Load balancing in a system of two queues with resequencing

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RÉPARTITION DE LA CHARGE DANS UN SYSTÈME DE DEUX FILES D'ATTENTE EN PARALLÈLE AVEC RESÉQUENCEMENT

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RÉSUMÉ

Nous étudions un système constitué de deux serveurs ./M/1 en parallèle. Les clients appartiennent à deux classes, et sont dirigés vers une des deux files selon un processus de Bernoulli dont le paramètre ne dépend que de leur classe. Nous supposons que les clients doivent être *reséquencés* classe par classe avant de quitter le système, et nous déterminons les valeurs des probabilités de routage qui minimisent le temps de réponse moyen du système, avec ou sans reséquencement.

ABSTRACT

We consider a system of two independent servers in parallel. The arriving customers can be directed to either of the two waiting lines. We suppose that there are several classes of customers, routed according to Bernoulli processes (with rates depending on class only), and that they have to be resequenced before leaving the system. We determine the routing probabilities that optimize the mean sojourn time in the system (resequencing time included).

KEYWORDS Resequencing - Random Routing - Optimal Control.

1/INTRODUCTION

The present paper addresses the problem of optimal load allocation in a two server queuing system.

Two independent queues with exponentially distributed service times and infinite waiting rooms with *first in first out* (FIFO) queuing discipline are placed in parallel. We assume that both service rates are equal to μ. Customers belong to two classes and arrive according to independent Poisson processes with class-dependent rates λ_1 and λ_2. The sum $\lambda = \lambda_1 + \lambda_2$ is the total arrival rate of customers in the system.

An incoming class i customer joins queue 1 (resp. 2) with a fixed probability p_i (resp $q_i = 1 - p_i$). It is assumed that these two Bernoulli switchings are instantaneous, mutually independent, and *do not* depend on the system state. Furthermore, all service times and arrival processes are assumed to be mutually independent.

The important requirement of the model is that class i customers must leave the system in the order of their arrivals. For this, each customer enters a "Resequencing Box" (RB) after completion of its service, where he eventually waits until *all* customers *of his own class* that entered the system before him have been served (Fig. 1). We furthermore assume that the time token by this resequencing process is negligible: if this customer has no one to wait for, he leaves instantaneously the system. On the other hand, he leaves as soon as the last customer he waits for enters the RB.

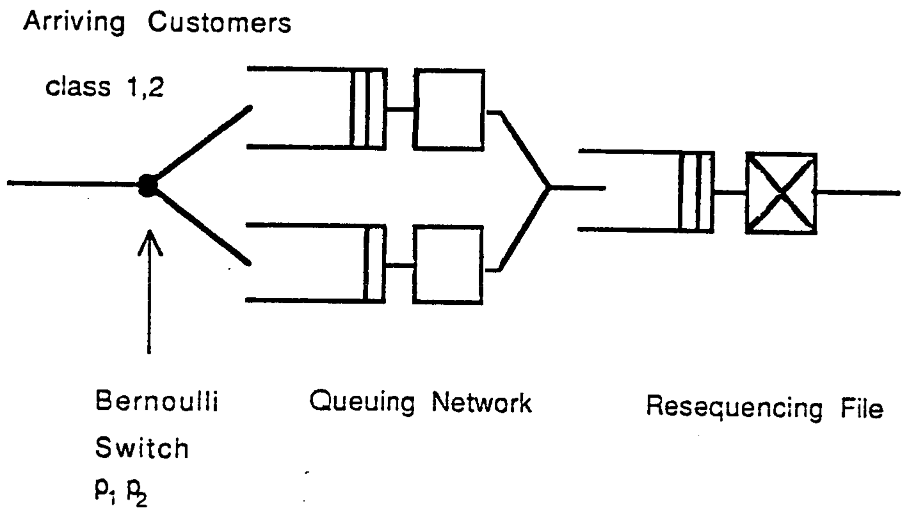

Figure 1 Switched Input Network

When the system is in steady state, let $\overline{S}_i$ denote the average total time spent in the network (queuing time plus service time plus the time spent in the RB) by class i customers. We assume that μ, λ_1 and λ_2 are fixed parameters. Our objective is to find the couples (p_1, p_2) that minimize the mean sojourn time of an arbitrary customer, namely:

$$\overline{S}(p_1, p_2) = \frac{1}{\lambda}\left(\lambda_1 \overline{S}_1(p_1, p_2) + \lambda_2 \overline{S}_2(p_1, p_2)\right). \tag{1.1}$$

The problem of optimal customer allocation to two queues has been already studied in the literature. If both servers are identical and exponential, and if the *waiting time* is the criterion to minimize, it is known that the optimal policy is to "Join the Shorter Queue". This policy has been analyzed by Flatto in [4]. Of course, this solution assumes that the mechanism that routes the arriving customers knows the state of the queues.

This paper addresses the case of routing mechanisms which do not have any information on the system state. We only assume that it knows the arrival rates of each class. This is often encountered in computer systems and in particular in distributed systems, or remote communication networks. Then, the simplest thing to do is to route customers randomly to either of the subsystems, as we assume in the model.

A new problem arises if the *arrival order* of customers has to be preserved within the output process. In many applications, such as voice or data transmission, remote computations, databases manipulations, the sequences of packets/data/customers have to be processed in the order of their emission. In a single queue with the FIFO queuing rule, there is no overtaking of customers. But in the model we consider here, the fact of switching arriving customers to either of the queues creates eventually a disorder in the output stream. The right order is restored with the use of the resequencing box. The problem of resequencing has been studied by several authors, e.g. in [5] and [2]. In these papers, the authors assume that the disorder is created by *independent* delays, which is not the case here, since customers share a common queue.

The paper is organized as follows: in section 2/, we define the variables and notations used in the analysis. In section 3/, we derive the values of the response and resequencing times (propositions 3.2 and 3.4) which are used to compute the total sojourn time. The minimization of this sojourn time is analyzed in section 4/. We obtain in theorem 4.3 the uniqueness of the minimum. Section 5/ concentrates some special cases, and section 6/ discusses extensions of this model.

2/DEFINITIONS AND NOTATIONS

For every fixed triple $(\lambda_1, \lambda_2, \mu)$, we say that (p_1, p_2) is *admissible* if the system is stable when (p_1, p_2) are the Bernoulli parameters. Let $\mathcal{D}_{(\lambda_1, \lambda_2, \mu)}$ be the set of these admissible parameters, that we shall call the "admissible domain" in the sequel. From now on, we shall assume that μ is constant, so that (λ_1, λ_2) will be the only parameters of the model. For more convenience, we shall also drop the indices of $\mathcal{D}$.

For any customer C, define:

T (response time) the random variable (in short RV) representing the time spent by C in the $./M/1$ queuing system, including the service time,

S (sojourn time) the RV representing the total time spent by C in the system,

$R = S - T$ (resequencing time) the time spent by C in the RB.

Finally, the system with switching parameters $p_1 = 1$ and $p_2 = 0$ we be called the "initial system". It corresponds to the case where a server is reserved for each class.

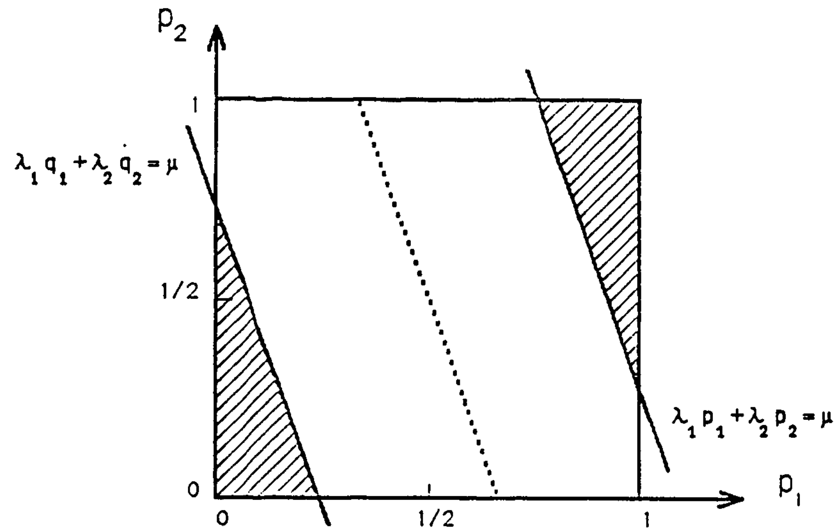

Figure 2 The admissible domain $\mathcal{D}$

3/ANALYSIS OF THE MODEL

The mean system response time $\overline{S}$ is the sum of the mean response time $\overline{T}$ and of the mean resequencing time $\overline{R}$. We shall compute separately these two values.

3.1/Preliminary results. Stability conditions

The result of the switching of a Poisson process of intensity l by an independent Bernoulli toss of parameter p is a couple of independent Poisson processes of respective intensities lp and $l(1-p)$. Therefore, the two input processes in queue 1 are independent Poisson processes of rates $\lambda_1 p_1$ and $\lambda_2 p_2$. Since both classes have identically distributed service times, the two queues behave as classical $M/M/1$ queues, independent of each other, with respective input rates λ'_1 and λ'_2 defined by:

$$\lambda'_1 = \lambda_1 p_1 + \lambda_2 p_2,$$
$$\lambda'_2 = \lambda_1 q_1 + \lambda_2 q_2 = \lambda - \lambda'_1.$$

This leads to the

Proposition 3.1 : the admissible domain
The admissible domain $\mathcal{D}$ is given by:

$$\mathcal{D} = \{(p_1, p_2) \mid 0 \leq p_1 \leq 1,\ 0 \leq p_2 \leq 1,\ \lambda - \mu < \lambda_1 p_1 + \lambda_2 p_2 < \mu \}.$$

The region $\mathcal{D}$ (Fig. 2) is nonempty iff $\lambda < 2\mu$, and it is the whole square iff $\lambda < \mu$. More precisely,

if $\lambda < \mu$, any choice of (p_1, p_2) leads to a stable system,

if $\mu \leq \lambda < 2\mu$, only a careful choice of (p_1, p_2) leads to a stable system; in particular, the initial system may be unstable but it can be stabilized by a convenient choice of the p_i's: we call this situation *Partial Instability*,

if $2\mu \leq \lambda$, the system is never stable (*Complete Instability*).

Note also that $\mathcal{D}$ possesses a center of symmetry: the point $(\frac{1}{2}, \frac{1}{2})$. This symmetry simply exchanges the p_i's with the q_i's, thus exchanging the roles played by the two queues.

3.2/The response time

Proposition 3.2

The expected value of T is given by:

$$\lambda \overline{T} = \frac{\lambda_1'}{\mu - \lambda_1'} + \frac{\lambda_2'}{\mu - \lambda_2'} . \tag{3.1}$$

Proof

At steady state, the response time of a customer in a $M/M/1$ system with arrival rate l and service rate m is known to be exponentially distributed with mean $1/(m - l)$ ([6]). Then, (3.1) follows easily.

□

3.3/The resequencing time

To compute the expected value of the resequencing time, we first compute its sample path value, then its distribution.
Define the function of three variables:

$$f(a, b, \theta) = \frac{a\ \theta}{b\ (b+a)\ (b+\theta)} .$$

Let R_1^1 be the resequencing time of class 1 customers routed to queue 1. Its expected value, $\overline{R}_1^1$ is given by

Proposition 3.3

$$\overline{R}_1^1 = f(\mu - \lambda_1', \mu - \lambda_2', q_1\lambda_1) . \tag{3.2}$$

To prove this proposition, we first state the following lemma.

Lemma 3.4

Assume that the system is at steady state. Let C be a customer of class 1 entering queue 1 at time $t = 0$. If

A is the response time of C,

B is the response time of the last customer of class 1 that that entered queue 2 before time $t = 0$,

τ is the date of the last arrival of a class 1 customer in queue 2 before time $t = 0$,

$[x]^+ = \max(0, x)$,

then the resequencing time of C is:

$$R_1^1 = [B - (\tau + A)]^+ . \tag{3.3}$$

Proof of Lemma 3.4

Consider a tagged customer C, arriving in queue 1 at time $t = 0$. Let C' be the *last* customer of C's class that entered queue 2 before $t = 0$. At time $t = A$, C joins the RB. Owing to the FIFO discipline along each route, it only has to wait until C' finishes its service. This event occurs at time

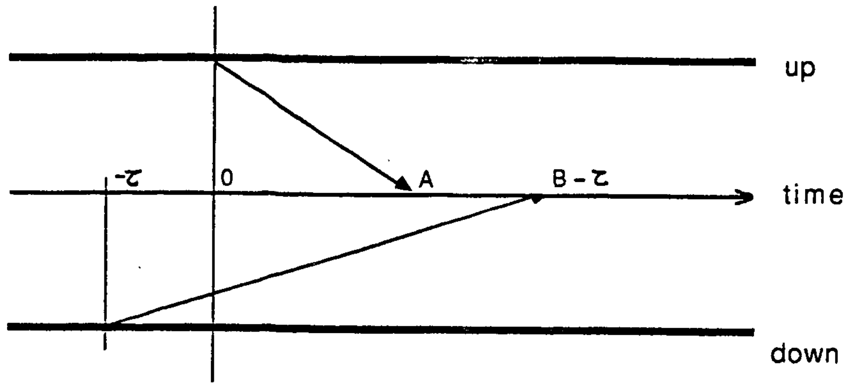

Figure 3 Resequencing time

$B - \tau$ (see Fig. 3). If $B - \tau < A$, then C' has already left the system, and R_1^1 is zero. Otherwise, $R_1^1 = B - \tau - A$.

□

Proof of Proposition 3.3

We shall first compute the distribution of R_1^1. To begin with, we prove the mutual independence of the three RVs A, B and τ. For this, we shall use the Marked Point Processes formalism (see for instance [1]).

We represent the arrival process of class 1 customers in queue i ($i = 1, 2$) by a time-stationary point process $(M^{(i)}, \theta_t, \mathbb{P}^{(i)})$. These processes are defined on a common probability space, and are marked by the successive network response times, noted $\{k_n^{(i)}\}_{-\infty}^{\infty}$. The arrival instants are the sequences $\{t_n^{(i)}\}_{-\infty}^{\infty}$, so that $M^{(i)} = \sum \delta_{t_n^{(i)}}$. Finally, we shall denote by $\mathbb{P}_0^{(i)}$ the Palm probability associated with the process $(M^{(i)}, \theta_t, \mathbb{P}^{(i)})$.

As remarked in 3.1/, the two input processes and service demand sequences are independent, so that these two processes are also independent. Therefore, at steady state any customer of class 1 "sees" the process $\mathbb{P}^{(2)}$ in its stationary state. More formally, if $\mathbb{P}^{(+)} = \mathbb{P}^{(1)} \otimes \mathbb{P}^{(2)}$ is the superposition of the two point processes, then the Palm distribution of $\mathbb{P}^{(+)}$ associated with $\mathbb{P}^{(1)}$ is ([1] p.19):

$$\mathbb{P}_0^{(1)} \otimes \mathbb{P}^{(2)} .$$

This implies that

- $\tau = -t_{-1}^{(2)}$ is exponentially distributed with the same intensity than $\mathbb{P}^{(2)}$, i.e. λq,
- the evolution of $\mathbb{P}_0^{(1)}$ is not affected by the arrival dates of $\mathbb{P}^{(2)}$ and so, A and τ are independent, and so are A and B.

Now, we prove the independence of τ and B. This would be clear if $t = 0$ was an arrival instant in queue 2. As we shall see, because $\mathbb{P}^{(2)}$ is a Poisson process, we can do "as if" there were really an arrival at this moment. Let us compute: $\mathbb{P}^{(2)}\{\tau \le t, B \le b\}$. We have the sequence of equalities :

$$\mathbb{P}^{(2)}\{\tau \le t,\ B \le b\} = \mathbb{P}^{(2)}\{-t_0^{(2)} \le t,\ k_0^{(2)} \le b\} , \qquad (3.4a)$$

by definition of $\mathbb{P}^{(2)}$,

$$= \mathbb{P}_0^{(2)}\{-t_{-1}^{(2)} \leq t,\ k_{-1}^{(2)} \leq b\}\,, \tag{3.4b}$$

because, as $\mathbb{P}^{(2)}$ is a Poisson process and as the marks $\{k_n^{(2)}\}$ depend only on the past, the point processes $M^{(2)}$ and $M^{(2)} + \delta_0$ have the same law, and we can add a point to $\mathbb{P}^{(2)}$ in $t = 0$,

$$= \mathbb{P}_0^{(2)}\{t_1^{(2)} \leq t,\ k_0^{(2)} \leq b\} \tag{3.4c}$$

using the fact that $\mathbb{P}_0^{(2)}$ is θ_{t_1}-stationary,

$$= \mathbb{P}_0^{(2)}\{t_1^{(2)} \leq t\}\ \mathbb{P}_0^{(2)}\{k_0^{(2)} \leq b\} \tag{3.4d}$$

by independence of the future arrival process with the past,

$$= \mathbb{P}_0^{(2)}\{-t_{-1}^{(2)} \leq t\}\ \mathbb{P}_0^{(2)}\{k_{-1}^{(2)} \leq b\} \tag{3.4e}$$

using backwards the θ_{t_1}-invariance,

$$\mathbb{P}^{(2)}\{\tau \leq t,\ B \leq b\} = \mathbb{P}^{(2)}\{\tau \leq t\}\ \mathbb{P}^{(2)}\{B \leq b\}\,, \tag{3.4f}$$

with the Poisson property again. Equation (3.4f) proves the desired independence.

We already used in 3.2/ the fact that A is exponentially distributed with parameter $a = \mu - \lambda_1'$. Similarly, using the independence of τ and B, we obtain that B is exponentially distributed with parameter $b = \mu - \lambda_2'$. Therefore, from (3.3), we can compute, for any $x > 0$:

$$\begin{aligned}
\mathbb{P}\{R_1^1 \geq x\} &= \mathbb{P}\{B \geq x + \tau + A\} \\
&= \mathbb{E}(\ e^{-b(A+\tau+x)}\ ) \\
&= e^{-bx}\ A^*(b)\ \tau^*(b) \\
&= e^{-bx}\ \frac{a}{a+b}\ \frac{\theta}{\theta+b}\,.
\end{aligned} \tag{3.5}$$

Integrating (3.5) on the real axis, and replacing a, b and θ by their values, we obtain (3.2).

□

Note that in the case where there is a single class (i.e. $\lambda_2 = 0$), C' finishes its service at $t = W_2$, where W_2 is the RV representing the *residual workload* of queue 2. Therefore, $R_1^1 = [A - W_2]^+$, and the well known relation ([6]):

$$\mathbb{P}\{W_2 > x\} = \frac{q_1\lambda_1}{\mu}\ e^{-(\mu - q_1\lambda_1)x}\,,$$

provides a second way to obtain (3.2). However, this argument does not hold in the two-class case, because C' is not necessarily the last customer in queue 2.

The value of the expected resequencing time follows easily from proposition 3.3.

Proposition 3.5

$$\begin{aligned}
\lambda\overline{R} = {}& \lambda_1 p_1 f(\mu - \lambda_1', \mu - \lambda_2', q_1\lambda_1) + \lambda_1 q_1 f(\mu - \lambda_2', \mu - \lambda_1', p_1\lambda_1) \\
&+ \lambda_2 p_2 f(\mu - \lambda_1', \mu - \lambda_2', q_2\lambda_2) + \lambda_2 q_2 f(\mu - \lambda_2', \mu - \lambda_1', p_2\lambda_2)
\end{aligned} \tag{3.6}$$

4/OPTIMIZATION OF THE SYSTEM

We shall briefly solve the problem of finding the optimal values of the p_i's when the function to minimize is simply $\overline{T}$. Then, we shall study the more complex problem of minimizing $\overline{S}$.

4.1/Minimizing the response time

Notice first that $\overline{T}$ is only a function of λ_1', since $\lambda_2' = \lambda - \lambda_1'$. It is easy to see that this is a strictly convex function of λ_1', for values of λ_1 in the range $(\lambda - \mu, \mu)$, and that it reaches its minimum when

$$\lambda_1' = \lambda_2' = \frac{\lambda}{2} ,$$

whatever p_1 and p_2 may be. This is obtained, in particular, when $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$. This minimal value is :

$$\overline{T}_{min} = \frac{1}{\mu - \frac{\lambda}{2}}$$

This corresponds to the case where the load is evenly balanced. We shall call this situation "*Equi-stream*". This proves that, omiting the resequencing phenomenon, the solution of our optimization problem is to choose the switching probabilities that minimize the maximum of the mean queue lengths, which is a stochastic version of the "join the shorter queue" rule.

The set of (p_1, p_2) for which the network is in Equi-Stream is represented by the dashed line in Fig. 2.

4.2/Minimizing the sojourn time

Our next objective is to find

$$\min_{(p_1,p_2)\in \mathcal{D}} \overline{S}(p_1, p_2) ,$$

as well as the value(s) of (p_1, p_2) for which this minimum is reached. Note that if the initial system is in partial instability, then any choice of (p_1, p_2) in the admissible domain will improve the performance of the network. On the other hand, the resequencing time can be greater than the gain on response time. Actually, when $\lambda_1 = \lambda_2$, the initial system is already in Equi-Stream (i.e. nothing can be done to decrease $\overline{T}$) and any re-routing creates a positive $\overline{R}$, increasing therefore $\overline{S}$.

We shall proceed in two steps, first by reducing the problem to the boundary of $\mathcal{D}$, then by solving the problem on this simplified domain. We shall assume in the sequel (without loss of generality), that $\lambda_1 \geq \lambda_2$.

4.2.1/Reduction of the domain

In general, $\mathcal{D}$ possesses two kinds of boundaries: those given by $p_i = 0$ or 1 (the "natural boundaries"), and those given by $\lambda_1' = \mu$ or $\lambda - \mu$ (the "instability boundaries"). For more convenience, we decompose $\mathcal{D}$ into segments which are parallel to its instability boundaries (Fig. 4):

$$\mathcal{D}_l \triangleq \{(p_1, p_2) \quad p_1\lambda_1 + p_2\lambda_2 = l\} \quad \text{for } \lambda - \mu < l < \mu , \quad 0 \leq l \leq \lambda .$$

If $\lambda \neq 0$, then $\lambda_1 \neq 0$ and $\mathcal{D}_l$ admits p_1 for parameter. We first prove the

Lemma 4.1

The minimum of $\overline{S}$ is located on the natural boundary of $\mathcal{D}$.

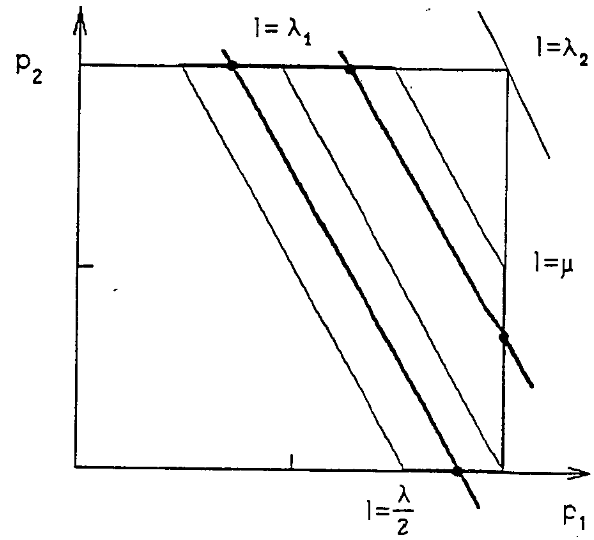

Figure 4 Decomposition of $\mathcal{D}$

Proof

If (p_1, p_2) is in $\mathcal{D}_l$, then $\lambda_1' = l$ and $\lambda_2' = \lambda - l$, and therefore are *constants*, as well as $\overline{T}$. Using theorem A1.1 in appendix 1, we obtain that, for each class, the mean resequencing time is a concave function of p_1 in $\mathcal{D}$. Consequently, $\overline{S}$ is also concave, and it is minimum on the boundary of $\mathcal{D}_l$.

□

This proposition shows that when $\overline{S}$ is minimum, the class with the lowest arrival rate will use only one queue.

Now, we have to specify on which end of $\mathcal{D}_l$ the minimum of $\overline{S}$ is located. According to the symmetry of $\mathcal{D}$, we can restrict our study of $\overline{S}$ to couples (p_1, p_2) belonging to $\mathcal{D}_l$, with $\lambda/2 \leq l < \mu$, $l \leq \lambda_1$.

Lemma 4.2

The minimum of $\overline{S}$ is located on one of the two boundaries $\mathcal{D} \cap \{p_2 = 0\}$ or $\mathcal{D} \cap \{p_1 = 1\}$.

Proof

As shown in Fig. 4, two cases may arise, depending on which edges of the square meet $\mathcal{D}_l$.

1/ If $\lambda/2 \leq l \leq \lambda_1$, $\mathcal{D}_l$ meets the two edges $p_2 = 0$ and $p_2 = 1$, at points $(l\lambda_1, 0)$ and $((l - \lambda_2)/\lambda_1, 1)$ respectively.

Recall that the mean response times of the two queues are constant. Their values are $1/a$ and $1/b$ respectively, with:

$$a = \mu - l \quad \text{and} \quad b = \mu - (\lambda - l) .$$

The two respective values of $\overline{S}$ are then $\overline{S}_0$ and $\overline{S}_1$, where:

$$\begin{cases} (2\mu - \lambda)\, \lambda\, \overline{S}_0 = l\, (\lambda_1 - l) \left(\dfrac{b}{a\, \mu} + \dfrac{a}{b\, (\mu - \lambda_2)} \right) , \\ (2\mu - \lambda)\, \lambda\, \overline{S}_1 = (l - \lambda_2)\, (\lambda - l) \left(\dfrac{b}{a\, (\mu - \lambda_2)} + \dfrac{a}{b\, \mu} \right) . \end{cases} \tag{4.1}$$

We compute the difference, and obtain that

$$\overline{S}_0 - \overline{S}_1 = \frac{\lambda_2}{ab\lambda\mu} \frac{2l - \lambda}{(2\mu - \lambda)(\mu - \lambda_2)} \left[l(2\mu - \lambda)(l - \lambda_1) - \mu(\mu - \lambda + l)^2 - (\mu - \lambda_2)(\mu - l)^2 \right] ,$$

which turns out to have the sign of $\lambda_2(2l-\lambda)$. Therefore,

$$\forall\, \lambda/2 \le l \le \lambda_1\,, \qquad \overline{S}_0 \;\le\; \overline{S}_1\,,$$

the equality being possible only if $l = \lambda/2$ or $\lambda_2 = 0$. Thus, the minimum of $\overline{S}$ restricted to $\mathcal{D}_l$ is obtained at $p_2 = 0$.

2/ If $\lambda_1 \le l \le \lambda$ then $\mathcal{D}_l$ meets the two edges $p_1 = 1$ and $p_2 = 1$, at points $(1, (l-\lambda_1)/\lambda_2)$ and $((l-\lambda_2)/\lambda_1, 1)$. The respective values of $\overline{S}$ are $\overline{S}'_1$ and $\overline{S}'_2$, where:

$$\begin{cases} (2\mu-\lambda)\,\lambda\,\overline{S}'_1 \;=\; (l-\lambda_1)\,(\lambda-l)\,\left(\dfrac{a}{b\,\mu} + \dfrac{b}{a\,(\mu-\lambda_1)}\right)\,, \\ (2\mu-\lambda)\,\lambda\,\overline{S}'_2 \;=\; (l-\lambda_2)\,(\lambda-l)\,\left(\dfrac{a}{b\,\mu} + \dfrac{b}{a\,(\mu-\lambda_2)}\right)\,, \end{cases} \tag{4.2}$$

with the same values $a = \mu - l$ and $b = \mu - \lambda + l$.

As in 1/, we compute the difference and obtain :

$$\forall\, \lambda_1 \le l \le \min(\mu, \lambda) \qquad \overline{S}'_1 \;\le\; \overline{S}'_2\,,$$

the equality being possible only if $l = \lambda$. Therefore, the minimum of $\overline{S}$ restricted to $\mathcal{D}_l$ is obtained at $p_1 = 1$.

□

4.2.2/Minimization on the reduced domain

We shall now study the behavior of $\overline{S}$ on the two edges given by lemma 4.2.

Rewriting (3.1) as a function of l, we obtain:

$$\lambda\overline{T} \;=\; \frac{l}{\mu-l} + \frac{\lambda-l}{\mu+l-\lambda}\,. \tag{4.3}$$

As mentioned in 4.2.1, this function is strictly convex and has an unique minimum at point $l = \lambda/2$. The behavior of $\overline{R}$ is not so nice, and clearly, this function reaches its minimum (which is 0) when there is no switching, which is precisely where $\overline{T}$ has a local maximum. We shall see in 5/ that $\overline{R}$ can actually have a non-monotone behavior. However, the behavior of $\overline{S}$ is quite regular, and we have:

Theorem 4.3

The function $\overline{S}$ possesses an unique minimum on $\mathcal{D}$, at point $(p_{opt}, 0)$. When p_{opt} is equal to one, it is obtained as a root of a fifth degree polynomial.

We obtain the proof of this theorem in two steps, first by proving the exitence and the uniqueness of a minimum on the edge $\{p_2 = 0\}$, then by showing that the minimum on the edge $\{p_1 = 1\}$ is obtained when $p_2 = 0$.

First, let us assume that $p_2 = 0$. The formulas of the mean response, resequencing and sojourn times are respectively given by (4.3) and

$$\lambda\overline{R} = l\,(\lambda_1 - l)\left(\frac{\mu-\lambda+l}{\mu\,(\mu-l)} + \frac{\mu-l}{(\mu-\lambda+l)\,(\mu-\lambda_2)}\right), \tag{4.4}$$

$$(2\mu-\lambda)\lambda\overline{S} = \frac{\lambda_1}{\mu-l} + \frac{\lambda}{\mu-\lambda+l} - \frac{l(\lambda_1-l)}{2\mu-\lambda}\left(\frac{1}{\mu} + \frac{1}{\mu-\lambda_2}\right) - \frac{\lambda_1-l}{\mu} - \frac{l}{\mu-\lambda_2}. \tag{4.5}$$

Let us make a change of parameters, calling: $\alpha = \mu - \lambda/2$ and $x = l - \lambda/2$. The conditions on p_1 are translated into: $0 \leq x < \alpha$ and $x \leq (\lambda - \lambda_1)/2$.
With these new variables, we get:

$$\lambda\frac{d\overline{S}}{dx} = P_{(\lambda,\lambda_2)}(x)\left[\mu(\mu-\lambda_2)(2\mu-\lambda_2)(\alpha-x)^2(\alpha+x)^2\right]^{-1} \tag{4.6}$$

which has clearly the sign of

$$\begin{aligned} P_{(\lambda,\lambda_2)}(x) &= 2x(2\mu-\lambda_2)(\alpha-x)^2(\alpha+x)^2 + \lambda_1\lambda_2(\alpha-x)^2(\alpha+x)^2 \\ &+ 2\alpha\mu(\mu-\lambda_2)(\lambda_1(\alpha+x)^2 - \lambda(\alpha-x)^2)\,. \end{aligned} \tag{4.7}$$

Now, assume that $p_1 = 1$. This case is derived from the previous one by exchanging the two queues. We then can use the same formulas, provided that we interchange λ_1 and λ_2, and that we transform l in $\lambda - l$ (i.e. x in $-x$), since the arrival rate in queue 1 is now $\lambda - l$. We then obtain that $d\overline{S}/dx$ has the sign of

$$Q_{(\lambda,\lambda_2)}(x) = -P_{(\lambda,\lambda-\lambda_2)}(-x)\,.$$

Theorem A2.2 in Appendix 2 implies then that the minimum of $\overline{S}$ on the edge $\{p_1 = 1\}$ of $\mathcal{D}$ is reached for $p_2 = 0$. As for the edge $\{p_2 = 0\}$, theorem A2.1 ensures that $P_{(\lambda,\lambda_2)}$ possesses an unique root x_0 in the interval $[0, \alpha)$, which corresponds to a number $p_0 > \frac{1}{2}$. If $p_0 < 1$, then the minimum of $\overline{S}$ on this edge is obtained for $p_1 = p_0$. Otherwise, $\overline{S}$ decreases until $p_1 = 1$, which turns out to be its minimum. In every case, the minimum is unique, which proves theorem 4.3.

□

4.2.3/Computation of p_{opt}

Now, we know that $\overline{S}$ has a minimal value at point $(p_{opt}, 0)$, where

$$p_{opt} = \min(1, p_0)\,,$$

and where p_0 is the root of $P_{(\lambda,\lambda_2)}$, given by theorem A2.1. Therefore, in order to find the value of p_{opt}, one has to:
1/ compute $P_{(\lambda,\lambda_2)}((\lambda-\lambda_1)/2)$; if it is negative, then $p_{opt} = 1$, else
2/ compute (numerically) the root p_0 of $P_{(\lambda,\lambda_2)}$.

We have displayed on Fig. 5 the curve $P(\lambda_1 - \lambda/2) = 0$ (the continuous line). The coordinates are λ and λ_2. The interesting values of (λ, λ_2) lie inside the triangle limited by the lines $\{\lambda = 2\mu\}$ and $\{\lambda_2 = \lambda/2\}$ (the "Equi-Stream line"). The thin band located between this Equi-Stream line and the curve is the region where $p_{opt} = 1$: when (λ, λ_2) is in in this region, the initial system is not sufficiently out of balance to let the gain of response time be greater than the resequencing time.

We have also drawn on Fig. 5 the curve (in dotted line) of equation:

$$\overline{S}(1,0) = \overline{S}(\frac{2\lambda_1}{\lambda}, 0)\,.$$

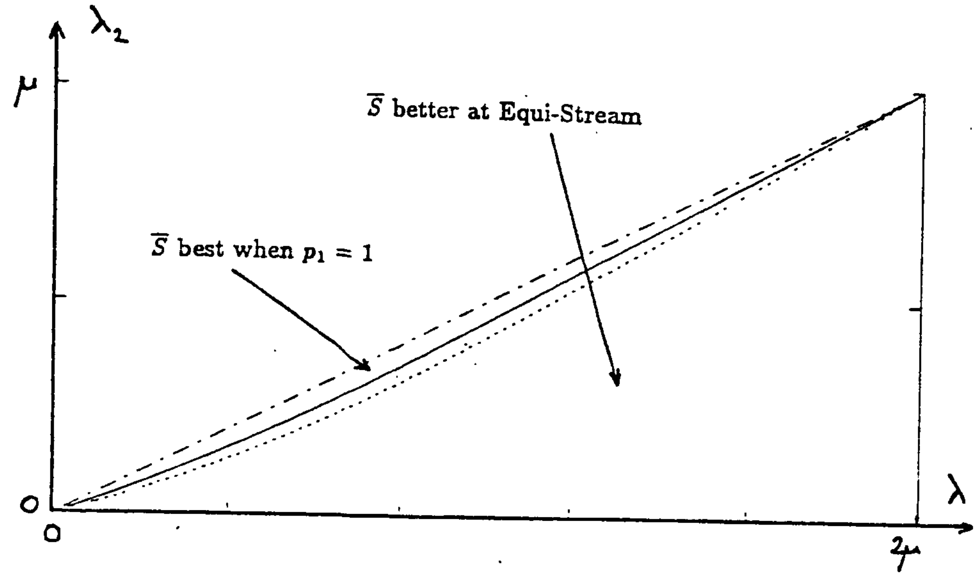

Figure 5 Sign of $P(\lambda_1 - \lambda/2)$

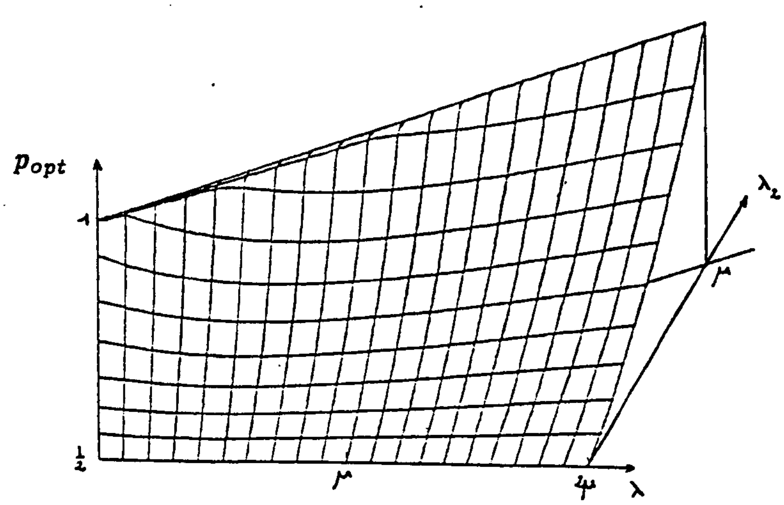

Figure 6 The optimal probability

It represents the values of (λ, λ_2) for which the mean initial sojourn time (when there is no switching) is equal to the sojourn time obtained at Equi-Stream state. This shows that in most cases, the Equi-Stream position improves the performance of the network. This might save the exact computation of p_{opt}.

Finally, we have displayed in Fig. 6 the surface $z = p_{opt}(\lambda, \lambda_2)$.

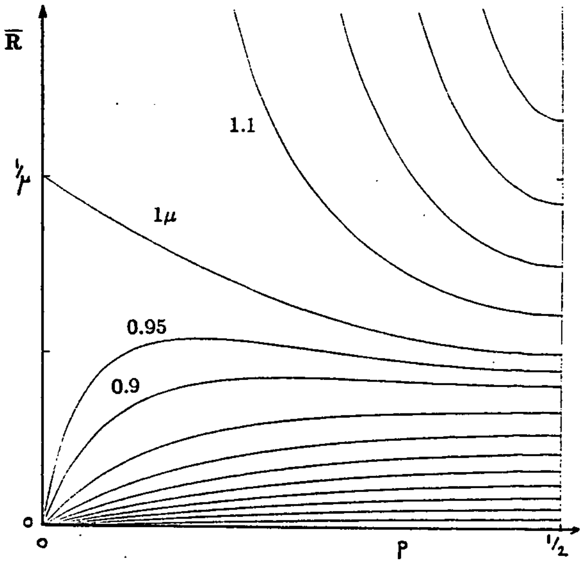

Figure 7 $\overline{R}$ in function of p and λ

5/SPECIAL CASES

5.1/$\lambda_2 = 0$

This is the case of a two-queue system with servers of same power, and an unique class of customers. From (4.4) and (4.5), we have expressions for the resequencing time and the total response time, that is:

$$\begin{aligned} \lambda\overline{R} &= \frac{\lambda - l}{\mu - l} + \frac{l}{\mu - \lambda + l} - \frac{l(\lambda - l)}{2\mu - \lambda}\frac{2}{\mu} - \frac{\lambda}{\mu}, \\ \lambda\overline{S} &= \frac{\lambda}{\mu - l} + \frac{\lambda}{\mu - \lambda + l} - \frac{l(\lambda - l)}{2\mu - \lambda}\frac{2}{\mu} - \frac{\lambda}{\mu}. \end{aligned} \tag{5.1}$$

The numerator of the derivative of $\overline{S}$ reduces to:

$$P(x) = 4\mu x(\alpha - x)^2(\alpha + x)^2 + 8\alpha^2\mu^2\lambda x,$$

which is always positive if $x > 0$. Thus, the total response time is minimum at Equi-Stream state, that is when $p_1 = \frac{1}{2}$ (see Fig. 6). This shows that the gain on the global service time is always greater than the loss due to resequencing, and that the policy consisting in minimizing the maximum mean length of the queues remains the best one.

Note that this is the only case where a system in Equi-Stream state is also optimal: in all other cases, $P(0) < 0$ (see Appendix 2).

In order to illustrate a remark we made in 4/, we show the behavior of the resequencing time $\overline{R}$ versus the switching probability. We have displayed in Fig. 7, the functions $\overline{R}(p)$, for values of λ ranging from 0 to 1.4μ with increments of 0.1μ (plus the value 0.95μ). There exists a limit value λ_{lim} (which turns out to be $2(\sqrt{2} - 1)\mu \simeq 0.828\mu$) such that,

- for all $\lambda \leq \lambda_{lim}$, $\overline{R}$ increases monotonously,
- for all $\lambda_{lim} < \lambda < \mu$, $\overline{R}$ possesses a local maximum, and a local minimum when $p = \frac{1}{2}$.

When $\lambda > \mu$ (partial instability), $\overline{R}$ decreases monotonously and has always a minimum at $p = \frac{1}{2}$. Note the interesting behavior when $\lambda = \mu$: when p tends to zero, the mean resequencing time tends to $1/\mu$, although the mean response time tends to infinity.

5.2/$\lambda_1 = \lambda_2$

We have previously mentioned this case, where the initial system is already in Equi-Stream state. From the general results of 3/, and in particular the study of $\overline{S}$ on the edge $p_1 = 1$, we see that nothing can be done to improve the total response time, since any small variation of the input streams will increase the service time, and will create a resequencing time.

6/THE MULTI-CLASS PROBLEM

A natural generalization of the studied network is a system with K inputs, each input having its own Bernoulli switch. Moreover, each input stream may contain several subclasses, submitted to an internal ordering constraint. This system models the behavior of a switching system commuting the K inputs to 2 outputs.

Let us show that the performance of the network is worst when there is a single class.

Let n_k be the number of subclasses in the kth input stream. Let $\lambda_k^1, \ldots, \lambda_k^{n_k}$ be their respective arrival rates, and $\lambda_k = \sum_1^{n_k} \lambda_k^i$.

The value of $\overline{T}$ is still given by (3.1), the values λ_1' and λ_2' being now:

$$\lambda_1' = \sum p_k \lambda_k \text{ and } \lambda_2' = \sum q_k \lambda_k \ ,$$

regardless of the subdivision into classes. The incidence of this subdivision will appear in the value of the resequencing time.

We can compute $\overline{R}$ class by class, as in 3/, and setting $a = \mu - \lambda_1'$ and $b = \mu - \lambda_2'$, we get:

$$\lambda \overline{R} = \sum_{k=1}^{K} \sum_{i=1}^{n_k} \lambda_k^i \, r_k^i \tag{6.1}$$

where

$$\begin{aligned} \overline{R_k^i} &= p_k \, f(a, b, q_k \lambda_k^i) \ + \ q_k \, f(b, a, p_k \lambda_k^i) \\ &= \frac{p_k q_k}{(a+b)} \left(\frac{a \lambda_k^i}{b(b + q_k \lambda_k^i)} + \frac{b \lambda_k^i}{a(a + p_k \lambda_k^i)} \right) . \end{aligned} \tag{6.2}$$

If a, b, and p_k are fixed, each $\lambda_k^i \overline{R_k^i}$ is a strictly convex function of λ_k^i. Consequently, it is easily checked that the function $x \mapsto x^2/(cx + d)$ is strictly convex and increasing with respect to x for every $x, c, d > 0$. Since $\overline{R_k^i}$ is the sum of two such functions, it is strictly convex. Consequently, each sum $\sum \lambda_k^i r_k^i$ is strictly convex on the domain $\{\sum \lambda_k^i = \lambda_k, \ \lambda_k^i \geq 0\}$, and is maximal at an extremal point: when $\lambda_k^i = \lambda_k$ for some i.

Similarly, the same argument holds for the domain $\{\sum p_k \lambda_k = a, \sum q_k \lambda_k = b \ \lambda_k \geq 0\}$ and we obtain that : given a, b and the $\{p_k\}$'s, the mean resequencing time is greater at the boundary of this domain. This fact allows us to reduce the complexity of the study, by removing some classes: if a choice of the p_k's improves the performance of a simpler model, it will also improve those of a more complex one.

Using backwards the convexity argument, and using the obvious symmetry between all the input rates, we conjecture that the resequencing time will be smaller when all the input rates are equal.

In the particular case of a single input, the mean resequencing time tends to zero as the the number of subclasses grows. That is natural, since there are less and less customers in each subclass. Since we have showed in 4/ that in the case of a single subclass, the optimal probability is $p = \frac{1}{2}$, this fact will hold if we increase the number of subclasses.

7/CONCLUDING REMARKS

This simple model shows that in the framework of distributed processing where the messages have to keep their ordering, the resequencing time may not be neglected, especially when the loads of the subsystems are equivalent.

This model raises some interesting questions and leads to possible extensions. The critical assumptions are the FIFO discipline and the existence of Poisson inputs and Bernoulli switches, which insures the independence of the queues. For instance, the method developed here extends to the M/G/1 case. Thus, the same analysis can lead to formulas for both T and R. However, it is very unlikely that closed form solutions can be derived for the optimal value of $\overline{T}$ and *a fortiori* of $\overline{R}$.

Other generalizations concerning the number of queues and the service rates can be investigated. If formulas of 4/ could be extended to a multi-class / multi-queue problem to obtain the expected sojourn time, the optimization problem becomes much harder to solve when the number of parameters grows. Only the case of a single class with several queues (with possibly different service rates) can be studied ([7]).

APPENDIX 1

We prove here that in the case of a constant disordering environment, the resequencing time is a convex function of the switching probability. Actually, in our case, this can be obtained by a direct computation but this is a more general result that is worth exposing.

Consider a system of two independent subsystems in parallel. Customers arrive according to a Poisson process and are switched according to an independent Bernoulli process of parameter p. We keep the notations of 3.3/, but we now assume that A and B have a probability distribution which is *independent of the value of* p. If $\bar{r}$ is the expected resequencing time of a customer, we have:

Theorem A1.1

$\bar{r}$ is a concave function of p.

Proof :

From (3.3), the resequencing time of a customer is:

$$r \;=\; p\,[B-(\tau+A)] \;+\; q\,[A'-(\tau'+B')]\;, \tag{A1.1}$$

where A', B' and τ' play the same role as A, B, τ for customers that have been routed to the second subsystem. A and A', and B and B' are identically distributed. τ and τ' are still exponentially distributed, but with respective means: $1/ql$ and $1/pl$.

Let $C = [B-A]^+$, and C^* its Laplace transform. Conditioning by the value of C, we have:

$$\begin{aligned}
\mathbb{E}([B-(\tau+A)]^+) &= \mathbb{E}\left(\int_0^C qle^{-qlt}(C-t)dt\right)\\
&= \mathbb{E}(C) - \mathbb{E}\left(\frac{1-e^{-qlC}}{ql}\right)\\
&= \mathbb{E}(C) - \frac{1-C^*(ql)}{ql}\;.
\end{aligned}$$

Introducing $D = [A-B]^+$, we obtain:

$$\bar{r} \;=\; pE(C) \;+\; qE(D) \;-\; \left(p\frac{1-C^*(ql)}{ql} \;+\; q\frac{1-D^*(pl)}{pl}\right)\;. \tag{A1.2}$$

The functions

$$G(x) \;=\; \frac{1-C^*(x)}{x} \quad \text{and} \quad H(x) \;=\; \frac{1-D^*(x)}{x}$$

are strictly convex and decreasing with respect to x. This is because C^* and D^* are Laplace transforms of probability distributions, and therefore are completely monotone (see [3]). Then $G(lq)$ is strictly convex and increasing in p (remember that $q = 1-p$). But the product of two positive strictly convex identically monotone functions (both increasing or decreasing) is strictly convex. Therefore, $pG(qp)$ and $qH(pl)$ are strictly convex functions of p, and so is the last term of (A1.2). Its opposite is then concave, and since the linear term in (A1.2) can be considered as concave, Theorem A1.1 is proved.

□

Corollary :

If A and B have the same distribution, then $\bar{r}$ has a single maximum when $p = q = \frac{1}{2}$.

Proof :

If A and B have the same distribution, so do C and D. Then $\bar{r}$ is symmetrical in p and q, and has an extremum for $p = q$. It is a maximum and it is unique because $\bar{r}$ is concave.

□

APPENDIX 2

In this appendix, we intend to outline the proof of the two following theorems. Define the set $\mathcal{L} = \{(\lambda, \lambda_2) \mid 0 \leq \lambda < 2\mu, 0 \leq \lambda_2 \leq \lambda/2\}$.

Theorem A2.1

For all (λ, λ_2) in $\mathcal{L}$, the polynomial $P_{(\lambda,\lambda_2)}$ possesses an unique root in the range $[0, \alpha)$.

Theorem A2.2

For all (λ, λ_2) in $\mathcal{L}$, the polynomial $Q_{(\lambda,\lambda_2)}$ is positive for $\alpha > x \geq \lambda_1 - \lambda/2$ (i.e. $p_2 \geq 0$).

Analyzing $P_{(\lambda,\lambda_2)}$ and $Q_{(\lambda,\lambda_2)}$ is not easy because of their degree 5, and also the two parameters (λ, λ_2). There exist methods to relate the number of roots of a polynomial to conditions on its coefficients, but all lead to the study of high-degree equations in λ and λ_2. Instead, we use a method based on bounds on $P_{(\lambda,\lambda_2)}$ and $P'_{(\lambda,\lambda_2)}$ that use polynomials in x whose coefficients are of degree 2 in one of the variables λ or λ_2. This method allows us to prove entirely "by hand" both theorems. However, the complete proof is too long to be reproduced here, so that we shall only expose the main steps. See [8] for more details.

Outline of the proof of theorem A2.1.

First, we prove the existence of the root: one may check that

- $P_{(\lambda,\lambda_2)}(\mu - \lambda/2) = P(\alpha) = 8\alpha^3\lambda_1\mu(\mu - \lambda_2)$ is always positive, and
- $P_{(\lambda,\lambda_2)}(0) = \lambda_2\alpha^3((\lambda - \lambda_2)(\mu - \lambda/2) - 2\mu(\mu - \lambda_2))$ is always negative. Actually, the last factor in $P(0)$ is an increasing linear function of λ_2. But λ_2 is smaller than $\lambda/2$, where this function takes the value $\alpha(2\mu - \lambda/2)$, which is strictly negative. So is $P(0)$.

The problem is now to prove that this root is unique in the considered interval. First, rewrite $P_{(\lambda,\lambda_2)}$ and compute $P'_{(\lambda,\lambda_2)}$ as follows:

$$
\begin{aligned}
P_{(\lambda,\lambda_2)}(x) &= Cg_{(\lambda,\lambda_2)}(x) + (\alpha^2 - x^2)^2 (Ax + B) , \\
P'_{(\lambda,\lambda_2)}(x) &= Cg'_{(\lambda,\lambda_2)}(x) + (\alpha^2 - x^2)\, h_{(\lambda,\lambda_2)}(x) .
\end{aligned}
$$

where A, B and C are positive, $g(x) = -\lambda_2 x^2 + 2\alpha(2\lambda - \lambda_2)x - \lambda_2\alpha^2$ and $h_{(\lambda,\lambda_2)}(x) = 2(2\mu - \lambda_2)(\alpha^2 - 5x^2) - 4x\lambda_2(\lambda - \lambda_2)$. Then, introduce the number $\gamma = 2\mu(\mu - \lambda_2)(4\lambda - 3\lambda_2)$. We can rewrite $P'_{(\lambda,\lambda_2)}$ again as

$$P'_{(\lambda,\lambda_2)}(x) = \left(Cg'_{(\lambda,\lambda_2)}(x) - \gamma(\alpha^2 - x^2)\right) + (\alpha^2 - x^2)(h_{(\lambda,\lambda_2)}(x) + \gamma).$$

One may check that:

- for all $(\lambda, \lambda_2) \in \mathcal{L}$ and $x \in [0, \alpha)$, $Cg'_{(\lambda,\lambda_2)}(x) - \gamma(\alpha^2 - x^2) > 0$,
- if $g^+(\lambda, \lambda_2)$ is the greatest root of $g_{(\lambda,\lambda_2)}$, then $\forall x \geq g^+(\lambda, \lambda_2)$, $P_{(\lambda,\lambda_2)}(x) > 0$,
- if $k^+(\lambda, \lambda_2)$ is the greatest root of $k_{(\lambda,\lambda_2)} = h_{(\lambda,\lambda_2)} + \gamma$, then $\forall x \leq k^+(\lambda, \lambda_2)$, $P'_{(\lambda,\lambda_2)}(x) > 0$.

In order to prove the theorem, it would be sufficient to show that for all (λ, λ_2), $g^+(\lambda, \lambda_2) < k^+(\lambda, \lambda_2)$, for then would we have: P increases strictly until $x = k^+(\lambda, \lambda_2)$, then is positive, and has therefore an unique root. Unfortunately, is is too complicated to compare analytically these two values. Instead, we compare simple bounds. We first check that:

$$\begin{aligned} k^+(\lambda, \lambda_2) &\leq \frac{\alpha\lambda_2}{4\lambda - 3\lambda_2} \triangleq \overline{k}(\lambda, \lambda_2) , \\ g^+(\lambda, \lambda_2) &> \frac{2\alpha}{5} - \frac{\lambda_2(\lambda - \lambda_2)}{5(2\mu - \lambda_2)} \triangleq \underline{g}(\lambda, \lambda_2) . \end{aligned}$$

Then, we compute the two domains $\mathcal{L}_1 = \{(\lambda, \lambda_2), k^+(\lambda, \lambda_2) > \alpha\}$ and $\mathcal{L}_2 = \{(\lambda, \lambda_2), \underline{g}(\lambda, \lambda_2) \leq \overline{k}(\lambda, \lambda_2)\}$. In each of these domains, P satisfies the desired property. The equations of the boundaries of $\mathcal{L}_1$ and $\mathcal{L}_2$ are simple enough to let us check that $\mathcal{L}_1 \cup \mathcal{L}_2$ covers the whole domain L (see Fig. A1), thus proving A1.1.

□

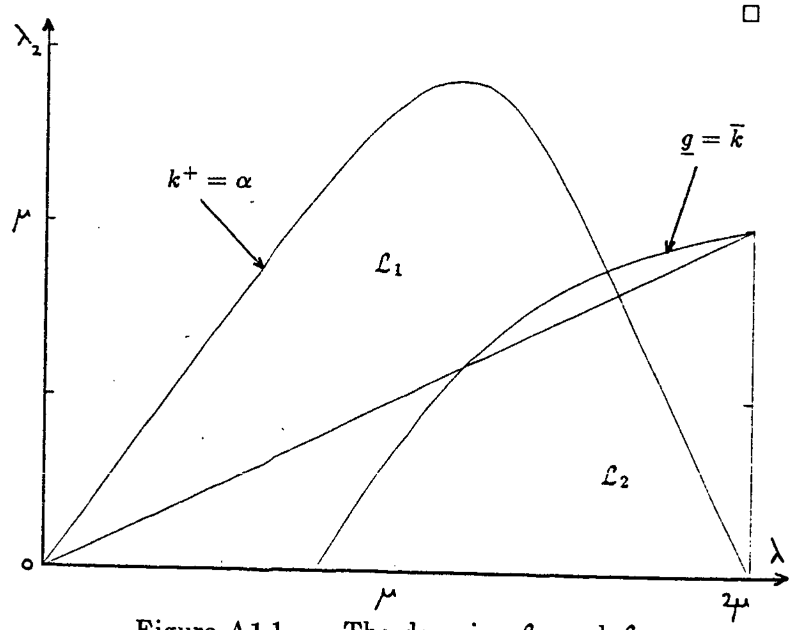

Figure A1.1 The domains $\mathcal{L}_1$ and $\mathcal{L}_2$.

Outline of the proof of theorem A2.2.

The proof of Theorem A2.2 is simpler. As before we may rewrite $Q_{(\lambda,\lambda_2)}$ as

$$Q_{(\lambda,\lambda_2)}(x) = (\alpha^2 - x^2)^2 \, (Ax - B) + g_{(\lambda,\lambda_2)}(x) \, ,$$

where A and B are positive, and compute its derivative. It turns out that Q increases up to $x = B/A$, then is positive up to $x = \alpha$. The only thing to do is to prove that $Q(\lambda_1 - \lambda/2) > 0$. This unequality is of minimal degree 3 (in λ), but here again, we can use bounds with second order polynomials that allow us to check that it is verified for all admissible (λ, λ_2).

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