sensitivity analysis of Kirchhof plate with obstacle

M.R. Rao, Jan Sokolowski

INRIA

UNITE DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P.105
78153 Le Chesnay Cedex
France
Tél.:(1) 39 63 55 11

Rapports de Recherche

N° 771

SENSITIVITY ANALYSIS OF KIRCHHOF PLATE WITH OBSTACLE

Murali RAO
Jan SOKOLOWSKI

DECEMBRE 1987

SENSITIVITY ANALYSIS OF
KIRCHHOFF PLATE WITH OBSTACLE.

ANALYSE DE SENSITIVITE
D'UNE PLAQUE DE KIRCHHOFF
AVEC OBSTACLE

Murali RAO
Department of Mathematics
University of Florida
Grainesville, Florida 32611
USA

and

Jan SOKOLOWSKI (*)
(Seminar presented at INRIA on october 29th, 1987).
Université des Sciences et Techniques
du Languedoc
Laboratoire de physique
mathématique
Place Eugene Bataillon
34060 Montpellier Cedex.

(*) permanent adress : System Research Institute,
Polish Academy of Sciences, ul. Newelska 6,
01 - 447 Warszawa, POLOGNE.

Abstract

The report is concerned with the Sensitivity analysis of simply supported Kirchhoff plate with obstacle. The conical differential of the displacement with respect to data is derived.

Résumé

Ce rapport traite de l'analyse de sensitivité d'une plaque de Kirchhoff simplement appuyée, en présence d'un obstacle. On obtient la différentielle conique du déplacement par rapport aux données.

1. Introduction :

We are concerned with sensitivity analysis of the following variational inequality.

$$w \in K(\Omega)$$
$$\int_\Omega \Delta w \, \Delta(\phi - w)\, dx \geq \int_\Omega f(\phi - w)\, dx \qquad \forall \phi \in K \qquad (1.1).$$

where :

$$K = \{ \phi \in H^2(\Omega) \mid \phi|_\Gamma = 0 \ , \ \phi \geq \psi \text{ in } \Omega \} \qquad (1.2).$$

$\Omega \subset R^n$ is a given domain with the boundary $\Gamma = \partial\Omega$, f, ψ are given elements such that $K \neq \phi$.

We show that the unique solution of (1.1) is conically differentiable with respect to the data i.e., f, ψ and in an appropriate way with respect to the perturbations of the domain Ω.

Sensitivity analysis of unilateral problems in the Sobolev space $H^1(\Omega)$ provided by F. Mignot [11] is based on the potential theory in the so-called Dirichlet space. If there is given a coercive bilinear form a(.,.) on $H^1(\Omega)\times H^1(\Omega)$ such that :

$$a\,(y^+,y^-) \leq 0 \ , \ \forall\, y \in H^1(\Omega) \qquad (1.3).$$

where :

$$y^+ = \max\,\{o,y\},\ y^- = \max\,\{0,-y\}$$

then the a-projection in $H^1(\Omega)$ onto a nonempty convex set $\{ y \geq \psi \} \subset H^1(\Omega)$ is conically differentiable [11]. Here we must obviously assume that :

$$y^+,\ y^- \in H^1(\Omega) \ , \quad \forall\, y \in H^1(\Omega) \qquad (1.4).$$

which can be verified.

However for an element $y \in H^s(\Omega)$, $s > 3/2$ it does not follow in general that $y^+ \in H^s(\Omega)$, therefore the results of Mignot [11] cannot be directly applied for the variational inequality (1.1).

We will show that the set $K \subset H^2(\Omega) \cap H^1_0(\Omega)$ is polyhedric [7,11,18,26] and therfore the metric projection in the space $H^2(\Omega) \cap H^1_0(\Omega)$ onto K is conically differentiable.

Let us recall the notion of polyhedric convex set in the Hilbert space. Let H be a Hilbert space , $a(.,.) : H \times H \to R$ a continuous and coercive bilinear form, i.e.

$$a\,(y,y) \geq \alpha \|y\|_H^2 \;,\; \alpha > 0 \;,\; \forall\, y \in H \qquad (1.5).$$

$$|a\,(y,z)| \leq M \|y\|_H \|z\|_H, \; \forall\, y,\, z \in H \qquad (1.6).$$

where $\alpha > 0$, M are given constants. We assume for simplicity, that the bilinear form $a(.,.)$ is symmetric : $a\,(y,z) = a\,(z,y)$, $\forall\, y,z \in H$.

Let us denote by $\Pi f = \Pi_K f$ a-projection in H of an element $f \in H$ onto a convex, closed set $K \subset H$. The element Πf satisfies variational inequality :

$$\Pi f \in K$$

$$a\,(\Pi f - f,\, z - \Pi f) \geq 0 \;,\; \forall\, z \in K \qquad (1.7).$$

It can be shown that mapping $\Pi : H \to K$ is Lipschitz continuous :

$$\|\Pi f_1 - \Pi f_2\|_H \leq \frac{M}{\alpha} \|f_1 - f_2\|_H \;,\; \forall\, f_1, f_2 \in H \qquad (1.8).$$

For a given element $y \in K$ we denote by :

$$C_K\,(y) = \{\, z \in H \,|\, \exists\, \tau > 0 \text{ such that } y + \tau z \in K \} \qquad (1.9).$$

the tangent cone.

Furthermore for a given element $f \in H$ we denote :

$$S_K\,(f) = \{\, z \in \overline{C_K(\Pi f)} \,|\, a\,(f - \Pi f,\, z) = 0 \} \qquad (1.10).$$

where $\overline{C_K(y)}$ is the closure in H of tangent cone $C_K(y)$. It can be verified that the set $S_K(f)$ is a closed and convex cone.

<u>Definition</u> :

The set K is called polyhedric if the following condition is verified for all f :

$$S_K\,(f) = \overline{\{\, z \in C_K(\Pi f) \,|\, a\,(f - \Pi f, z) = 0 \}} \qquad (1.11).$$

<u>Theorem 1 [7,11]</u> :

Let us assume that the set K is polyhedric, then for $\tau > 0$, τ small enough:

$$\forall\, h \in H : \; \Pi_K(f + \tau h) = \Pi_K f + \tau\, \Pi_S h + 0(\tau) \qquad (1.12).$$

where $S = S_K(f)$, $\|0(\tau)\|_H / \tau \to 0$ with $\tau \downarrow 0$ uniformly with respect to h on compact subsets of H.

We will apply Theorem 1 to variational inequality (1.1) since in section 2 we show that the set $K \subset H = H^2(\Omega) \cap H^1_0\,(\Omega)$ enjoys the property (1.11).

This result combined with the material derivative method [28-35] is used in section 3 to derive the form of so-called shape derivative $w' \in H^2(\Omega)$ of the solution w to (1.1) in the direction of a vector field $\underset{\sim}{V}(.,.)$.

The method of sensitivity analysis used here is proposed in [18] in the case of the set $\{\phi \geq \psi\} \subset H_0^2(\Omega)$ with the slightly different proof of property (1.11). The conical differentiability of metric projection in the space $H^2(\Omega) \cap H_0^1(\Omega)$ onto a subset with pointwise constraints is used in [19] for the shape sensitivity analysis of state constrained optimal control problems for elliptic systems. The proof of property (1.11) presented here is simplified in some technical details compared to that of [19].

We refear the reader to monograph's [3-5] for the general results on variational inequalities shape sensitivity analysis of boundary value problems is considered by many authors, we refer the reader to e.g. [2,6,8,12-17,20,21,27-35] for the related results and the applications. The related results on the sensitivity analysis of variational inequalities can be found in [1,7,11,12,22-26]. The standard notation is used throughout the present paper [9].

2. Sensitivity Analysis of Simply Supported Plate with Obstacle :

Let $\Omega \subset R^n$ be a given domain with smooth boundary $\Gamma = \partial\Omega$, $n = 2,3$.

Variational inequality (1.1) can be used as the mathematical model of simply supported thin elastic plate subjected to perpendicular force $f(x)$, $x \in \Omega \subset R^2$. The displacement w of the plate satisfies the imposed nonpenetration condition $w(x) \geq \phi(x)$, $x \in \Omega$, where ϕ describes the obstacle. We denote :

$$a(y,z) = \int_\Omega \Delta y \, \Delta z \, dx, \ \forall\, y,z \in H \qquad (2.1).$$

$$H = H^2(\Omega) \cap H_0^1(\Omega)$$

$H' = (H^2(\Omega) \cap H_0^1(\Omega))'$ denotes dual of H.

By standard elliptic regularity results for the Laplace equation it follows that bilinear form (2.1) satisfies (1.5) (1.6). Therefore the solution $w = \Pi f$ of (1.1) is unique and it is Lipschitz continuous with respect to $f \in H'$:

$$||\Pi f_1 - \Pi f_2||_H \leq C\, ||f_1 - f_2||_{H'}, \ \forall\, f_1, f_2 \in H'.$$

In order to show that the condition (1.11) is satisied for the set (1.2) we derive the form of the closure of tangent cone $\overline{C_K(u)}$ for an arbitrary element $u \in K$. We denote : $H^2 \cap H_0^1 = H^2(\Omega) \cap H_0^1(\Omega)$.

<u>Theoreme 2</u> :

Let $u \in K$ be a given element, denote :

$$\Xi = \Xi(u) = \{x \in \Omega \,|\, u(x) = \psi(x)\} \qquad (2.2).$$

and assume that $\psi \in H^2 \cap H_0^1$, Ξ is compact

Then $C_K(u) = \{ \phi \in H^2 \cap H_0^1 \,|\, \phi \geq 0 \text{ q.e. on } Z \}$ (2.3).

A proof very similar to that of theorem 2 gives also the following.

<u>Corollary</u> 1 : $\overline{C_K(w)} \cap [F - w]^\perp = \overline{C_K(w) \cap [F - w]^\perp}$ (2.4).

here we denote :

$$F \in H^2 \cap H_0^1 : \int_\Omega \Delta F \Delta\phi \, dx = \int_\Omega \phi f dx, \ \forall\, \phi \in H^2 \cap H_0^1$$

w is the metric projection of F onto K and $[F - w]^\perp$ is the subspace of $H^2 \cap H_0^1$ orthogonal to $F - w \in H^2 \cap H_0^1$.

For the convenience of the reader we provide the proof of theorem 2 [19]. Before we proceed let us establish the framework. It is not difficult to see that :

$$H^2 \cap H^1_0 = \{ Gf | f \in L^2(\Omega) \} \qquad (2.5).$$

where G is the Green function of Ω i.e. $G = (-\Delta)^{-1}$. We define the inner product in $H^2 \cap H^1_0$ by :

$$(Gf,Gg) = \int_\Omega f(x)\, g(x)\, dx \qquad (2.6).$$

We note that the corresponding topology is that inherited from H^2. For purpose of this paper we define the C_2 - capacity C(F) of a compact set $F \subset \Omega$ by :

$$C_2\ (F) = \inf \{ ||f||^2_{L^2} \mid Gf \geq 1 \text{ on } F \}$$

we extend this definition to all Borel sets by :

$$C_2\ (B) = \sup \{ C_2(F) \mid \text{compact } F \subset B' \}$$

a statement holds q.e when it holds except for a G - polar set i.e a set of C_2 - capacity zero.Observe that convergence of a sequence in $H^2 \cap H^1_0$ implies pointwise convergence (for a subsequence) off a G- polar set. For more on Capacities see [10].

Proof a Theorem 2 : We start off by observing that $C_K(u)$ (and in particular also its closure) has the following properties :

1. it contains all non-negative elements of $H^2 \cap H^1_0$.
2. if $\phi_i \in C_K(u)$, $a_i \geq 0$ then $\sum a_i\phi_i \in C_K(u)$.
3. $\phi \in C_K(u)$, $0 \leq \xi \in C^\infty$ then : $\xi\phi \in C_K(u)$
4. $\phi = 0$ in a neighbourhood of Z then $\phi \in C_K(u)$.

These properties are simple consequence of the definition of the tangent cone. Property 4 above is immediate if ϕ is bounded and for general ϕ is by taking limits.

Since convergence in $H^2 \cap H^1_0$ implies q.e. convergence for a subsequence, it is clear that the left side of (2.3) is a subset of the right side.

Let $V \in H^2 \cap H^1_0$ and suppose $V \geq 0$ q.e. on $\Xi = \{u = \psi\}$. Our object is to show that $V \in C_K(u)$.To this end let ϕ_0 denote the unique element of $\overline{C_K(u)}$ such that :

$$||V-\phi_0||_H = \inf ||V-\phi||_H \quad \phi \in C_K(u) \qquad (2.7)$$

using simple arguments we see that (2.7) implies :

$$(\phi_0-V,\phi) \geq 0 \ , \ \phi \in \overline{C_K(u)} \qquad (2.8)$$

For simplicity let us define the linear map :

$$L\phi = (\phi_0 - V, \phi), \quad \phi \in H^2 \cap H_0^1 \tag{2.9}$$

Let $f_0 \in L^2$ be such that :

$$\phi_0 - V = Gf_0 \tag{2.10}$$

If $g \geq 0$, then $\phi = Gg \geq 0$ and hence belongs to $C_K(u)$. Using (2.9) we see that $\int f_0 g \geq 0$. This says that $f_0 \geq 0$ a.e. If $0 \leq \phi \in C_0^\infty$ then again using (2.9) we see :

$$\int f_0 \Delta\phi \leq 0, \quad 0 \leq \phi \in C_0^\infty$$

i.e. that f_0 is superharmonic. By Riesz decomposition we way write :

$$f_0 = G\mu + h_0 \tag{2.11}$$

where μ is a positive Radon-measure and h_0 is positive harmonic in Ω. For clarity we break up the proof into small steps.

<u>Step 1</u> : For all $\phi \in H^2 \cap H_0^1$:

$$\int |\phi| \, d\mu \leq \| L \| \| \phi \|_{H^2 \cap H_0^1} \tag{2.12}$$

Indeed let $0 \leq f \in L^2$. There is a sequence of non-negative elements of C_0^∞ which <u>increases</u> pointwise to Gf.

$$Gf = \lim \phi_n, \quad 0 \leq \phi_n \in C_0^\infty$$

From (2.11) and (2.8) :

$$L(Gf) \geq L(\phi_n) = \int \phi_n d\mu$$

By monotone convergence we get :

$$\int (Gf) d\mu = \lim \int \phi_n d\mu \leq L(Gf)$$

Now it $\phi = Gf$ then :

$$\int |\phi| \, d\mu \leq \int (G|f|) d\mu \leq L(G|f|) \leq \| L \| \| f \|_{L^2} = \| L \| \| \phi \|_{H^2 \cap H_0^1} \tag{2.13}$$

(2.13) in particular tels us that if ϕ_n convergens to ϕ in $H^2 \cap H_0^1$, it also converges in $L^1(\mu)$.

<u>Step 2</u> : If $\phi \in H^2 \cap H_0^1$ has compact support then :

$$\int \phi d\mu = L\phi \tag{2.14}$$

Indeed for such ϕ, there is a sequence $\phi_n \in C_0^\infty$ converging to ϕ in $H^2 \cap H_0^1$. Then from Step 1 , ϕ_n also converges in L^1 (μ) to ϕ and, L agrees with μ on C_0^∞. Thus (2.14) is valid.

<u>Step 3</u> : If $\phi \in \overline{C_K(u)}$ then :

$$0 \le \int \phi d\mu \le L\phi \tag{2.15}$$

Indeed let $0 \le \xi \le 1$, $\xi \in C_0^\infty$. Then $\xi\phi \in \overline{C_K(u)}$ <u>and also</u> $(1 - \xi)\phi \in \overline{C_K(u)}$. this :

$$\int \xi\phi d\mu = L\ (\xi\phi) \le L\phi$$

because $L\phi = L(\xi\phi) + L\ (1 - \xi)\phi$

and the last term is non-negative.

Now we let ξ increase to 1 on Ω.

<u>Step 4</u> : μ is concentrated on Ξ. Indeed since $y - \psi \ge 0$, if $0 \le \phi \le 1$, $\phi \in C_0^\infty$ and $-1 \le t \le 1$, we have :

$$u - \psi + t\phi(u - \psi) \ge 0$$

in other words $t\phi\ (u - \psi) \in C_K(u)$. Using

$$\int t\phi(u - \psi)\ d\mu \ge 0 \tag{2.15}$$

$-1 \le t \le 1$ or that : $\int \phi(u-\psi)d\mu = 0$

Since $u > \psi$ off. Ξ. this can only be true if μ is concentrated on Ξ.

<u>Step 5</u> : $\mu = 0$. To show this note first that :

$$\int \phi_0\ d\mu = 0 \tag{2.16}$$

Indeed we know $L\phi_0 = 0$. So since $\phi_0 \ge 0$ on Ξ, from (2.15), (2.16) is seen to be valid. Now $\phi_0 - V = Gf_0$ and we knew that $f_0 \ge 0$.

So $\phi_0 - V$ is non-negative superharmonic and so either identifically zero are strictly positive everywhere. Since $\int(\phi_0 - V)d\mu = 0$, we must have $\mu = 0$.

<u>Step 6</u> : We claim $h_0 = 0$. For this we use property 4 of $\overline{C_K(u)}$. Let D be a relatively compact open set containing Z. Using proposition 1 in Appendix. We see that there is a, $0 \le f \in L^2$ such that $Gf \equiv 1$ on D . Let $\phi \in C_0^\infty$ s.t. $\phi \equiv 1$ on D. Then $\phi - Gf$ vanishes on D and hence $\phi - Gf \in C_K(u)$. Hence $L(\phi\text{-}Gf) = 0$, But $L(\phi\text{-}Gf) = \int f_0 \Delta(\phi\text{-}Gf) = \int h_0[\Delta\phi + f] = \int h_0 f$. Because h_0 is harmonic. Since $f \ge 0$ we get $h_0 \equiv 0$. Thus $L = 0$ or that $V \in \overline{C_K(u)}$ which completes the proof of Theorem 2.

We are now in the position to derive the form of the conical differential of solution to (1.1) with respect to the right-hand side of this variational inequality.

Theorem 3 :

Assume that $\psi \in H_0^2(\Omega)$, let $w = \Pi f$ denotes the solution of (1.1), then for any $h \in H' = (H^2 \cap H_0^1)'$ and for $\epsilon > 0$, ϵ small enough

$$\Pi(f+\epsilon h) = \Pi f + \epsilon \Pi' h + O(\epsilon) \qquad (2.17).$$

where $||O(\epsilon)||_{H^2(\Omega)}/ \to 0$ with $\epsilon \downarrow 0$.

The element $Q = \Pi' h$ is given by the unique solution of the following variational inequality :

$$Q \in S(\Omega) = \{ \phi \in H^2 \cap H_0^1 \mid \phi \geq 0 \text{ q.e. on } Z, \int \phi d\mu = 0 \} \qquad (2.18).$$

$$\int_\Omega \Delta Q \, \Delta(\phi - Q) dx \geq \int_\Omega h \, (\phi - Q) \, dx, \quad \forall \, \phi \in S(\Omega)$$

Proof :

From Corollary 1 it follows that the set (1.2) is polyhedric, hence Theorem 3 follows from Theorem 1.

3. Shape Sensitivity Analysis.

We derive the form of so-called shape (Lagrange) derivative of the solution to (1.1) in the direction of a vector field V(.,.). First, we define a family of domains : $\{ \Omega_\epsilon \} \subset R^n$, $\epsilon \in [0,\delta]$, depending on a given vector field V(.,.).

3.1. Family $\{\Omega_\epsilon\}$.

let $V(.,.) \in C([0,\delta) ; C^1(R^n;R^n))$ be a given vector field. We denote by :

$$T_\epsilon(V) : R^n \to R^N, \ \epsilon \in (0,\delta) \qquad (3.1).$$

the mapping defined as follows :

$$T_\epsilon(V)(X) = x(\epsilon), \ \epsilon \in (0,\delta) \qquad (3.2).$$

where $x(\epsilon)$ is given by the unique solution of the following system of ordinary differential equation.

$$\frac{d}{dt} x(t) = V(t,x(t)), \qquad t \in (0,\delta) \qquad (3.3).$$

$$x(0) = X \qquad (3.4).$$

we denote :

$$\Omega_\epsilon = T_\epsilon(V)(\Omega) = \{X \epsilon R^n \mid \exists \, X \epsilon \Omega \text{ such that } x(0) = X \text{ and } x(\epsilon) = X\} \qquad (3.5).$$

In particular we have for $\epsilon = 0$:

$$\Omega = T_0(V)(\Omega) \qquad (3.6).$$

We will denote by $DT_\epsilon(X)$ the Jacobian of the mapping (3.2) evaluated at $X \in R^n$, by $DT_\epsilon^{-1}(X)$ inverse of matrix $DT_\epsilon(X)$ and by $^*DT_\epsilon^{-1}(X)$ the transpose of $DT_\epsilon^{-1}(X)$.

3.2 Shape Derivative :

Let us recall that the shape derivative $w' = w'(\Omega)$ of the solution $w = w(\Omega)$ to variational inequality (1.1) in the direction of a vector field $V(.,.)$ is defined as follows :

$$w' = \dot{w} - \nabla w . V \ , \ V = V(0,.) \tag{3.7}$$

where
$$\dot{w} = \lim_{\epsilon \downarrow 0} (w_\epsilon o T_\epsilon - w)/\epsilon \tag{3.8}$$

here $w_\epsilon \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon)$ denotes the unique solution of variational inequality (1.1) defined in Ω_ϵ, $\epsilon \in [0,\delta]$.

$$w_\epsilon \in K(\Omega_\epsilon) = \{ \phi \in H^2(\Omega_\epsilon) \cap H_0^1(\Omega_\epsilon) \mid \phi \geq \psi \text{ in } \Omega \} \tag{3.9}$$

$$\int_{\Omega_\epsilon} \Delta w_\epsilon \ \Delta(\phi - w_\epsilon) dx \geq \int_{\Omega_\epsilon} f(\phi - w_\epsilon) dx \quad \forall \phi \in K(\Omega_\epsilon) \tag{3.10}$$

where $f \in L^2(R^n)$ is a given element. We assume $\psi \in H^3(R^n)$, $n = 2,3$, supp $\psi \subset \Omega$, therefore for $\epsilon > 0$, ϵ small enough $K(\Omega_\epsilon)$ is nonempty, convex closed subset.

Let $\tilde{w}_\epsilon$ be an extension of w_ϵ to an open neighbourhood of Ω.

Theorem 4 :

For $\epsilon > 0$, ϵ small enough

$$\tilde{w}_\epsilon|_\Omega = w + \epsilon w' + 0(\epsilon), \tag{3.11}$$

where $\| 0(\epsilon) \|_{H^2(\Omega)}/\epsilon \to 0$ with $\epsilon \downarrow 0$.

The shape derivative $w' \in H^2(\Omega)$ is the unique solution of the following variational inequality.

$$w' \in S_V(\Omega)$$

$$\int_\Omega \Delta w' \ \Delta(\phi - w') dx \geq - \int_{\partial\Omega} v \frac{\partial}{\partial n} (\Delta w) \frac{\partial}{\partial n} (\phi - w') d\Gamma, \quad \forall \phi \in S_V(\Omega) \tag{3.12}$$

here we denote by :

$$v(x) = \langle V(0,x), n(x) \rangle, \ x \in \partial\Omega \tag{3.13}$$

the normal component on $\Gamma = \partial\Omega$ of the vector field $V(0,.)$,

$$S_V(\Omega) = \{ \phi \in H^2(\Omega) \mid \phi = - v \frac{\partial w}{\partial n} \text{ on } \partial\Omega, \tag{3.14}$$
$$\phi \geq 0 \text{ q.e. on } Z, \int \phi \, d\mu = 0 \}$$

Proof :

First, we transport variational inequality (3.9), (3.10) to the fixed domain Ω using the mapping (3.1). It follows that the element :

$$w^\epsilon \equiv w_\epsilon o T_\epsilon \in H^2(\Omega), \ \epsilon \in [0,\delta) \tag{3.15}$$

satisfies :

$$w^\epsilon \in K^\epsilon = \{\phi \in H^2(\Omega)\cap H_0^1(\Omega) \mid \phi \geq \psi^\epsilon \text{ in } \Omega\} \quad (3.16).$$

$$a^\epsilon\ (w^\epsilon,\phi-w^\epsilon) \geq \int_\Omega f^\epsilon(\phi-w^\epsilon)dx, \quad \forall\ \phi \in K^\epsilon \quad (3.17).$$

here $\psi^\epsilon \equiv \psi o T_\epsilon$, for $\epsilon > 0$, ϵ small enough supp $\psi^\epsilon \subset \Omega$, $f^\epsilon = \gamma_\epsilon f o T_\epsilon$, $\gamma_\epsilon = \det\ (DT_\epsilon)$,

$$a^\epsilon(z,\phi) \equiv \int_\Omega \gamma_\epsilon^{-1} \operatorname{div}\ (A_\epsilon.\nabla z)\ \operatorname{div}\ (A_\epsilon.\nabla\phi)dx \quad (3.18).$$

$$\text{with } A_\epsilon \equiv \gamma_\epsilon\ DT_\epsilon^{-1}.\ {}^*DT_\epsilon^{-1} \quad (3.19).$$

we denote :

$$Z^\epsilon = w^\epsilon - \psi^\epsilon,\ \xi^\epsilon(\phi) = \operatorname{div}\ (A_\epsilon.\nabla\phi)$$

$$Z^\epsilon \in K_o \equiv \{\phi \in H^2(\Omega)\ \cap H_0^1(\Omega) \mid \phi \geq 0 \text{ in } \Omega\} \quad (3.22).$$

$$a^\epsilon(Z^\epsilon,\phi - Z^\epsilon) \geq \int_\Omega f^\epsilon(\phi - Z^\epsilon)\ dx - a^\epsilon(\psi^\epsilon,\ \phi - Z^\epsilon) \quad (3.23).$$

$$\forall\ \phi \in K_o$$

By application of Theorem 3 combined with Theorem 1 of [29] it follows that for $\epsilon > 0$, ϵ small enough :

$$Z^\epsilon = Z^o + \epsilon\dot{Z} + 0(\epsilon),\ \text{in } H^2(\Omega) \quad (3.24).$$

where $\dot{Z} \in H^2(\Omega)$ satisfies the following variational inequality.

$$\dot{Z} \in S(\Omega)$$

$$\int_\Omega \Delta\dot{Z}\ \Delta\ (\phi - \dot{Z})dx \geq \int_\Omega \dot{f}\ (\phi-\dot{Z})dx - \dot{a}\ (Z,\phi-\dot{Z}) \quad (3.25).$$

$$- \dot{a}\ (\psi,\phi-\dot{Z}) - a(\dot{\psi},\phi-\dot{Z}) \quad \forall\ \phi \in S\ (\ \Omega)$$

In view of (3.21) $\dot{Z} = \dot{w} - \dot{\psi}$, hence :

$$\dot{w} \in S(\Omega)$$

$$\int_\Omega \Delta\ \dot{w}\ \Delta(\phi-\dot{w})dx \geq \int_\Omega \dot{f}(\phi-\dot{w})dx - \dot{a}(w,\phi-\dot{w}) \quad (3.26).$$

$$- a(\dot{\psi},\ \phi-\dot{Z}), \quad \forall\ \phi \in S(\Omega)$$

where we denote :

$$\dot{\psi} = \nabla\psi.\ V \in H^2(\Omega) \quad (3.27).$$

$$\dot{f} = \operatorname{div}\ (fV) \quad (3.28).$$

$$\dot{a}(Z,\phi) \equiv \int_\Omega \{- \operatorname{div}V\ \Delta Z\ \Delta\phi + \dot{\xi}(Z)\ \Delta\phi + \Delta Z\dot{\xi}(\phi)\}\ dx \quad (3.29).$$

$$\forall\ Z,\ \phi \in H^2(\Omega)$$

$$\dot{\xi}\ (\phi) = \operatorname{div}\ (A'.\nabla\phi) \quad (3.30).$$

$$A' = \operatorname{div}\ VI - DV - {}^*DV \quad (3.31).$$

Since the shape derivative w' depends actually on the normal component v of the vector field $V(.,.)$ on $\partial\Omega$, hence for any vector field $V(.,.)$ such that $v(x) \equiv 0$ on $\partial\Omega$ it follows :

$$\dot{w} = \nabla w.V \quad (3.32).$$

and from (3.26) we obtain the following Green formula :

$$\int_\Omega \Delta\ (\nabla w.V)\ \Delta\phi\ dx = \int_\Omega \dot{f}\phi\ dx - \dot{a}\ (w,\phi) - a(\dot{\psi},\phi) \quad (3.33).$$

which holds for all $\phi \in \{S(\Omega)\ -\ S(\Omega)\}$ and all $V(.,.)$ such that $v(x) \equiv 0$

For an arbitrary vector field V(.,.) and the test function ϕ smooth enough, integration by parts, in view of (3.33), leads to :

$$- \int_\Omega \Delta \, (\nabla w.V)\Delta\phi \, dx + \int_\Omega \dot{f}\phi \, dx - \dot{a}(w,\phi) - a(\dot{\psi},\phi)$$

$$= - \int_{\partial\Omega} v \frac{\partial}{\partial n} (\Delta w)^{\cdot} \frac{\partial\phi}{\partial n} d\Gamma$$

therefore, in view of (3.7); (3.26), it follows that :

$$\int_\Omega \Delta w' \; \Delta(\phi - w')dx \geq -\int_{\partial\Omega} v \frac{\partial}{\partial n} (\Delta w) \frac{\partial}{\partial n} (\phi - w') \, d\Gamma$$

funthermore :

$$w' \in \{ \eta | \, \eta = \phi - \nabla w.V, \; \phi \in S(\Omega) \} \equiv S_V(\Omega)$$

since we can select V(0,.) with the support in a small open neighbourhood of Ω, which completes the proof of Theorem 4.

Appendix :

Proposition 1 :

Let $K \subset \Omega$ be a compact subset of the bounded domain Ω. Let G be the Green function of Ω. Then there exists an element $f \in L^\infty(\Omega)$, $f \geq 0$ such that $Gf \equiv 1$ on K.

Proof :

Let D be open relatively compact, $D \supset K$. Then we know $\exists \, \mu$ finite measure on ∂D such that $G\mu \equiv 1$ on D.

Let $2\delta = \text{dist } (K,\partial D)$. Let ϕ be radial, $\phi \in C^\infty$, vanishes off $B(0,\delta)$ and $\int\phi(x)dx = 1$.

Let $x \in K$ fixed and $y \in \partial D$. Then G(x,Z) is harmonic in $B(y,\delta)$.

So : for all $x \in K$ and $y \in \partial D$:

$$G(x,y) = \int G \, (x,Z) \; \phi(y-Z)dZ$$

therefore integrating relative to μ :

for all $x \in K$:

$$1 = \int G(x,y) \; \mu(dy) = \int G(x,Z) \int\phi(y-Z) \; \mu(dy)$$

and $\int\phi(y-Z) \; \mu(dy)$ is C^∞ with compact support in Ω.

q.e.d.

Références :

[1] **Bendsoe M.P., Olhoff N., Sokolowski J.. (1985)** : "Sensitivity analysis of problems of elasticity with unilateral constraints. J. Struct. Mech. 13(2), 201-222".

[2] **Benedict R., Sokolowski J., Zolesio J.P. (1984)** : "Shape optimization for contact problems. In : Thoft-Christensen P. (ed) System modelling and optimization, LNCIS vol.59, Sringer Verlag, 790-799".

[3] **Duvaut G., Lions J.L. (1972)** : "Les inéquations en mécanique et en physique. Dunod, Paris".

[4] **Fichera G. (1972)** : "Boundary value problems of elasticity with unilateral constraints, In : Handbuch der physik, band 6a/2, Springer Verlag".

[5] **Friedman A. (1982)** : "Variational principles and free boundary problems, J. Wiley and Sons, New-York".

[6] **Haug E.J., Choi K.K., Kombov V. (1986)** : "Design sensitivity analysis of structural systems, Academic Press, New-York".

[7] **Haraux A. (1977)** : "How to differentiate the projection on a convex set in Hilbert space. Some application to variational inequalities. J. Math. Soc. Japan, 29(4) 615-631".

[8] **Hadamard J.**: "Mémoire un le probème d'analyse relatif à l'équilibre des plaques élastiques encastrées. Mémoire des savants étrangers (1908)".

[9] **Lions J.L., Magenes E. (1968)** : "Problèmes aux limites non homogènes et applications. Vol. 1, Dunod, Paris".

[10] **Meyers N.G.** : "Theory of Capacities, Math. Scand. 26(1970)".

[11] **Mignot F. (1976)** : "Contrôle dans les inéquations variationnelles elliptiques. J. Functional analysis, 22, 130-185".

[12] **Neittaanmaki P., Sokolowski J., Zolesio J.P.** : "Optimization of the domain in elliptic variational inequalities, Appl. Math. & Optim. (to appear)".

[13] **Murat F., Simon J.** : "Sur le contrôle par un domaine géométrique, Université de Paris 6, Publication du L.A. 189 (1976)".

[14] **Pironneau O.** : "Optimal shape design for elliptic systems, Springer Verlag, New-York (1985)".

[15] **Rousselet B.** "Quelques résultats en optimisation de domains, thèse, Université de Nice (1982)".

[16] **Simon J.** : "Variation par rapport au domaine dans des problems aux limites, publication du L.A. 189, Université de Paris 6 (1980)".

[17] **Simon J.** : "Differentiability with respect to the domain in boundary value problems, Numer. Funct. Anal. and Optimiz. 2 (7&8), 649-687 (1980)".

[18] **Rao M., Sokolowski J.** : "Sensitivity analysis of unilateral problems in $H^2(\Omega)$ and applications (to appear)".

[19] **Rao M., Sokolowski J.** : "Shape sensitivity analysis of state constrained optimal control problems for distributed parameter systems, to appear in : Optimal Control of Distributed Parameter Systems, ed. A. Bermudez, Sringer Verlag".

[20] **Sokolowski J.** : "Shape sensitivity analysis of boundary optimal control problems for parabolic systems. SIAM J. Control and Optimization (to appear)".

[21] **Sokolowski J.** : " Sensitivity analysis of control constrained optimal control problems for distributed parameter systems. SIAM J. Control and Optimization 25 (6), November 1987.

[22] **Sokolowski J. (1981)** : " Sensitivity analysis for a class of variational inequalities. In : Haug E.J., Cea J. (eds), Optimization of distributed parameter structures, vol.2, Sijthoff & Noordhoff Alphen aan den Rijn, the Netherlands, 1600-1609".

[23] **Sokolowski J.** : " Sensitivity analysis of contact problems with prescribed friction, to appear in : Appl. Math. Optim".

[24] **Sokolowski J. (1983)** : "Optimal control in coefficients of boundary value problems with unilateral constraints. Bulletin of the Polish Academy of Sciences, Technical Sciences, vol.31 (1-12) : 71-81".

[25] **Sokolowski J.** : "Sensitivity analysis of Signorini variational inequality. In : Bojarski B. (ed) Banach Center Publications, Polish Scientific Publisher, Warsaw (to appear)".

[26] **Sokolowski J. (1985)** : "Differential stability of solutions to constrained optimization problems. Appl. Math. Optim. 13 : 97-115".

[27] **Sokolowski J.** : "Shape sensitivity analysis of nonsmooth variational problems to appear in : Boundary Control & Boundary Variations, ed. J.-P. Zolesio, Springer Verlag".

[28] **Sokolowski J., Zolesio J.P. (1985)** : "Dérivée par rapport au domaine de la solution d'un problème unilatéral. C.R. Acad. Sc. Paris, t.301, série 1, n°4 : 103-106".

[29] **Sokolowski J., Zolesio J.P.** : "Shape sensitivity analysis of unilateral problems. SIAM Journal of Mathematical Analysis Vol.18 n°5 (1987).

[30] **Sokolowski J., Zolesio J.P. (1985)** : "Shape sensitivity analysis of an elastic-plastic torsion problem. Bulletin of the polish Academy of Sciences. Technical Sciences. Vol.33, n°11-12, pp.579-586".

[31] **Sokolowski J., Zolesio J.P.** : "Shape design sensitivity analysis of plates and plane solids under unilateral constraints, JOTA. Vol.54 n°2 (1987) pp. 361-382".

[32] **Sokolowski J., Zolesio J.P. (1987)** : "Shape sensitivity analysis of contact problem with prescribed friction, Journal of Nonlinear Analysis : Theory, Methods, and Applications (to appear)".

[33] **Zolesio J.P. (1984)** : "Shape controlability for free boundaries. In Thoft-Christensen P. (ed) System modelling and Optimization, LNCIS vol.59, Springer Verlag, 354-361".

[34] **Zolesio J.P. (1979)** : "Identification de domaines par déformations. Thèse d'état, Université de Nice.

[35] **Zolesio J.P. (1980)** : "The Material derivative (or speed method) for shape optimization, in "Optimization of distributed parameter structures", E.J. Haug and J. Céa eds. pp. 1457-1473 Sijjthoff and Noordhof, Alphan aaan den Rijn, Netherlands, 1980".

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique