,\theta,\phi), y_0 = B(\psi,\theta,\phi), z_0 = C(\psi,\theta,\phi)$
degenerate congruence	$x_0 = A(\psi,\theta,\phi), y_0 = B(\psi,\theta,\phi), z_0 = C(\psi,\theta,\phi)$
hyperbolic congruence (first case)	$x_0 = A(z_0,\psi,\theta,\phi), y_0 = B(z_0,\psi,\theta,\phi)$
hyperbolic congruence (second case)	$y_0 = A(x_0,z_0,\psi,\theta,\phi)$ $,F(x_0,z_0,\psi,\theta,\phi) = 0$
general complex (first case)	$\theta = \phi = 0 \quad \psi = \pm\frac{\pi}{2}$
general complex (second case)	$\theta = \pm\frac{\pi}{2}$ or $\psi = \phi$ $A(x_0,y_0,\psi,\theta,\phi)z_0^3 + B(x_0,y_0,\psi,\theta,\phi)z_0^2+$ $C(x_0,y_0,\psi,\theta,\phi)z_0 + D(x_0,y_0,\psi,\theta,\phi) = 0$
special complex	$x_0 = A(z_0,\psi,\theta,\phi), y_0 = B(z_0,\psi,\theta,\phi)$

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5) Configuration 3c for the TSSM

In this case line 3 and 5 are coplanar. This yields to the conditions:

$$\mathbf{A_5B_5} \wedge \mathbf{B_1B_5} = \mathbf{0}$$

$$\mathbf{A_3B_3} \wedge \mathbf{B_1B_3} = \mathbf{0}$$

which in turn yield four independent equations. Among these equations (which are linear in x_0, y_0, z_0) we choose three of them to find x_0, y_0, z_0. The last equation give the condition to be fulfilled by the orientation parameters. This last equation is:

$$-(y_1 - y_3)(-v3xa_3 - v3xa_4 - v6ya_3 + v6ya_4)(v7x_3 + v8y_1 - v8y_3)(v7x_3 - v8y_1 + v8y_3) = 0$$

The last three terms can be equal to zero. However it can be shown that the second term must be equal to zero.

In term of the Euler's angles this is equivalent to :

$$sin\theta(-(xa_3 + xa_4)sin\psi - (ya_4 - ya_3)cos\psi) = 0$$

If $\theta = 0$ we get $z_0 = 0$, the mobile and the base are coplanar. Otherwise we found :

$$tan\psi = \frac{ya_3 - ya_4}{xa_3 + xa_4}$$

The values of x_0, y_0, z_0 can be then computed if we fixed θ.

```
comment: line 3,5 are coplanar with the mobile


comment coordinates of the articulation points on the base and the mobile$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$

comment : rotation matrix$
rot:=mat((v1,v2,v3),(v4,v5,v6),(v7,v8,v9))$

b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$

b1b3:=b3-b1$b1b5:=b5-b1$

nn:=mat((a3b3(2,1)*b1b3(3,1)-b1b3(2,1)*a3b3(3,1)),
(a3b3(3,1)*b1b3(1,1)-b1b3(3,1)*a3b3(1,1)),
(a3b3(1,1)*b1b3(2,1)-b1b3(1,1)*a3b3(2,1)))$

nn1:=mat((a5b5(2,1)*b1b5(3,1)-b1b5(2,1)*a5b5(3,1)),
(a5b5(3,1)*b1b5(1,1)-b1b5(3,1)*a5b5(1,1)),
(a5b5(1,1)*b1b5(2,1)-b1b5(1,1)*a5b5(2,1)))$
```

```
eq1:=nn(1,1)$
eq2:=nn(2,1)$
eq3:=nn1(1,1)$
eq4:=nn1(2,1)$
solve(lst(eq1,eq2,eq3),x0,y0,z0)$
let x0=soln(1,1)$
let y0=soln(1,2)$
let z0=soln(1,3)$
eqfin:=num(eq4)$
array w(10)$
factorize(eqfin,w)$
let v1*v8-v2*v7=-v6$
let v4*v8-v5*v7=v3$
w(0);
(-1)
w(1);
 - v3*xa3 - v3*xa4 - v6*ya3 + v6*ya4
w(2);
v7*x3 + v8*y1 - v8*y3
w(3);
v7*x3 - v8*y1 + v8*y3
w(4);
y1 - y3

let w(1)=0$
x0;
            2                2                     2
( - v1 v7 x3  ya3 + v1 v7 x3  ya4 - 2 v2 v3 x3 y1  + 2 v2 v3 x3 y1 y3
            2                2
  + v2 v8 y1  ya3 - v2 v8 y1  ya4 - 2 v2 v8 y1 y3 ya3 + 2 v2 v8 y1 y3
               2                2
 ya4 + v2 v8 y3  ya3 - v2 v8 y3  ya4 + 2 v3 x3 xa3 y1 - 2 v3 x3 xa3 y3

  + v6 x3 y1 ya3 - v6 x3 y1 ya4 - v6 x3 y3 ya3 + v6 x3 y3 ya4)/(2 v3

   x3 (y1 - y3))
y0;
                 2
( - 2 v3 v5 x3 y1  + 2 v3 v5 x3 y1 y3 + v3 x3 y1 ya3 + v3 x3 y1 ya4 -
                                      2               2
 v3 x3 y3 ya3 - v3 x3 y3 ya4 - v4 v7 x3  ya3 + v4 v7 x3  ya4 + v5 v8
   2                2
 y1  ya3 - v5 v8 y1  ya4 - 2 v5 v8 y1 y3 ya3 + 2 v5 v8 y1 y3 ya4 + v5
      2                2
 v8 y3  ya3 - v5 v8 y3  ya4)/(2 v3 x3 (y1 - y3))
z0;
                 2                        2   2           2   2
( - 2 v3 v8 x3 y1  + 2 v3 v8 x3 y1 y3 - v7  x3  ya3 + v7  x3  ya4 +
   2   2          2   2           2                 2                2
 v8  y1  ya3 - v8  y1  ya4 - 2 v8  y1 y3 ya3 + 2 v8  y1 y3 ya4 + v8
   2          2   2
 y3  ya3 - v8  y3  ya4)/(2 v3 x3 (y1 - y3))
end;
```

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6) Singularity of type 4d for the TSSM

We assume here that B_3 lie in the plane spanned by lines 1,2 and that line 5 intersects B_3. If B_3 lie in the plane P_{12} then :

$$\mathbf{A_1B_3}.(\mathbf{A_1B_1} \wedge \mathbf{A_2B_1}) = 0 \tag{16}$$

If line 5 intersects B_3 then :

$$\mathbf{A_5B_5} \wedge \mathbf{A_5B_3} = 0 \tag{17}$$

Equation 16 is a linear equation in term of x_0, y_0, z_0. Equation 17 yields two independent equations which are linear in term of x_0, y_0, z_0. Thus we get a system of three linear equations which permit to find x_0, y_0, z_0 according to ψ, θ, ϕ.

```
comment coordinates of the articulation points on the base and the mobile$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$

comment : rotation matrix$
rot:=mat((v1,v2,v3),(v4,v5,v6),(v7,v8,v9))$
let v4*v8-v5*v7=v3$
let v2*v7-v1*v8=v6$
let v1*v5-v2*v4=v9$
let v1*v3+v4*v6+v3*v9=0$

b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$


b1b3:=b3-b1$
b1b5:=b5-b1$

comment: the normal to the flat pencils 1-2,3-4,5-6$
n12:=mat((a1b1(2,1)*a2b2(3,1)-a2b2(2,1)*a1b1(3,1)),
(a1b1(3,1)*a2b2(1,1)-a2b2(3,1)*a1b1(1,1)),
(a1b1(1,1)*a2b2(2,1)-a2b2(1,1)*a1b1(2,1)))$
a5b3:=b3-a5$
a1b3:=b3-a1$
eq1:=n12(1,1)*a1b3(1,1)+n12(2,1)*a1b3(2,1)+n12(3,1)*a1b3(3,1);
nn:=mat((a5b5(2,1)*a5b3(3,1)-a5b3(2,1)*a5b5(3,1)),
(a5b5(3,1)*a5b3(1,1)-a5b3(3,1)*a5b5(1,1)),
(a5b5(1,1)*a5b3(2,1)-a5b3(1,1)*a5b5(2,1)))$
eq2:=nn(1,1);
eq3:=nn(2,1);
solve(lst(eq1,eq2,eq3),x0,y0,z0)$
on gcd;
let x0=soln(1,1)$
let y0=soln(1,2)$
let z0=soln(1,3)$
x0;
y0;
z0;
         2                2                2                2
```

```
( - v1*v7 *x3*ya1 + v1*v7 *x3*ya4 + v2*v7 *y1*ya1 - v2*v7 *y1*ya4 - v2

   2                2
 *v7 *y3*ya1 + v2*v7 *y3*ya4 + v3*v7*v9*x3*y1 - v3*v7*v9*x3*y3 - v3*v7

                                                         2
 *xa4*y1 + v3*v7*xa4*y3 + v3*v8*v9*y1*y3 - v3*v8*v9*y3  + v4*v6*v7*x3*

                                                      2
 y1 - v4*v6*v7*x3*y3 + v5*v6*v7*y1*y3 - v5*v6*v7*y3  - v6*v7*y1*ya1 +

 v6*v7*y1*ya4 + v6*v7*y3*ya1 - v6*v7*y3*ya4)/(v3*v7*(y1 - y3))
```

```
                                                         2
( - v3*v4*x3*y1 + v3*v4*x3*y3 - v3*v5*y1*y3 + v3*v5*y3  + v3*y1*ya1 -

 v3*y3*ya1 - v4*v7*x3*ya1 + v4*v7*x3*ya4 + v5*v7*y1*ya1 - v5*v7*y1*ya4

  - v5*v7*y3*ya1 + v5*v7*y3*ya4)/(v3*(y1 - y3))
```

```
                                                         2     2
( - v3*v7*x3*y1 + v3*v7*x3*y3 - v3*v8*y1*y3 + v3*v8*y3  - v7 *x3*ya1

      2
  + v7 *x3*ya4 + v7*v8*y1*ya1 - v7*v8*y1*ya4 - v7*v8*y3*ya1 + v7*v8*y3

 *ya4)/(v3*(y1 - y3))
```

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7 Singularity of type 4b for the TSSM, first case

In this case B_3 belongs tp P_{12} and B_1 to P_{34}. This yields two linear equations which enable to calculate x_0, y_0 according to z_0, ψ, θ, ϕ. We get:

$$y_0 = \frac{(v5y3 - v5y1 + v4x3)z0 + (v8y3 - v8y1 + v7x3)ya1 + v3x3y1}{v8y3 - v8y1 + v7x3}$$

and

$$\begin{aligned} x_0 \quad = \quad & ((v2y3 - v2y1 + v1x3)(ya4 - ya3)z0 \\ & +(v8y3 - v8y1 + v7x3)(xa3ya4 - xa4ya3) + x3y1(v6ya3 - v6ya4) \\ & +(xa4 - xa3)(v8y3 - v8y1 + v7x3)ya1) \\ & /((v8y3 - v8y1 + v7x3)(ya4 - ya3)) \end{aligned}$$

```
(c3) /* solve the problem of singularities of type 4b
for the TSSM.
a line D1 belongs to plane 1,2 and
B3 belongs to D1
a line D2 belongs to plane 3,4 and
B1 belongs to D2
*/

/* rotation matrix of the mobile */
```

```
rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$

(c16) /* position of the center of the mobile */

cen:matrix([x0],[y0],[z0])$

(c31) /*u12: normal to plane 1-2*/

(c32) cross(a1b1,a2b2,u12)$

(c34) /*u34: normal to plane 3-4*/

(c35) cross(a3b3,a4b4,u34)$

(c37) /* a point M belongs to P12 if A1M.u12=0*/

m:matrix([x],[y],[z])$

(c39) eq1:expand(dot(a1m,u12))$

(c41) /*a point M belongs to  P34 if A3M.u34=0*/

(c42) eq2:expand(dot(a3m,u34))$

(c44) eq3:ev(eq1,x=b3[1],y=b3[2],z=b3[3])$

(c46) eq4:ev(eq2,x=b1[1],y=b1[2],z=b1[3])$

linsolve([eq3[1][1],eq4[1][1]],[x0,y0]),globalsolve:true$

(c58) y0;
      (v5 y3 - v5 y1 + v4 x3) z0 + (v8 y3 - v8 y1 + v7 x3) ya1 + v3 x3 y1
(d58) ------------------------------------------------------------------
                              v8 y3 - v8 y1 + v7 x3
x0;

(d123) ((v2 y3 - v2 y1 + v1 x3) (ya4 - ya3) z0

 + (v8 y3 - v8 y1 + v7 x3) (xa3 ya4 - xa4 ya3) + x3 y1 (v6 ya3 - v6 ya4)

 + (xa4 - xa3) (v8 y3 - v8 y1 + v7 x3) ya1)

/((v8 y3 - v8 y1 + v7 x3) (ya4 - ya3))
```

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8) Singularity 4b for the TSSM, second case

In this case we have :

$$D_1 = P_{12} \cap P_{34}$$

and

$$D_2 = B_1 B_3$$

This last condition means that line 5 and the mobile are coplanar. We get:

$$y_0 = -\frac{b9z_0 - b6ya_4 + b3xa_4 + b3x_0}{b6}$$

The first condition yields four linear equations in term of the coordinates x, y, z of the intersection point of D_1 and line 5. These equations are denoted $eq1, eq2, eq3, eq4$ in the following MACSYMA program A numerical procedure may be used to get x, y, z and verify the last equation.

```
(c3) /* solve the problem of singularities of type 4b
for the TSSM.
D1=P12 inter P34
D2=B1B3 D1,D2 intersects line 5
*/

/* rotation matrix of the mobile */

rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$

(c16) /* position of the center of the mobile */

cen:matrix([x0],[y0],[z0])$

(c31) /*u12: normal to plane 1-2*/

(c32) cross(a1b1,a2b2,u12)$

(c34) /*u34: normal to plane 3-4*/

(c35) cross(a3b3,a4b4,u34)$

(c37) /*the mobile and A5B5 are coplanar
=> A5B5.(B1B3^B1B5)=0*/

(c40) cross(b1b5,b1b3,pvec1)$

(c41) eq6:dot(pvec1,a5b5)$

(c49) eq6:apply1(eq6,ru1,ru2,ru9,ru10,ru6,ru5)$

(c50) linsolve(eq6,y0),globalsolve:true$

(c51) /* equation of D1*/

m:matrix([x],[y],[z])$

(c53) eq1:ratsimp(dot(a1m,u12))$

(c56) eq2:ratsimp(dot(a4m,u34))$


(c58) /*equation of A5B5*/

(c59) cross(a5b5,a6b6,pvec3)$

(c61) eq3:ratsimp(dot(a5m,pvec3))$

(c62) eq4:ratsimp(dot(a5m,b1b3))$

(c64) y0;
                              v9 z0 - v6 ya4 + v3 xa4 + v3 x0
(d64)                       - -------------------------------
                                            v6
```

```
(c65) eq1;
            2 xa2 (v9 z0 - v6 ya4 + v3 xa4 + v3 x0)
(d65) [z (- --------------------------------------- - 2 xa2 ya1 + 2 v5 xa2 y1)
                              v6

                  + (2 xa2 ya1 - 2 xa2 y) z0 + 2 v8 xa2 y1 ya1 - 2 v8 xa2 y y1]

(c66) eq2;
            (xa4 - xa3) (v9 z0 - v6 ya4 + v3 xa4 + v3 x0)
(d66) [z (- ---------------------------------------------
                                 v6

 + (- v2 y3 + xa3 - v1 x3 - x0) ya4 + (v2 y3 - xa4 + v1 x3 + x0) ya3

 + (v5 xa4 - v5 xa3) y3 + v4 x3 xa4 - v4 x3 xa3)

 + ((x - xa3) ya4 + (xa4 - x) ya3 + (xa3 - xa4) y) z0

 + ((v8 x - v8 xa3) y3 - v7 x3 xa3 + v7 x x3) ya4

 + ((v8 xa4 - v8 x) y3 + v7 x3 xa4 - v7 x x3) ya3 + (v8 xa3 - v8 xa4) y y3

 + (v7 x3 xa3 - v7 x3 xa4) y]

(c68) eq3;
(d68) [((- xa3 - x) ya4 + (xa4 + x) ya3 + (xa3 - xa4) y) z0

 + ((v2 y3 + xa3 - v1 x3 + x0) ya4 + (- v2 y3 - xa4 + v1 x3 - x0) ya3

 + (v5 xa4 - v5 xa3) y3 + (xa4 - xa3) y0 - v4 x3 xa4 + v4 x3 xa3) z

 + ((- v8 xa3 - v8 x) y3 + v7 x3 xa3 + v7 x x3) ya4

 + ((v8 xa4 + v8 x) y3 - v7 x3 xa4 - v7 x x3) ya3 + (v8 xa3 - v8 xa4) y y3

 + (v7 x3 xa4 - v7 x3 xa3) y]

(c70) eq4;
(d70) [(v8 y3 - v8 y1 + v7 x3) z + (- v5 y3 + v5 y1 - v4 x3) ya4

 + (v5 y + v2 xa4 + v2 x) y3 + (- v5 y - v2 xa4 - v2 x) y1 + v4 x3 y

 + v1 x3 xa4 + v1 x x3]
```

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9) The TSSM as a general complex,first case

3 lines belonging to the the flat pencils spanned by 1-2,4-3,5-6 lying on the mobile must intersect the same point m if the 6 lines belong to a general complex. We assume there is only rotation around the z axis.

We get the condition:

$$2v1z0(-x_3ya_3 + x_3ya_4 - xa_3y_1 + xa_3y_3 + xa_4y_1 - xa_4y_3) = 0$$

which is true if $v1 = cos\psi = 0$ and therefore $\psi = \pm\frac{\pi}{2}$ (the second term will be zero if the edge of the mobile and the base were parallel).

```
comment 3 lines belonging to the the flat pencils spanned by
1-2,4-3,5-6 lying on the mobile must intersect the same point m
```

```
if the 6 lines belong to a general complex. We assume there
is only rotation around the z axis$

comment coordinates of the articulation points on the base and the mobile$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$

comment : rotation matrix$
rot:=mat((v1,v2,0),(-v2,v1,0),(0,0,0))$
let v4*v8-v5*v7=v3$
let v2*v7-v1*v8=v6$
let v1*v5-v2*v4=v9$
let v1*v3+v4*v6+v3*v9=0$

comment: the axis vector of the link$
b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$

b1b3:=b3-b1$
b1b5:=b5-b1$

comment: the normal to the flat pencils 1-2,3-4,5-6$
n12:=mat((a1b1(2,1)*a2b2(3,1)-a2b2(2,1)*a1b1(3,1)),
(a1b1(3,1)*a2b2(1,1)-a2b2(3,1)*a1b1(1,1)),
(a1b1(1,1)*a2b2(2,1)-a2b2(1,1)*a1b1(2,1)))$
n34:=mat((a4b4(2,1)*a3b3(3,1)-a3b3(2,1)*a4b4(3,1)),
(a4b4(3,1)*a3b3(1,1)-a3b3(3,1)*a4b4(1,1)),
(a4b4(1,1)*a3b3(2,1)-a3b3(1,1)*a4b4(2,1)))$
n56:=mat((a6b6(2,1)*a5b5(3,1)-a5b5(2,1)*a6b6(3,1)),
(a6b6(3,1)*a5b5(1,1)-a5b5(3,1)*a6b6(1,1)),
(a6b6(1,1)*a5b5(2,1)-a5b5(1,1)*a6b6(2,1)))$

comment: normal to the mobile$

normal:=mat((b1b3(2,1)*b1b5(3,1)-b1b5(2,1)*b1b3(3,1)),
(b1b3(3,1)*b1b5(1,1)-b1b5(3,1)*b1b3(1,1)),
(b1b3(1,1)*b1b5(2,1)-b1b5(1,1)*b1b3(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(z0))$

b1m:=m-b1$
b3m:=m-b3$
b5m:=m-b5$

comment intersection condition $
eq1:=b1m(1,1)*n12(1,1)+b1m(2,1)*n12(2,1)+b1m(3,1)*n12(3,1)$
eq2:=b3m(1,1)*n34(1,1)+b3m(2,1)*n34(2,1)+b3m(3,1)*n34(3,1)$
```

```
eq3:=b5m(1,1)*n56(1,1)+b5m(2,1)*n56(2,1)+b5m(3,1)*n56(3,1)$

comment: the two equations eq1,eq2 enable to find the coordinates x,y
of the intersection point$
solve(lst(eq1,eq2),x,y)$
let x=soln(1,1)$
let y=soln(1,2)$
comment : intersection condition$
eq3;
2*v1*z0*( - x3*ya3 + x3*ya4 - xa3*y1 + xa3*y3 + xa4*y1 - xa4*y3)
end;
```

18 Appendix

10) The TSSM as a general complex,second case

3 lines belonging to the the flat pencils spanned by 1-2,4-3,5-6 lying on the mobile must intersect the same point m if the 6 lines belong to a general complex.

We write the following equations :

$$
\begin{aligned}
eq1 &= B_1M \wedge N_{12} = 0 && (18)\\
eq2 &= B_1M \wedge N_m = 0 && (19)\\
eq3 &= B_3M \wedge N_{34} = 0 && (20)\\
eq4 &= B_3M \wedge N_m = 0 && (21)\\
eq5 &= B_5M \wedge N_{56} = 0 && (22)\\
eq6 &= B_4M \wedge N_m = 0 && (23)
\end{aligned}
$$

If we combine equations 19 and 21 we get :

$$v9x_3(v3 - v7) = 0$$

which is true if :

$$\theta = 0 \quad \theta = \pm\frac{\pi}{2} \quad \psi = \phi$$

The first case has been treated in the previous appendix.

```
comment coordinates of the articulation points on the base and the mobile$
a1:=mat((xa1),(ya1),(0))$b1r:=mat((0),(y1),(0))$
a2:=mat((-xa1),(ya1),(0))$b2r:=mat((0),(y1),(0))$
a3:=mat((xa3),(ya3),(0))$b3r:=mat((x3),(y3),(0))$
a4:=mat((xa4),(ya4),(0))$b4r:=mat((x3),(y3),(0))$
a5:=mat((-xa4),(ya4),(0))$b5r:=mat((-x3),(y3),(0))$
a6:=mat((-xa3),(ya3),(0))$b6r:=mat((-x3),(y3),(0))$

comment coordinates of the articulation points on the mobile$
b1r:=mat((0),(y1),(0))$
b2r:=mat((0),(y1),(0))$
b3r:=mat((x3),(y3),(0))$
b4r:=mat((x3),(y3),(0))$
b5r:=mat((-x3),(y3),(0))$
b6r:=mat((-x3),(y3),(0))$

comment position of the center of the mobile$
cen:=mat((x0),(y0),(z0))$

comment : rotation matrix$
rot:=mat((v1,v2,v3),(v4,v5,v6),(v7,v8,v9))$
let v4*v8-v5*v7=v3$
```

```
let v2*v7-v1*v8=v6$
let v1*v5-v2*v4=v9$
let v1*v3+v4*v6+v3*v9=0$

comment: the axis vector of the link$

b1:=cen+rot*b1r$a1b1:=b1-a1$
b2:=cen+rot*b2r$a2b2:=b2-a2$
b3:=cen+rot*b3r$a3b3:=b3-a3$
b4:=cen+rot*b4r$a4b4:=b4-a4$
b5:=cen+rot*b5r$a5b5:=b5-a5$
b6:=cen+rot*b6r$a6b6:=b6-a6$

b1b3:=b3-b1$
b1b5:=b5-b1$

comment: the normal to the flat pencils 1-2,3-4,5-6$
n12:=mat((a1b1(2,1)*a2b2(3,1)-a2b2(2,1)*a1b1(3,1)),
(a1b1(3,1)*a2b2(1,1)-a2b2(3,1)*a1b1(1,1)),
(a1b1(1,1)*a2b2(2,1)-a2b2(1,1)*a1b1(2,1)))$
n34:=mat((a4b4(2,1)*a3b3(3,1)-a3b3(2,1)*a4b4(3,1)),
(a4b4(3,1)*a3b3(1,1)-a3b3(3,1)*a4b4(1,1)),
(a4b4(1,1)*a3b3(2,1)-a3b3(1,1)*a4b4(2,1)))$
n56:=mat((a6b6(2,1)*a5b5(3,1)-a5b5(2,1)*a6b6(3,1)),
(a6b6(3,1)*a5b5(1,1)-a5b5(3,1)*a6b6(1,1)),
(a6b6(1,1)*a5b5(2,1)-a5b5(1,1)*a6b6(2,1)))$

comment: normal to the mobile$

normal:=mat((b1b3(2,1)*b1b5(3,1)-b1b5(2,1)*b1b3(3,1)),
(b1b3(3,1)*b1b5(1,1)-b1b5(3,1)*b1b3(1,1)),
(b1b3(1,1)*b1b5(2,1)-b1b5(1,1)*b1b3(2,1)))$

comment coordinates of the intersection point $
m:=mat((x),(y),(z))$

b1m:=m-b1$
b3m:=m-b3$
b5m:=m-b5$

comment intersection condition $
eq1:=b1m(1,1)*n12(1,1)+b1m(2,1)*n12(2,1)+b1m(3,1)*n12(3,1)$
eq2:=b1m(1,1)*normal(1,1)+b1m(2,1)*normal(2,1)+b1m(3,1)*normal(3,1)$
eq3:=b3m(1,1)*n34(1,1)+b3m(2,1)*n34(2,1)+b3m(3,1)*n34(3,1)$
eq4:=b3m(1,1)*normal(1,1)+b3m(2,1)*normal(2,1)+b3m(3,1)*normal(3,1)$
eq5:=b5m(1,1)*n56(1,1)+b5m(2,1)*n56(2,1)+b5m(3,1)*n56(3,1)$
eq6:=b5m(1,1)*normal(1,1)+b5m(2,1)*normal(2,1)+b5m(3,1)*normal(3,1)$

let v2*v3+v5*v6+v8*v9=0$

eq1;

2*xa1*( - v5*y1*z - v8*y1*ya1 + v8*y1*y - y0*z - ya1*z0 + ya1*z + y*z0)

eq2;
v3*x0 - v3*x + v6*y0 - v6*y + v9*z0 - v9*z

eq3;
 - v1*x3*ya3*z + v1*x3*ya4*z - v2*y3*ya3*z + v2*y3*ya4*z + v4*x3*xa3*z
```

```
 - v4*x3*xa4*z + v5*xa3*y3*z - v5*xa4*y3*z + v7*x3*xa3*ya4 - v7*x3*xa3

*y - v7*x3*xa4*ya3 + v7*x3*xa4*y + v7*x3*x*ya3 - v7*x3*x*ya4 + v8*xa3*

y3*ya4 - v8*xa3*y3*y - v8*xa4*y3*ya3 + v8*xa4*y3*y + v8*x*y3*ya3 - v8*

x*y3*ya4 - x0*ya3*z + x0*ya4*z + xa3*y0*z + xa3*ya4*z0 - xa3*ya4*z -

xa3*y*z0 - xa4*y0*z - xa4*ya3*z0 + xa4*ya3*z + xa4*y*z0 + x*ya3*z0 - x

*ya4*z0

eq4;

v3*v9*x3 - v3*x0 + v3*x - v6*y0 + v6*y - v7*v9*x3 - v9*z0 + v9*z

eq5;
 - v1*x3*ya3*z + v1*x3*ya4*z + v2*y3*ya3*z - v2*y3*ya4*z - v4*x3*xa3*z

 + v4*x3*xa4*z + v5*xa3*y3*z - v5*xa4*y3*z - v7*x3*xa3*ya4 + v7*x3*xa3

*y + v7*x3*xa4*ya3 - v7*x3*xa4*y + v7*x3*x*ya3 - v7*x3*x*ya4 + v8*xa3*

y3*ya4 - v8*xa3*y3*y - v8*xa4*y3*ya3 + v8*xa4*y3*y - v8*x*y3*ya3 + v8*

x*y3*ya4 + x0*ya3*z - x0*ya4*z + xa3*y0*z + xa3*ya4*z0 - xa3*ya4*z -

xa3*y*z0 - xa4*y0*z - xa4*ya3*z0 + xa4*ya3*z + xa4*y*z0 - x*ya3*z0 + x

*ya4*z0

eq2+eq4;

v9*x3*(v3 - v7)
end;
```

19 Appendix

11) Singularity of type 5b for the TSSM

We suppose here that a line D intersects the 6 segments. This line is the intersection of the planes spanned by 1,2 and 3,4 and B_5 belongs to this line.

The point B_5 belongs to the plane spanned by 1,2 if:

$$\mathbf{A_1B_5}.(\mathbf{A_1B_1} \wedge \mathbf{A_2B_2}) = 0 \tag{24}$$

In a similar way B_5 belongs to the plane spanned by 3,4 if:

$$\mathbf{A_3B_5}.(\mathbf{A_3B_3} \wedge \mathbf{A_4B_4}) = 0 \tag{25}$$

This system may be solved to give the expression of x_0, y_0 according to z_0, ψ, θ, ϕ. We get:

$$y_0 = \frac{(b5y_3 - b5y_1 - b4x_3)z_0 + (b8y_3 - b8y_1 - b7x_3)ya_1 - b3x_3y_1}{b8y_3 - b8y_1 - b7x_3} \tag{26}$$

The expression yielding to x_0 is a little bit more complicated :

$$\begin{aligned} x_0 \quad = \quad & ((v1(v8y_3 - v8y_1 - v7x_3)(ya_4 - ya_3) - 2v3(xa_4 - xa_3)y_3)z0 \\ & -(v8y_3 - v8y_1 - v7x_3)(v7(-xa_3ya_4 + xa_4ya_3 + (xa_3 - xa_4)ya_1) \\ & +v6y_3(ya_4 - ya_3)) - v3(xa_4 - xa_3)(y_3 - y_1)(v8y_3 - v7x_3)) \\ & /(v7(v8y_3 - v8y_1 - v7x_3)(ya_4 - ya_3)) \end{aligned}$$

```
(c3) /* solve the problem of singularities of type 5b
for the TSSM.
a line D is the intersection of plane 1,2 and
plane 3,4 and D5 intersects B5
*/

/* rotation matrix of the mobile */

rot:matrix([v1,v2,v3],[v4,v5,v6],[v7,v8,v9])$

cen:matrix([x0],[y0],[z0])$

(c31) /*u12: normal to plane 1-2*/

(c32) cross(a1b1,a2b2,u12)$

(c34) /*u34: normal to plane 3-4*/

(c35) cross(a3b3,a4b4,u34)$

(c37) /* a point M belongs to D if A1M.u12=0 and
A3M.u34=0 */

m:matrix([x],[y],[z])$

(c39) eq1:expand(dot(a1m,u12))$

(c41) eq2:expand(dot(a3m,u34))$

(c56) linsolve(eq1[1][1],y0),globalsolve:true$

(c63) linsolve(eq2[1][1],x0),globalsolve:true$

(c65) x0;
(d65)
((v1 (v8 y3 - v8 y1 - v7 x3) (ya4 - ya3) - 2 v3 (xa4 - xa3) y3) z0

-(v8 y3 - v8 y1 - v7 x3) (v7 (-xa3 ya4 + xa4 ya3 + (xa3 - xa4) ya1)

+v6 y3 (ya4 - ya3)) - v3 (xa4 - xa3) (y3 - y1) (v8 y3 - v7 x3))

/(v7 (v8 y3 - v8 y1 - v7 x3) (ya4 - ya3))


(c66) y0;
      (v5 y3 - v5 y1 - v4 x3) z0 + (v8 y3 - v8 y1 - v7 x3) ya1 - v3 x3 y1
(d66) -------------------------------------------------------------------
                             v8 y3 - v8 y1 - v7 x3
```

20 Study of the SSM

We will deal now with the case of the SSM (Figure 38)

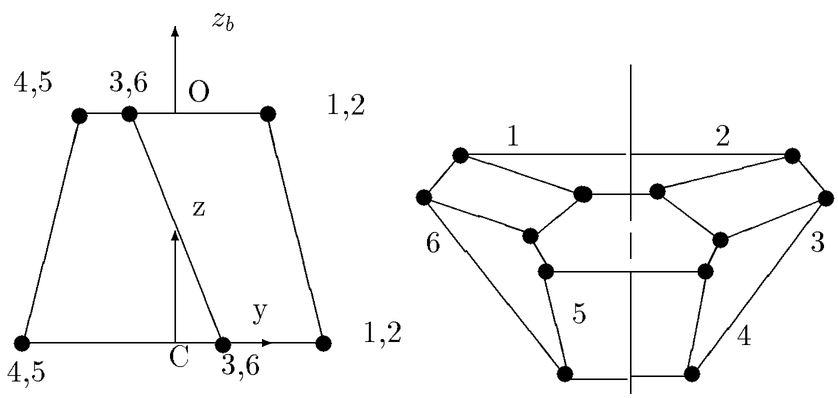

Figure 38: the simplified symmetric manipulator (SSM)

.

20.1 Subsets of 2,3 bars

20.1.1 Subset of 2 bars

In the case of 2 bars we have only to count if there is 6 bars in our mechanism. Thus this verification is trivial.

20.1.2 Subset of 3 bars

There is no set of three lines which have collinear articulation points. Thus the only case where three segments are in the same plane is obtained when the base and the mobile are in the same plane. In this case all the lines are in this plane.

20.2 Subsets of 4 bars

20.2.1 Degeneracy of type 3d

We have seen in the preceding part that 4 segments will lie in a plane if and only if the mobile and the base are in the same plane. Thus we have a singular configuration if the mobile and the base are in the same plane.

20.2.2 Degeneracy of type 3c

Let us say to begin with that lines 1,2,3,4 have a common point M. In this case we have:

$$\mathbf{A_1M} = \lambda_1 \mathbf{n_1} \qquad \mathbf{A_2M} = \lambda_2 \mathbf{n_2} \tag{27}$$

where λ_1 denotes the distance from the articulation points A_1 to the intersection point M. This yield to :

$$\mathbf{A_1A_2} \wedge \mathbf{n_2} = \lambda_1 \mathbf{n_1} \wedge \mathbf{n_2} \tag{28}$$

where $\wedge$ denotes the cross product. In the same manner we have:

$$\mathbf{A_1A_3} \wedge \mathbf{n_3} = \lambda_1 \mathbf{n_1} \wedge \mathbf{n_3} \qquad \mathbf{A_1A_4} \wedge \mathbf{n_4} = \lambda_1 \mathbf{n_1} \wedge \mathbf{n_4} \tag{29}$$

If λ_1' denote the distance between M and the articulation point B_1 we get in the same manner:

$$\mathbf{B_1B_2} \wedge \mathbf{n_2} = \lambda_1' \mathbf{n_1} \wedge \mathbf{n_2} \qquad \mathbf{B_1B_3} \wedge \mathbf{n_3} = \lambda_1' \mathbf{n_1} \wedge \mathbf{n_3} \tag{30}$$

$$\mathbf{B_1B_4} \wedge \mathbf{n_4} = \lambda_1' \mathbf{n_1} \wedge \mathbf{n_4} \tag{31}$$

We calculate λ_1, λ_1' by:

$$\lambda_1 = \frac{||\mathbf{A_1A_2} \wedge \mathbf{n_2}||}{||\mathbf{n_1} \wedge \mathbf{n_2}||} \qquad \lambda_1' = \frac{||\mathbf{B_1B_2} \wedge \mathbf{n_2}||}{||\mathbf{n_1} \wedge \mathbf{n_2}||} \tag{32}$$

where $||\ ||$ denotes the euclidian norm of a vector. Let us calculate now $\mathbf{A_1B_1}$:

$$\mathbf{A_1B_1} = \rho_1 \mathbf{n_1} = \mathbf{A_1A_2} + \mathbf{A_2B_2} + \mathbf{B_2B_1} \tag{33}$$

which yield to:

$$\rho_1 \mathbf{n_1} \wedge \mathbf{n_2} = (\mathbf{A_1A_2} + \mathbf{A_2B_2} + \mathbf{B_2B_1}) \wedge \mathbf{n_2} = (\mathbf{A_1A_2} + \mathbf{B_2B_1}) \wedge \mathbf{n_2} \tag{34}$$

and therefore

$$\rho_1 = \frac{||(\mathbf{A_1A_2} + \mathbf{B_2B_1}) \wedge \mathbf{n_2}||}{||\mathbf{n_1} \wedge \mathbf{n_2}||} \tag{35}$$

but $\rho_1 = \lambda_1 - \lambda_1'$ and therefore a necessary condition for the intersection of the segment 1 and 2 is:

$$||(\mathbf{A_1A_2} + \mathbf{B_2B_1}) \wedge \mathbf{n_2}|| = ||\mathbf{A_1A_2} \wedge \mathbf{n_2}|| + ||\mathbf{B_2B_1} \wedge \mathbf{n_2}|| \tag{36}$$

This mean that either A_1A_2 is parallel to B_1B_2 or B_1B_2 is collinear to either A_1B_1 or A_2B_2. In the former case M is either B_2 or B_1. The same reasoning can be made for point 3 and 4 and therefore A_3A_4 is either parallel to B_3B_4 or B_3B_4 is collinear to either A_3B_3 or A_4B_4.

α) B_3B_4 parallel to A_3A_4

This yields to the necessary condition :

$$\psi = \theta = \phi = 0 \tag{37}$$

which mean that the base and the mobile are parallel. From this point we develop equations (2)(3) we find the necessary and sufficient conditions:

$$if \quad \psi = \theta = \phi = 0 \qquad xa_2 = xa_4 \qquad 2ya_3 - ya_1 - ya_4 = 0 \tag{38}$$

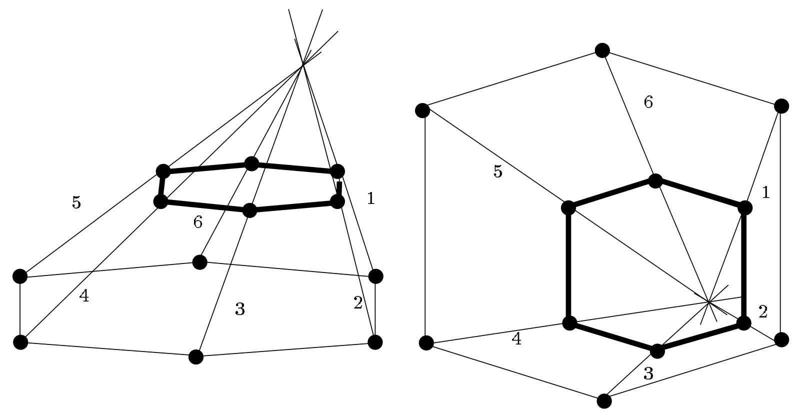

Figure 39: Perspective and top view of a singular configuration of dimension 4

then for every x_o, y_o, z_o we have a singular configuration. Appendix 12 in section 21, page 21, gives the details of the calculation which is valid whatever is the choice of points. This yields to rule 1.

Rule 1: The articulation points of a SSM much satisfy the relation $xa_2 \neq xa_4$ **or** $2ya_3 - ya_1 - ya_4 \neq 0$

It is easy to design a manipulator where these relations are not satisfied. Figure 39 gives an example of such configuration. We may notice that rule 1 is true if the articulation points are equally distributed on the circles. It may be noted that, for symmetry reasons, if 4 lines have a common intersection point then all 6 lines have also this common intersection point. In this latter case we can find rather easily the relation between the links . Let us calculate the difference between ρ_1^2 and ρ_4^2 . We get:

$$||\mathbf{A_1B_1}||^2 = ||\mathbf{A_1C}||^2 + ||\mathbf{CO}||^2 + ||M\mathbf{OB_{1r}}||^2 + 2\mathbf{CO}.\mathbf{A_1C} + 2\mathbf{CO}.M\mathbf{OB_{1r}} \quad (39)$$

$$+ 2\mathbf{A_1C}.M\mathbf{OB_{1r}} \quad (40)$$

which yields to:

$$\rho_1^2 - \rho_4^2 = 2\mathbf{CO}.\mathbf{A_1A_4} + 2\mathbf{CO}.M\mathbf{B_4B_1} \quad (41)$$

In the same manner we get:

$$\rho_3^2 - \rho_6^2 = 2\mathbf{CO}.\mathbf{A_3A_6} + 2\mathbf{CO}.M\mathbf{B_6B_3} \quad (42)$$

Due to the symmetry of the position of the articulation point we have:

$$\mathbf{B_1B_4} + \mathbf{B_3B_6} = \mathbf{B_2B_5} \qquad \mathbf{A_1A_4} + \mathbf{A_3A_6} = \mathbf{A_2A_5} \quad (43)$$

which yield to

$$\rho_1^2 - \rho_4^2 + \rho_3^2 - \rho_6^2 = \rho_2^2 - \rho_5^2 \quad (44)$$

Or:

$$\rho_1^2 + \rho_3^2 + \rho_5^2 = \rho_2^2 + \rho_4^2 + \rho_6^2 \tag{45}$$

β) B_3B_4 collinear to either A_3B_3 or A_4B_4

In this case M is either B_4 or B_3. If the lines A_1B_1,A_2B_2 pass through this point ,B_1, B_2, B_3, B_4 being coplanar then all the lines are coplanar. Thus the base and the mobile are in the same plane and we know that this is a singular configuration.

It is clear that the other cases are equivalent to the preceding three cases.

20.2.3 Degeneracy of type 3b

In this case 4 lines constitute two flat pencils having a line in common but lying in distinct planes.

We will consider lines 1,2,3,4 without loss of generality. Among these four lines three of them must be coplanar. But there is not three articulation points which are co-linear. Thus to have three coplanar lines the mobile and the base must be coplanar and this a classical singular configuration.

Degeneracy of type 3a The problem is to find 4 lines which are on the same regulus.

We will consider without loss of generality the regulus defined by lines 1,2,3 and use the homogeneous coordinates with the reference frames X,Y be lines 1,2 and the origin be the point A_3. The four homogeneous coordinates $\lambda_3, \lambda_1, \lambda_2, 1$ of a point M are such that:

$$\mathbf{A_3M} = \lambda_1\mathbf{A_3A_1} + \lambda_2\mathbf{A_3B_1} + \lambda_3\mathbf{A_3A_2} + 1.\mathbf{A_3B_2} \tag{46}$$

For any point M which belongs to line 3 it exists α such that:

$$\mathbf{A_3M} = \alpha\mathbf{A_3B_3} \tag{47}$$

Therefore we get:

$$\begin{aligned} \lambda_1 &= \alpha\lambda_1(B3) \\ \lambda_2 &= \alpha\lambda_2(B3) \\ \lambda_3 &= \alpha\lambda_3(B3) \end{aligned} \tag{48}$$

Using the matrix notation we may write:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{\lambda_{1B_3}}{\lambda_{3B_3}} & 0 \\ \frac{\lambda_{2B_3}}{\lambda_{3B_3}} & 0 \end{pmatrix} \begin{pmatrix} \lambda_3 \\ 1 \end{pmatrix}$$

or:

$$y = A_3x \tag{49}$$

For any point M which belongs to line 4 it exists β such that:

$$\mathbf{A_4M} = \beta \mathbf{A_4B_4} \Leftrightarrow \mathbf{A_3M} = \beta \mathbf{A_4B_4} + \mathbf{A_3A_4} \tag{50}$$

Therefore we get:

$$\begin{aligned} \lambda_1 &= \beta(\lambda_1(B_4) - \lambda_1(A_4)) + \lambda_1(A_4) \\ \lambda_2 &= \beta(\lambda_2(B_4) - \lambda_2(A_4)) + \lambda_2(A_4) \\ \lambda_3 &= \beta(\lambda_3(B_4) - \lambda_3(A_4)) + \lambda_3(A_4) \end{aligned} \tag{51}$$

Using the matrix notation we may write:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} \lambda_3 \\ 1 \end{pmatrix}$$

with

$$P = \frac{\lambda_1(B_4) - \lambda_1(A_4)}{\lambda_3(B_4) - \lambda_3(A_4)}$$
$$Q = -\lambda_3(A_4)(\lambda_1(B_4) - \lambda_1(A_4)) + \lambda_1(A_4)$$
$$R = \frac{\lambda_2(B_4) - \lambda_2(A_4)}{\lambda_3(B_4) - \lambda_3(A_4)}$$
$$S = -\lambda_3(A_4)(\lambda_2(B_4) - \lambda_2(A_4)) + \lambda_2(A_4)$$

We have seen that any generator of the regulus defined by the line 1,2,3 has a matrix equation:

$$y = \gamma A_3 x \tag{52}$$

Thus we may write a necessary condition for a line $i, i \neq 1,2,3$ to belong to the regulus by:

$$P = \gamma \frac{\lambda_{1B_3}}{\lambda_{3B_3}} \tag{53}$$
$$Q = 0 \tag{54}$$
$$R = \gamma \frac{\lambda_{2B_3}}{\lambda_{3B_3}} \tag{55}$$
$$S = 0 \tag{56}$$

From this relation we can get three independent constraint equations in term of the position parameters. These equations are third degree polynomials in term of z_0. We calculate these equations with the help of REDUCE but the resolution of these equations seems to be rather difficult.

20.3 Subsets of 5 bars

20.3.1 configuration 4d

All the five lines are in a plane or pass through one point of this plane. Thus we need to have at least three coplanar lines and this not possible for the SSM.

20.3.2 configuration 4b

In this case three lines must constitute a flat pencil of lines and therefore this configuration is not to be considered.

20.3.3 configuration 4c

Five lines must pass through two skew lines. It is clear that this is a difficult geometric problem in the most general case. However if the orientation angles are equal to zero we have seen that lines 1-2,2-3,3-4,4-5,5-6 have an intersection point and therefore are coplanar. Thus we can apply the reasoning made for the MSSM to state that there is no two skew lines which intersect five segments.

When the base and the mobile are not parallel this problem has not been solved yet.

20.4 Subsets of 6 bars

It si clear that we found again Hunt's singular configuration as a special complex. However in the most general case two lines of the SSM being in general no more coplanar the reasoning we use for the TSSM or the MSSM cannot be used here. Thus these configurations are difficult to study.

21 Appendix

12) Singular configuration of type 3d

```
SINGULARITY OF RANK 3 FOR THE SSM
(4 lines pass through the same point)

(c61) /* singularity of rank 4: links 1,2,3,4
have a common point.
It has been shown that a necessary condition for this
configuration is that the two plate have parallel sides.
Necessary and sufficient conditions are :
A1A2^n2=lambda1 n1^n2,A1A3^n3=lambda1 n1^n3
,A1A4^n2= lambda1 n1^n4
where n denote the unit vector of the links
,lambda1 a constant
and Ai the articulation point on the fixed plate
l is the homotetic factor between the base
and the mobile*/

/* rotation matrix */

rot: matrix([1,0,0],[0,1,0],[0,0,1])$

(c62) /* relative coordinates of the
articulation point of the
mobile plate ;the parallelism of the side
is expressed by the
fact that these coordinates are homotetic
```

```
to those of the fixed
plate*/

b4r: matrix([ xa2*l],[-ya1*l],[0])$
(c63) b5r: matrix([-xa2*l],[-ya1*l],[0])$
v(c64) b6r: matrix([-xa3*l],[-ya3*l],[0])$
(c65) b1r: matrix([-xa4*l],[-ya4*l],[0])$
(c66) b2r: matrix([ xa4*l],[-ya4*l],[0])$
(c67) b3r: matrix([ xa3*l],[-ya3*l],[0])$

(c68) /* absolute coordinates of the articulation
 point on the fixed
plate */

a1: matrix([-xa2],[ya1],[0])$
(c69) a2: matrix([xa2],[ya1],[0])$
(c70) a3: matrix([xa3],[ya3],[0])$
(c71) a4: matrix([xa4],[ya4],[0])$
(c72) a5: matrix([-xa4],[ya4],[0])$
(c73) a6: matrix([-xa3],[ya3],[0])$

(c74) /* unit vector of the link 1,2,3,4 */

matrix([x0-a1[1]],[y0-a1[2]],[z0])$
(c75) n1:(%+rot.b1r)/r1$

(c76) matrix([x0-a2[1]],[y0-a2[2]],[z0])$
(c77) n2:(%+rot.b2r)/r2$

(c78) matrix([x0-a3[1]],[y0-a3[2]],[z0])$
(c79) n3:(%+rot.b3r)/r3$

(c80) matrix([x0-a4[1]],[y0-a4[2]],[z0])$
(c81) n4:(%+rot.b4r)/r4$

(c82) /* the three first equations */

a1a2: matrix([a2[1]-a1[1]],[a2[2]-a1[2]],[a2[3]-a1[3]])$

(c83) vect31: matrix([a1a2[2]*n2[3]-a1a2[3]*n2[2]],
[a1a2[3]*n2[1]-
a1a2[1]*n2[3]],[a1a2[1]*n2[2]-a1a2[2]*n2[1]])$

(c84) matrix([n1[2]*n2[3]-n1[3]*n2[2]],[n1[3]*n2[1]-
n1[1]*n2[3]],[n1[1]*n2[2]-n1[2]*n2[1]])$

(c85) vect32:lambda1*%$

(c86) /* 1st equation*/

vect31[1]= ratsimp (expand(vect32[1]),x0,y0,z0);
(d86)                          [[[0]]] = [[[0]]]

  /* 2nd equation */
(c87) vect31[2]= ratsimp (expand(vect32[2]),x0,y0,z0);
                  2 xa2 z0           2 xa2 z0
(d87)        [[[- --------]]] = [[[- --------]]]
                    r2                 r2
```

```
 /* 3thd equation */

(c88) t3:vect31[3]= ratsimp (expand(ev(vect32[3])),x0,y0,z0);
         2 xa2 (- l ya4 - ya1 + y0)
(d88) [[[-------------------------]]] =
                     r2

              - 2 l xa2 ya4 - 2 xa2 ya1 + 2 xa2 y0
           [[[------------------------------------]]]
                                r2

(c89) /* the former equation give lambda1 */

/* the first 3 equations are always satisfied and enable
to calculate lambda1 */

lambda1: -xa2*r1/(l*xa4-xa2)$

(c90) /* verification of the third equation */

ratsimp (ev(t3));
           2 l xa2 ya4 + 2 xa2 ya1 - 2 xa2 y0
(d90) [[[- ----------------------------------]]] =
                          r2

              2 l xa2 ya4 + 2 xa2 ya1 - 2 xa2 y0
          [[[- ----------------------------------]]]
                            r2

(c91) a1a3: matrix([a3[1]-a1[1]],[a3[2]-a1[2]],
[a3[3]-a1[3]])$

(c92) a1a4: matrix([a4[1]-a1[1]],[a4[2]-a1[2]],
[a4[3]-a1[3]])$

(c93) vect31: matrix([a1a3[2]*n3[3]-a1a3[3]*n3[2]],
[a1a3[3]*n3[1]-
a1a3[1]*n3[3]],[a1a3[1]*n3[2]-a1a3[2]*n3[1]])$

(c94) matrix([n1[2]*n3[3]-n1[3]*n3[2]],[n1[3]*n3[1]-
n1[1]*n3[3]],[n1[1]*n3[2]-n1[2]*n3[1]])$

(c95) vect32:lambda1*%$

(c96) vect41: matrix([a1a4[2]*n4[3]-a1a4[3]*n4[2]],
[a1a4[3]*n4[1]-
a1a4[1]*n4[3]],[a1a4[1]*n4[2]-a1a4[2]*n4[1]])$

(c97) matrix([n1[2]*n4[3]-n1[3]*n4[2]],[n1[3]*n4[1]-
n1[1]*n4[3]],[n1[1]*n4[2]-n1[2]*n4[1]])$

(c98) vect42:lambda1*%$

(c99) /* we get the 6 equations */

vect31[1]-vect32[1]=0$

(c100) ev(%)$
```

```
(c101) t4: ratsimp (%);
  (l xa2 ya4 + (- l xa4 - l xa2) ya3 + l xa4 ya1) z0
- -------------------------------------------------- = 0
                  l r3 xa4 - r3 xa2

(c102) vect31[2]-vect32[2]=0$

(c103) ev(%)$

(c104) t5: ratsimp (%);
              (l xa3 xa4 - l xa2 xa3) z0
(d104)     [[[- --------------------------]]] = 0
                  l r3 xa4 - r3 xa2

(c105) vect31[3]-vect32[3]=0$

(c106) ev(%)$

(c107) t6: ratsimp (%,x0,y0,z0);
(d107) [[[(x0 (l xa2 ya4 + (- l xa4 - l xa2) ya3 + l xa4 ya1)

     2                                     2
 + (l  - l) xa2 xa3 ya4 + (2 l xa2 xa3 - 2 l  xa3 xa4) ya3

     2
 + (l  - l) xa3 xa4 ya1 + (l xa3 xa4 - l xa2 xa3) y0)/
(l r3 xa4 - r3 xa2)]]] = 0

(c108) vect41[1]-vect42[1]=0$

(c109) ev(%)$

(c110) t7: ratsimp (%);
 ((l xa4 - l xa2) ya4 + (l xa2 - l xa4) ya1) z0
------------------------------------------------ = 0
         l r4 xa4 - r4 xa2

(c111) vect41[2]-vect42[2]=0$

(c112) ev(%)$

(c113) t8: ratsimp (%);
                        2        2
                   (l xa4  - l xa2 ) z0
(d113)          [[[- --------------------]]] = 0
                    l r4 xa4 - r4 xa2

(c114) vect41[3]-vect42[3]=0$

(c115) ev(%)$

(c116) t9: ratsimp (%,x0,y0,z0);
(d116) [[[(x0 ((l xa2 - l xa4) ya4 + (l xa4 - l xa2) ya1)

        2                  2         2
 + ((- l  - l) xa2 xa4 + (l  + l) xa2 ) ya4

        2         2     2
 + ((- l  - l) xa4  + (l  + l) xa2 xa4) ya1 +
```

```
       2         2
(l xa4  - l xa2 ) y0)/(l r4 xa4 - r4 xa2)]]] = 0
```

22 Conclusion

We have proposed a systematic method to find all the singular configurations for a parallel manipulator. This geometrical approach, very different from the classical jacobian analysis, yields to very interesting results.

All the singular configurations for the MSSM and the TSSM are known. We plan to developed a software which will determine, for a given design, if the singular configurations are in the working area. Such tool will be useful for the design and the command of this kind of parallel manipulator.

For the SSM manipulator we found very difficult problems. It remain basically the following geometrical problems :

- 4 lines on a regulus
- the hyperbolic congruence
- the special complex
- the general complex

Our current prototype is a SSM. But this study shows that the use of TSSM seems to be more appropriate. We have some other reasons which justify the use of a TSSM like the possible determination of the jacobian matrix. Therefore our new prototype will be a TSSM.

The use of Grassmann geometry can be generalized to the design of other closed loop mechanisms such as robotics hands.

The last remark is that we use extensively symbolic computation tools in this paper, after the theoretical work. The classical tools like MACSYMA and REDUCE are not very convenient to deal with geometrical objects like lines. We hope that a geometric-oriented symbolic computation program will be available in the near future.

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