



Kinetic boundary layers, numerical simulations

Ph. Rostand

► To cite this version:

Ph. Rostand. Kinetic boundary layers, numerical simulations. RR-0728, INRIA. 1987. inria-00075824

HAL Id: inria-00075824

<https://inria.hal.science/inria-00075824>

Submitted on 24 May 2006

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



UNITÉ DE RECHERCHE
INRIA-ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
BP 105
78153 Le Chesnay Cedex
France

Tél. (1) 39 63 55 11

Rapports de Recherche

N° 728

**KINETIC BOUNDARY LAYERS,
NUMERICAL SIMULATIONS**

Philippe ROSTAND

OCTOBRE 1987

KINETIC BOUNDARY LAYERS, NUMERICAL SIMULATIONS

Philippe ROSTAND *

Abstract :

We take interest here in non equilibrium boundary layers for polytropic viscous flows. A quick physical introduction is given, then we propose a numerical scheme for the compressible Navier-Stokes equations with slip boundary conditions ; two-dimensional numerical simulations are presented.

COUCHES LIMITE CINETIQUES, SIMULATIONS NUMERIQUES

Résumé :

On s'intéresse ici aux couches limites cinétiques dans un écoulement visqueux compressible d'un gaz polytropique. Après une brève introduction physique, un schéma élément finis pour la résolution de l'équation de Navier-Stokes compressible avec des conditions aux limites de glissement est proposé ; on présente enfin quelques exemples de simulations bidimensionnelles.

* INRIA-MENUSIN Domaine de Voluceau Rocquencourt BP 105 78153 Le Chesnay Cedex FRANCE



Introduction :

The prediction of temperature on space shuttles, and more generally on supersonic and hypersonic airplanes, has raised the need for a precise description of physical phenomena within the proximity of obstacles.

Confrontation of numerical simulations to experimental data (see for example [1]) has shown that compressible Navier-Stokes equations with the usual no slip boundary conditions do not give an accurate shock in the case of rarefied gas flows, which is our concern for the study of a space vehicle entering the atmosphere.

This is due to the fact that the equations of fluid dynamic (compressible Euler or Navier-Stokes) are an approximation to the kinetic description given by the Boltzman equation, valid only when the Knudsen number is small ($Kn = \lambda/L$, where λ is the local mean free path and L a local characteristic length of the flow). This is not the case near an obstacle, because L is arbitrarily small, so one must consider a nonequilibrium boundary layer, about one mean free path thick, between the body and the fluid flow, and obtain from its study boundary conditions for the latter.

A simplified boundary layer model, which we will use here, has been obtained by Gupta, Scott and Moss in [2]. Coron, Golse and Sulem are working on a more elaborate model ; preliminary results are shown in [3].

In part I, we will give a short introduction to the Boltzmann equation, and to the derivation and validity of the fluid approximation. The boundary layer model, and the derivation of boundary conditions for the Navier-Stokes equations will be outlined in Part II (see [2] for more details). Slip and no-slip conditions are compared from a mathematical point of view in part III, an adaptation of existing numerical methods for the compressible Navier-Stokes equations is proposed in part IV ; flow simulations are shown in part V.

I. Boltzmann equation and fluid approximation

This is only a very quick and simplified introduction to a large subject ; further information can be found in the books by C. Bardos [8] for basic outlines, by R. Brun [4], S. Chapman and T. Cowling [5], Truesdell and Muncaster [6] or G.A. Bird [7] (among others) for extensive details.

At the molecular level, a gas is described by calculating the path of every molecule, taking into account it's interactions with all the others ; of course numerically this is an impossible task. Statistical models assume that the particles are numerous enough, i.e. that the gas is dense enough, to allow description by a smooth probability function $f(x,v,t)$, defined by:

$$m \, dn = f(x,v,t) \, d^3x \, d^3v \quad (1)$$

$x \in R^3$ position

$v \in R^3$ speed

$t \in R^+$ time

where m is the mass of one particle, dn the number of particles having their coordinates in an elementary cell $d^3x \otimes d^3v$ of the state space, here R^6 , at a given time t .

Then f verifies

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = C(f) \quad (2)$$

where C is a collision operator.

If we also assume that

- the particles have no electric charge,
- the gas is mono-atomic, so that the particles have only three degrees of freedom : the three speed components,
- the gas is not too dense, so that collisions involving more than two particles can be neglected, and the collisions are elastic,

then the collision operator is local in space, and quadratic :

$$C(f) = Q(f, f) = \int_{R^3} \int_{S^2} (f' f'_1 - f f_1) q(|v - v_1|, w) dw dv_1 \quad (3)$$

where

$$f = f(v, v_1, w) = f(x, v, t) \quad (4)$$

$$f_1 = f_1(v, v_1, w) = f(x, v_1, t) \quad (5)$$

$$f' = f'(v, v_1, w) = f(x, v + (v_1 - v)w, t) \quad (6)$$

$$f'_1 = f'_1(v, v_1, w) = f(x, v_1 - (v_1 - v)w, t) \quad (7)$$

$$S^2 = \{w \in R^3, \|w\| = 1\}$$

q is the diffusion cross section, it depends on the collision modelization chosen.

The Boltzmann equation ((2)-(7)) is difficult to solve because it is an integro-differential equation depending on six variables (7 if the problem is time dependent). Probabilistic and deterministic simulation methods exist (see [7], or [9] for a recent contribution), but they are very expensive. But if our concern is aerodynamics, we do not need the complete kinetic solution f , but only some of its moments : the density $\rho(x, t)$, the momentum $\rho u(x, t)$ and the total energy $E(x, t)$.

$$\rho(x, t) = \int_{R^3} f(x, v, t) d^3v \quad (8)$$

$$(\rho u)_i(x, t) = \int_{R^3} v_i f(x, v, t) d^3v \quad i = 1, 2, 3 \quad (9)$$

$$E(x, t) = \int_{R^3} \frac{|v|^2}{2} f(x, v, t) d^3v \quad (10)$$

The problem of fluid approximation is to find approximate equations giving ρ , ρu and E without solving the Boltzmann equation.

The collisions being elastic, mass momentum and energy are conserved in a collision.

This implies that:

$$\int Q(f, f) d^3v = 0 \quad (11)$$

$$\int Q(f, f) v_i d^3v = 0 \quad (12)$$

$$\int Q(f, f) \frac{|v|^2}{2} d^3v = 0 \quad (13)$$

as can be checked directly; $1, v_i, |v|^2/2$ are the summational invariants.

Integrating (2) over the velocity space, one obtains

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0 \quad (14)$$

Multiplying (2) by v_i and integrating, one obtains

$$\frac{\partial}{\partial t}(\rho u)_i + \sum_j \frac{\partial}{\partial x_j} \left(\int_{R^3} v_i v_j f d^3 v \right) = 0 \quad (15)$$

Multiplying by $|v|^2/2$, and integrating, one obtains

$$\frac{\partial}{\partial t} E + \sum_i \frac{\partial}{\partial x_i} \left(\int_{R^3} \frac{|v|^2}{2} v_i f d^3 v \right) = 0 \quad (16)$$

By using

$$u_i(x, t) = \frac{\int f(x, v, t) v_i d^3 v}{\int f(x, v, t) d^3 v} \quad (17)$$

and for a given x and t , $c_i = v_i - u_i$, one can rewrite (15) as

$$\frac{\partial(\rho u)}{\partial t} + \text{div}(\rho u \otimes u) + \text{div}(\tau) = 0 \quad (18)$$

where τ is the stress tensor, defined by

$$\tau_{i,j} = \int c_i c_j f(x, v, t) d^3 v \quad (19)$$

One can rewrite (16) as

$$\frac{\partial E}{\partial t} + \text{div}(Eu) + \text{div}(u \cdot \tau) + \text{div}q = 0 \quad (20)$$

where q is heat flux vector

$$q_i = \int \frac{|v|^2}{2} c_i f(x, v, t) d^3 v \quad (21)$$

These equations are the conservation laws, they involve ρ , ρu and E , but also other moments of f . It is necessary to obtain expressions of these moments depending only of ρ , ρu and E and of their derivatives : we want a closing relation for (14), (15), (16).

In order to do that, we go back to the Boltzmann equation. One can show by taking the continuous limit of a discrete model that the probability of a state $\{f(x, v, t), x \in R^3, v \in R^3\}$ is given by the opposite of Boltzmann's H function :

$$H(t) = - \int \int_{R^6} f \text{Log} f d^3 x d^3 v \quad (22)$$

H can be shown to be decreasing in time for an isolated system : it is an entropy. The state with the minimum entropy is the most probable state, it is necessarily a maxwellian :

$$M(x, v, t) = \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\left(-\frac{|v - u|^2}{2T}\right) \quad (23)$$

where ρ , u and T are the density, the average speed and the temperature.

Such a state is said to be at equilibrium ; one may suppose that f is locally "not far" from a Maxwellian when there are no perturbations (shocks, walls, initial layers, etc...) near the point under consideration. This is the base of the Chapman-Enskog expansion : one tries to find f as a maxwellian plus a small perturbation.

The zero order expansion is obtained by supposing that f is a maxwellian

$$f(x, v, t) = M(x, v, t) \quad (24)$$

Replacing f by M in (19), one obtains

$$\tau_{ij} = \rho(x, t)T(x, t)\delta_{i,j} = p(x, t)\delta_{i,j}$$

p : pressure

Replacing f by M in (21), one obtains

$$q = 0$$

So (14), (18) and (20) are now the compressible conservative Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad (25)$$

$$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) + \nabla p = 0 \quad (26)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot ((E + p)u) = 0 \quad (27)$$

To have a more accurate result, we suppose

$$f = M(1 + \epsilon \phi) \quad \phi = \phi(x, v, t) \quad (28)$$

where $\epsilon = Kn$

The adimensionalized Boltzman equation can be written as :

$$\frac{\partial f^*}{\partial t} + v^* \cdot \nabla_x f^* = \frac{1}{\epsilon} Q(f^*, f^*) \quad (29)$$

replacing f^* by $M^*(1 + \epsilon \phi)$, we obtain

$$-\frac{1}{\epsilon} [Q(M^*, M^*)] + \left(\frac{\partial M^*}{\partial t} + v^* \cdot \nabla_x M^* - 2Q(M^*, M^* \phi) \right) + \epsilon F(M) = 0$$

Supposing the factor of order $\frac{1}{\epsilon}$ to be zero, we obtain $Q(M^*, M^*) = 0$, which is equivalent to M^* being a maxwellian : we again find that the lowest order approximation is a maxwellian.

Supposing the factor of order 0 to be null, we obtain

$$\frac{\partial M^*}{\partial t} + v \cdot \nabla_x M^* - 2Q(M^*, M^* \phi) = 0 \quad (30)$$

This is a Fredholm integral equation of ϕ , the orthogonality of the left hand side expression to the null space of the operator gives the Euler equations.

The equation (30) is difficult to solve, and doesn't yield useful results because the expression for ϕ is too complicated. But one can obtain a good approximation of the solution by taking ϕ as a linear combination of two special eigenvectors of the operator : the two first Sonine polynomials.

After some calculations, one obtains

$$\phi = -\frac{1}{K} \frac{\sqrt{\pi}}{\rho} + \frac{5}{8} \sqrt{\frac{m}{kT}} \left(\frac{3}{2} \frac{1}{T} \nabla T \cdot c \left(\frac{mc^2}{2kT} - \frac{5}{2} \right) + \frac{m}{kT} (c \otimes c - \frac{1}{3} c^2 I) : (\text{gradu}) \right) \quad (31)$$

where m is the mass of each particle, k the Boltzman constant and $K=K(\epsilon)$ a normalization constant.

Equation (19) then gives :

$$\tau = \left(\int c_i c_j M(1 + \epsilon \phi) \right)_{i,j} = 2\mu \left[\frac{1}{2} (\nabla u + \nabla u^t) - \frac{1}{3} (\text{div} u) I \right] + P \quad (32)$$

where μ is the first viscosity,

$$\mu = -\frac{5}{8} \frac{\sqrt{\pi}}{K} m \left(\frac{kT}{m} \right)^{\frac{1}{2}} \quad (33)$$

for a Maxwellian interaction potential.

Equation (21) gives

$$q = \chi \nabla T \quad (34)$$

$$\chi = -\frac{75}{32} \frac{\sqrt{\pi}}{K} k \left(\frac{kT}{m} \right)^{\frac{1}{2}} \quad (35)$$

where χ is the heat diffusion coefficient ;(14), (18) and (20) are now the conservative compressible Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \quad (36)$$

$$\frac{\partial (\rho u)}{\partial t} + \nabla \cdot (\rho u \otimes u) + \nabla p = \nabla \cdot \left(-\mu ((\nabla u + \nabla u^t) - \frac{2}{3} \nabla \cdot u I) \right) \quad (37)$$

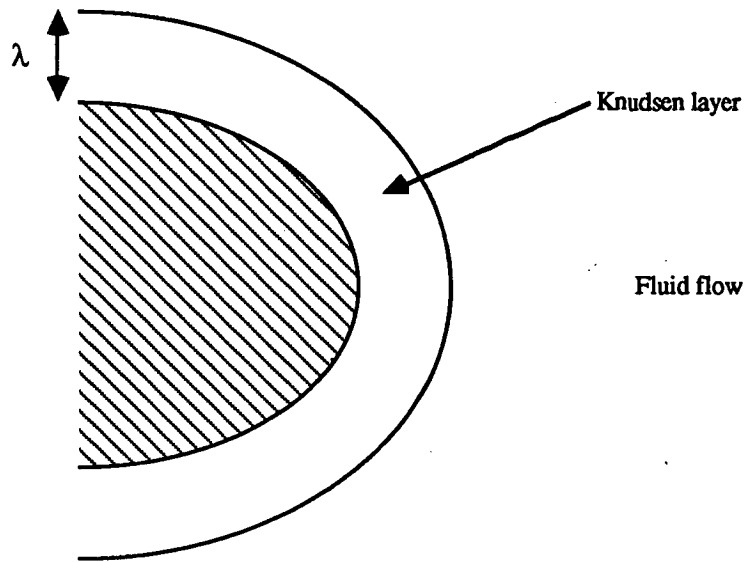
$$\frac{\partial E}{\partial t} + \nabla \cdot [(E + p)u] = \nabla \cdot [u \cdot (-\mu ((\nabla u + \nabla u^t) - \frac{2}{3} \nabla \cdot u))] + \nabla \cdot (-\chi \nabla T) \quad (38)$$

Remark :

These values for μ and λ are valid only for a monoatomic gas with a Maxwellian interaction potential. For a diatomic or a triatomic gas, a similar theory can be developped, but it is much more complicated because the diffusion cross section isn't spheric.

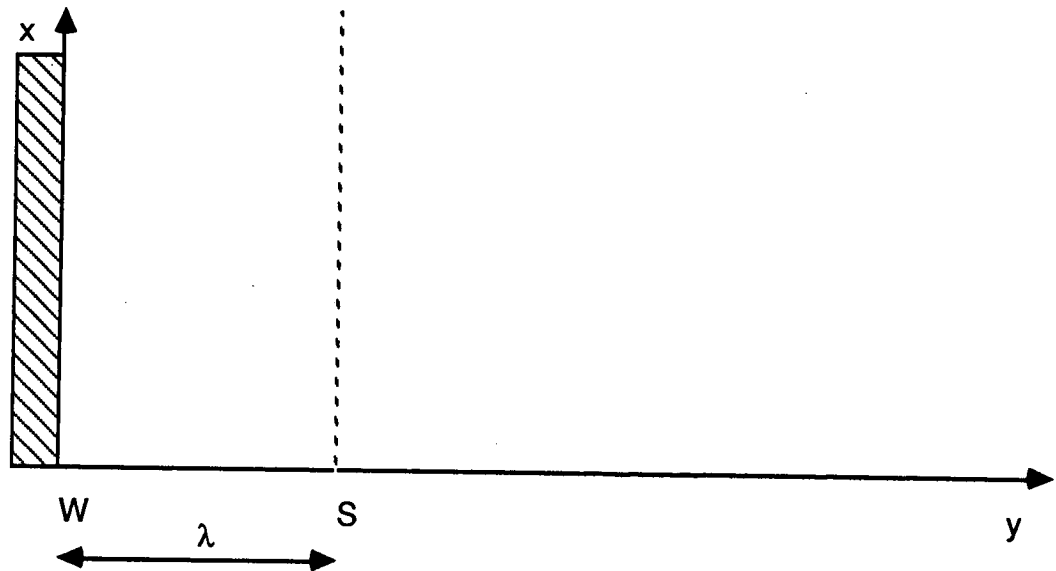
The Chapman-Enskog approximation is obviously not valid in the neighborhood of a wall, a sufficient reason for that beeing the spatial dissymetry.

II. Nonequilibrium boundary layer models



The general problem of boundary layers is to derive boundary conditions for the fluid flow from an assumed behavior of the particles when they hit the wall.

The usual formulation of this problem is the following :



Given a set of fluid variables at the edge of the Knudsen layer $\rho_s, (\rho u)_s, E_s$, the corresponding Navier-Stokes probability function $f_s = M_s(1 + \epsilon_s \phi_s)$ is known, and consequently the incident fluxes : for $\phi_i = 1$, $v_x, v_y, v_z, |v|^2/2$.

$$F_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_y \phi_i f_s(v) d^3v$$

Rescaling the problem by $1/\lambda$, the calculation of the reflected fluxes R_i is a classical half space problem (see [3] or [10]). The boundary conditions are given by equating the sum of the incident and reflected fluxes to the known boundary flux :

$$F_i + R_i = G_i \quad (39)$$

with

$$G_i = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_y \phi_i f_s(v) d^3v \quad (40)$$

To avoid resolving the half space problem, one makes assumptions on the behavior of f in the Knudsen layer.

The model provided by Gupta, Scott and Moss in [2] assumes that :

- a proportion θ of the particles hitting the wall is accomodated and re-emitted according to a maxwellian law, the rest is specularly reflected (θ is the accomodation coefficient),

- the fluxes are constant during a "one-way travel" through the Knudsen layer.

The second hypothesis is a very strong one ; it rules out shock-boundary layer interaction.

In this context,

$$R_i = (1 - \theta) \int_{-\infty}^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} v_y \phi_i f_s(v_x, -v_y, v_z) d^3v + \theta \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_y \phi_i f_w(v) d^3v \quad (41)$$

where

$$f_w(x, v, t) = \frac{\rho_w(x, t)}{(2\pi T_w(x, t))^{\frac{3}{2}}} e^{-\frac{|v|^2}{2T}} \quad (42)$$

Equation (39) then gives the following boundary conditions, when f_s is replaced by its value, and after simplifications :

$$u_x^s = \frac{2 - \theta}{\theta} \frac{5\pi}{16} \lambda_s n \cdot D(u_s) \cdot \tau_x \quad (43)$$

$$u_z^s = \frac{2 - \theta}{\theta} \frac{5\pi}{16} \lambda_s n \cdot D(u_s) \cdot \tau_z \quad (44)$$

where s indices refer to the values at the surface between the Knudsen layer and the fluid flow, n is the inward normalized unit vector, τ_x and τ_z the two tangential normalized vectors,

$$D(u) = \nabla u + \nabla u^t - 2/3 \operatorname{div} u \mathbf{I}$$

$$\frac{p^s}{p^w} = (1 - [\frac{2 - \theta}{\theta} \frac{15}{16} \frac{\lambda}{T} \frac{\partial T}{\partial n} + \frac{5}{12} \frac{\sqrt{\pi}}{2} \frac{\lambda}{\sqrt{RT_s}} \frac{\partial(u \cdot \tau_x)}{\partial x} + \frac{\partial(u \cdot \tau_z)}{\partial z} - 2 \frac{\partial(u \cdot n)}{\partial n}])^{-1} \quad (45)$$

$$\frac{T_s}{T_w} = \frac{1}{2} (\frac{P_y}{p^s} + 1) [-\frac{1}{2} + \frac{3}{4} (\frac{P_y}{p^s} + 1) - \frac{75\pi}{128} \frac{2 - \theta}{\theta} (\frac{\lambda}{T} \frac{\partial T}{\partial y})]^{-1} \quad (46)$$

where P_y is the normal momentum flux.

Only (40) and (41) are useful for the fluid calculation if T_w is unknown, (42) and (43) allow to calculate the pressure and temperature on the wall once the fluid flow is known.

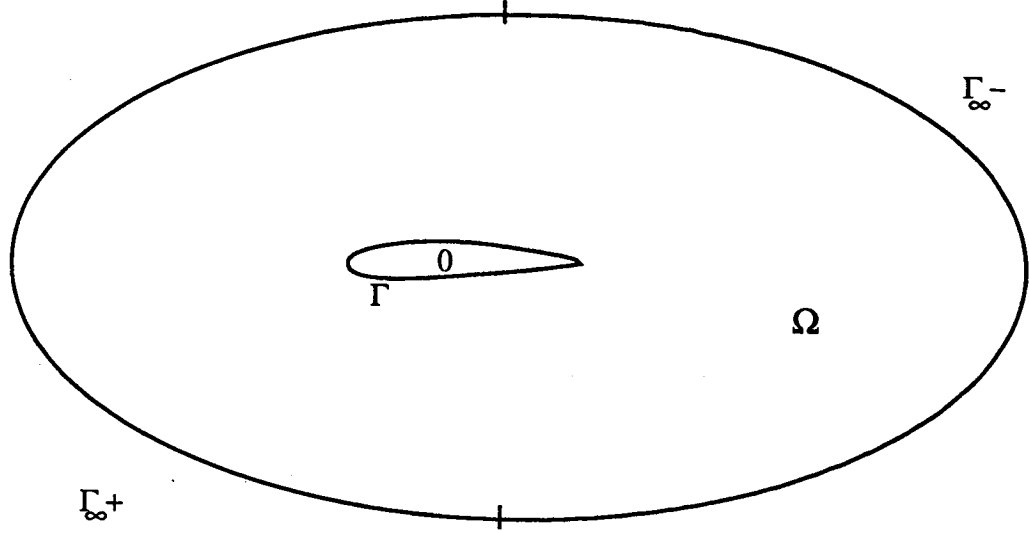
More accurate models are under investigation at the Ecole Normale Supérieure (Bardos and al.); they introduce a boundary layer term χ in the Chapman-Enskog expansion : if the wall is located at $x = 0$.

$$f = M(1 + \epsilon\phi) + \chi(x/\epsilon, v, t),$$

By rescaling the equation, one obtains a half-space problem for χ , the well posedness of which implying necessary conditions for $M(1 + \epsilon\phi)$, and consequently boundary conditions for the fluid flow.

III. The compressible Navier-Stokes equation with slip-boundary condition :

We restrict ourselves to the two-dimensional case for simplicity, and assume the viscosity and heat diffusion coefficients to be constant.



$O \in R^2$ represents the obstacle, Ω the flow domain, $\Gamma : \partial O$, $\Gamma_\infty = \partial\Omega/\Gamma$.
The adimensionalized Navier-Stokes system is then :

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho u) = 0 \quad (44)$$

$$\frac{\partial(\rho u)}{\partial t} + \text{div}(\rho u \otimes u) + \nabla p = \frac{1}{Re} \text{div}(D(u)) \quad (45)$$

$$\frac{\partial E}{\partial t} + \text{div}((E + p)u) = \frac{1}{Re} (\text{div}(u(D(u))) + \frac{\gamma}{Pr} \Delta T) \quad (46)$$

in Ω , where:

$u \in R^2$ is the speed
 $p = (\gamma - 1)\rho T$ is the pressure,
 Re is the Reynolds number,
 Pr is the Prandtl number,

with boundary conditions on Γ :

$$u \cdot n = 0 \quad (47)$$

$$n \cdot D(u) \cdot \tau = -A \rho u \cdot \tau \quad (48)$$

$$\frac{1}{Re} u \cdot D(u) \cdot n + \frac{\gamma}{Pr Re} \frac{\partial T}{\partial n} = 0 \quad (48)$$

where n is the outward normal unit vector to Γ , τ a unit vector tangential to Γ ,

$$D(u) = \nabla u + \nabla u^t - \frac{2}{3} \nabla \cdot u \, I$$

Equation (47) is a natural boundary condition, assuming there is no mass flux through the Knudsen layer

Equation (48) is given by (40) and (41), A is a positive constant, depending on the nature of the gas in consideration and on its density :

$$A = \rho_{\infty} c \quad (50)$$

where c is a physical constant, $c \cong 10^7$ U.S.I. for air, and ρ_{∞} is the physical density of the gas for away from the body, $\rho_{\infty} = 1,2 \text{ kg m}^{-3}$ at ground level, $\rho_{\infty} = 10^{-7} \text{ kg m}^{-3}$ at an altitude of 120 km.

Equation (49) is the consequence of the known adiabaticity of the wall. We assume the flow at infinity to be uniform :

$$u = (\cos \alpha, \sin \alpha) \quad (51)$$

$$\rho = 1 \quad (52)$$

$$T = \frac{1}{\gamma(\gamma - 1)M_o^2} \quad (53)$$

where M_o is the free stream mach number and α the angle of attack .

In fact, we enforce the four boundary conditions (51), (52), (53) at the inflow boundary Γ_{∞}^- , but only one of them at the outflow in the supersonic case, for an advection dominated problem.

Theoretical results on the compressible Navier-Stokes equations are scarce (see [11] for a local in time result), but one can check a few necessary conditions for the well posedness of the problem : decay of energy and entropy of a smooth solution, coercivity of the viscous term restricted to the speed space, number of boundary conditions compared to the signature of the convection Jacobian matrix.

We assume the external boundary Γ_{∞} to be sufficiently far of the body for the flow to be uniform in it's neighborhood ; consequently all derivatives of ρ , u and E are zero on Γ_{∞} .

The decay of energy is very easily obtained :

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} E \right) &= - \int_{\Omega} \nabla \cdot ((E + p)u) + \frac{1}{Re} \int_{\Omega} \nabla \cdot (u \cdot D(u)) + \frac{\gamma}{Re Pr} \int_{\Omega} \Delta T \\ &= - \int_{\Gamma \cup \Gamma_{\infty}} (E + p)(u \cdot n) + \int_{\Gamma \cup \Gamma_{\infty}} \left(\frac{1}{Re} u \cdot D(u) \cdot n + \frac{\gamma}{Re Pr} \frac{\partial T}{\partial n} \right) \end{aligned}$$

on Γ ,

$$u \cdot n = 0$$

$$\frac{1}{Re} u \cdot D(u) \cdot n + \frac{\gamma}{Pr Re} \frac{\partial T}{\partial n} = 0$$

so the boundary flux is zero.

On Γ_{∞} , $D(u) = 0$, $\partial T / \partial n = 0$, and $(E + p)(u \cdot n)$ is constant, so

$$\int_{\Gamma_{\infty}} (E + p)(u \cdot n) = 0$$

thus

$$\frac{d}{dt} \left(\int_{\Omega} E \right) = 0$$

The decay of entropy is slightly more difficult to obtain: the entropy per unit mass is ρs , with

$$s = \text{Log}\left(\frac{\rho^{\gamma-1}}{T}\right) \quad (54)$$

combining (44), (45) and (46), we obtain

$$\frac{\partial \rho s}{\partial t} + \nabla \cdot (\rho u s) = -\frac{1}{T \text{Re}} \left[\frac{\gamma}{\text{Pr}} \Delta T + F(\nabla u) \right] \quad (55)$$

with

$$F(\nabla u) = (\nabla u + \nabla u^t - \frac{2}{3} \nabla \cdot u) : \nabla u \quad (56)$$

or

$$F(\nabla u) = \frac{4}{3} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] + \left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \right)^2 - \frac{4}{3} \frac{\partial v}{\partial x} \frac{\partial w}{\partial y} \quad (57)$$

if $u = (v, w)$

$$F(\nabla u) = \frac{2}{3} \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] + \left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y} \right)^2 + \frac{2}{3} \left(\frac{\partial v}{\partial x} - \frac{\partial w}{\partial y} \right)^2 \quad (58)$$

$$F(\nabla u) \geq 0$$

$$\begin{aligned} \frac{d}{dt} \left(\int_{\Omega} \rho s \right) &= - \int_{\Omega} \nabla \cdot (\rho u s) - \frac{1}{\text{Re}} \int_{\Omega} \frac{F(\nabla u)}{T} - \int_{\Omega} \frac{1}{\text{Re}} \frac{\gamma}{\text{Pr}} \frac{\Delta T}{T} \\ &\leq - \int_{\Gamma \cup \Gamma_{\infty}} \rho s(u \cdot n) - \int_{\Omega} \frac{\gamma}{\text{RePr}} \frac{|\nabla T|^2}{T^2} - \int_{\Gamma \cup \Gamma_{\infty}} \frac{\gamma}{\text{RePr}} \frac{1}{T} \frac{\partial T}{\partial n} \end{aligned}$$

on Γ :

$$u \cdot n = 0$$

$$\frac{\gamma}{\text{RePr}} \frac{\partial T}{\partial n} = - \frac{1}{\text{Re}} u \cdot D(u) \cdot n$$

but on Γ , $u = (u \cdot \tau) \tau$, so that

$$\frac{\gamma}{\text{RePr}} \frac{\partial T}{\partial n} = - \frac{1}{\text{Re}} (u \cdot \tau) \tau \cdot D(u) \cdot n$$

using (48),

$$\frac{\gamma}{\text{RePr}} \frac{\partial T}{\partial n} = \frac{A}{\text{Re}} \rho(u \cdot \tau)^2 \geq 0 \quad (59)$$

on Γ_∞ , $\partial T / \partial n = 0$, $\rho s(u.n)$ is constant, so we obtain

$$\frac{d}{dt} \left(\int_{\Omega} \rho s \right) \leq 0 \quad (60)$$

Coercivity of the viscous term :

The following results are extensions of R. Verfurth's work on the incompressible case (see [12]). The question is on the existence and uniqueness of a solution $u \in (H^1(\Omega))$ of

$$\operatorname{div}(D(u)) = g \quad \text{in } \Omega \quad (61-a)$$

$$u.n = 0 \quad \text{in } \Gamma \quad (61-b)$$

$$n.D(u).\tau = -A(u.\tau) \quad \text{in } \Gamma \quad (61-c)$$

for Ω bounded domain of R^2 , $g \in L^2(\Omega)$, where A is a positive constant, n is the outward normal unit vector to $\partial\Omega$, τ is a tangential vector to $\partial\Omega$.

Let $V = \{u \in H^1(\Omega)^2, u.n = 0 \text{ on } \partial\Omega\}$
if we assume $\partial\Omega$ to be regular (say C^1), the function

$$n : \partial\Omega \rightarrow R^2$$

$$x \mapsto n(x)$$

is C^1 and consequently, the trace operator

$$H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$$

$$u \mapsto u.n$$

is continuous (see [12] for details).

So V with the $H^1(\Omega)$ norm, as a closed subspace of $H^1(\Omega)$, is a Hilbert space.

Let $\phi \in V$, (61) implies, integrating by parts :

$$\frac{1}{2} \int_{\Omega} D(u).D(\phi) + \int_{\Omega} \frac{2}{9} \nabla.u \nabla.\phi - \int_{\partial\Omega} \phi.D(u).\tau = \int_{\Omega} g\phi \quad (62)$$

$\int_{\partial\Omega} \phi.D(u).\tau$ is a well defined integral because $D(u)|_{\Gamma} \in H^{-1/2}(\Gamma)$

Equation (61-c) is equivalent to :

$$\forall \psi \in H^{1/2}(\Gamma), \quad \int_{\Gamma} (n.D(u).\tau + Au.\tau) \psi = 0$$

$$\Leftrightarrow \forall \phi \in V, \quad \int_{\Gamma} (n.D(u).\tau + Au.\tau).(\phi.\tau) = 0$$

Equation (62) then implies

$$\forall \phi \in V \quad \frac{1}{2} \int_{\Omega} D(u)D(\phi) + \frac{2}{9} \int_{\Omega} \nabla \cdot u \nabla \cdot \phi + A \int_{\Gamma} (\phi \cdot \tau)(u \cdot \tau) = \int_{\Omega} g \cdot \phi \quad (63)$$

conversely, it is well-known (see [12] for example) that (63) and $u \in V$ implies (61).

To use Lax-Milgram's theorem, we must prove the coercivity of the bilinear form ϕ :

$$\begin{aligned} \phi & : \quad V \rightarrow \mathbb{R} \\ u & \rightarrow \frac{1}{4} \int_{\Omega} |D(u)|^2 + \frac{1}{9} \int_{\Omega} (\nabla \cdot u)^2 + \frac{A}{2} \int_{\Gamma} (u \cdot \tau)^2 \end{aligned}$$

where we recall that A is a strictly positive constant.

$$|D(u)|^2 = |\nabla u + \nabla u^t - \frac{2}{3} \nabla \cdot u I|^2 = |\nabla u + \nabla u^t|^2 - \frac{16}{9} (\nabla \cdot u)^2$$

$$\frac{1}{4} |D(u)|^2 + \frac{1}{9} (\nabla \cdot u)^2 = \frac{1}{4} |\nabla u + \nabla u^t|^2 - \frac{1}{3} (\nabla \cdot u)^2$$

if $u = (v, w)$,

$$(\nabla \cdot u)^2 \leq 2 \left[\left(\frac{\partial v}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right]$$

$$|\nabla u + \nabla u^t|^2 \geq 4 \left(\frac{\partial v}{\partial x} \right)^2 + 4 \left(\frac{\partial w}{\partial y} \right)^2$$

so $\forall u \in H^1(\Omega)$

$$\frac{1}{4} |D(u)|^2 + \frac{1}{9} (\text{div} u)^2 \geq \frac{1}{12} |\nabla u + \nabla u^t|^2 \quad (64)$$

We will now recall two classical lemmas :

Lemma 1 : (Korn's inequality) : There are two positive constants, depending on Ω , such that

$$\forall u \in H^1(\Omega), \quad \frac{1}{2} \int_{\Omega} |\nabla u + \nabla u^t|^2 \geq C_1 \|u\|_{H^1(\Omega)}^2 - C_2 \|u\|_{L^2(\Omega)} \quad (65)$$

a proof is given in [13].

Lemma 2 : (Poincaré-Morrey) : There is a constant $C_3(\Omega)$, strictly positive, such that

$$\forall u \in H^1(\Omega), \quad C_3(\Omega) \|u\|_{L^2(\Omega)} \leq \int_{\Omega} |\nabla u + \nabla u^t|^2 + \int_{\Gamma} |u \cdot n|^2 + \int_{\Gamma} |u \cdot \tau|^2 \quad (66)$$

Proof :
See [12], lemma 2.2.

Proposition 1 : ϕ is coercive.

Proof :
(64),(65) and (66) imply $\phi(u) \geq C\|u\|_{H^1(\Omega)} \forall u \in V$, if $C \leq \frac{C_1}{12} (\frac{1}{2} + \frac{C_2}{C_3})$ and $C \leq A \frac{C_3}{C_2}$.
So, by Lax Milgram's theorem, (61) has a unique solution.

It is easy to check that no such result exists in three-dimensions.

Weak formulation of (44)-(49), (51)-(53)

It is well known that the boundary condition $u.n = 0$ cannot be enforced strongly on a polygonal domain: if for example u is piecewise linear, if s is a corner of the boundary, $u_s.n_1 = 0$, $u_s.n_2 = 0$, so $u_s = 0$, this means $u = 0$ on the boundary. Consequently, we will take a weak formulation in $(H^1(\Omega))^4$.

Let $\phi, \epsilon \in H^1(\Omega)$, $\psi \in H^1(\Omega)^2$, (44) – (49), (51) – (53) is equivalent to

$$\frac{\partial}{\partial t} \int_{\Omega} \rho \phi - \int_{\Omega} \rho u \cdot \nabla \phi + \int_{\Gamma_{\infty}} \rho_0(u_0.n)\phi = 0 \quad (67)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{\Omega} \rho u \cdot \psi - \int_{\Omega} \rho(u \cdot \nabla \psi \cdot u) + \int_{\Gamma_{\infty}} \rho_0(u_0.n)(u \cdot \psi) - \int_{\Omega} P \nabla \cdot \psi + \int_{\Gamma_{\infty}} P_0 \psi \cdot n + \int_{\Gamma} P \psi \cdot n \\ & = -\frac{1}{Re} \int_{\Omega} D(u) : \nabla \psi + \frac{1}{Re} \int_{\Gamma_{\infty}} n \cdot D(u) \cdot \psi - \frac{A}{Re} \int_{\Gamma} \rho(u \cdot \tau)(\psi \cdot \tau) + \frac{1}{Re} \int_{\Gamma} (n \cdot D(u) \cdot n)(\psi \cdot n) \end{aligned} \quad (68)$$

$$\begin{aligned} & \frac{\partial}{\partial t} \int_{\Omega} E \epsilon - \int_{\Omega} (E + p) u \cdot \nabla \epsilon + \int_{\Gamma_{\infty}} (E_0 + p_0)(u \cdot n) \epsilon \\ & = -\frac{1}{Re} \int_{\Omega} u \cdot D(u) \cdot \nabla \epsilon - \frac{\gamma}{Pr Re} \int_{\Omega} \nabla T \cdot \nabla \epsilon + \frac{1}{Re} \int_{\Gamma_{\infty}} (u \cdot D(u) \cdot n + \frac{\gamma}{Pr} \frac{\partial T}{\partial n}) \epsilon \end{aligned} \quad (69)$$

where $u_0, \rho_0, T_0, E_0, P_0$ are $u_{\infty}, \rho_{\infty}, T_{\infty}, E_{\infty}, p_{\infty}$ or u, ρ, T, E, P depending on the signature of the jacobian matrix of the convective terms (see [14] or [15] for details).

IV. Numerical methods :

Efficient finite element methods have been developed in the recent years for the compressible Navier-Stokes equations (see [14] and [15]). They rely on centered Galerkin approximation with artificial viscosity, or upwind finite volume solvers, for the Euler equations (see [16], [17]). We will only outline these methods here.

The functional space $(H^1(\Omega))^4$ is approximated by $(P^1(\Omega_h))^4$, where Ω_h is a polygonal approximation of Ω , and $P_1(\Omega_h)$ the space of piecewise linear functions on a standard triangulation of Ω_h .

The discretized variational formulation then reads

$$\frac{\partial}{\partial t} \int_{\Omega} w_h \phi_h = E(w_h, \phi_h) + N(w_h, \phi_h) \quad \forall \phi_h \in (P^1(\Omega_h))^4 \quad (70)$$

where

$$w_h = \{\rho_h, (\rho u)_h, (\rho v)_h, E_h\}$$

$E(w_h, \phi_h)$ stands for the convection terms, depending on the approximation chosen (we usually take an upwind Osher approximation, see (18)), $N(w_h, \phi_h)$ stands for the diffusion terms

$$N(w_h, \phi_h) = N_1(w_h, \phi_h) + N_2(w_h, \phi_h) + N_3(w_h, \phi_h) + N_4(w_h, \phi_h) \quad (71)$$

with

$$N_1(w_h, \phi_h) = 0 \quad (72)$$

$$\begin{aligned} N_2(w_h, \phi_h) + N_3(w_h, \phi_h) = & -\frac{1}{Re} \int_{\Omega_h} D(u_h) \cdot \nabla (\phi_h^2, \phi_h^3) - \frac{A}{Re} \int_{\Gamma_h} \rho_h(u_h \cdot \tau) ((\phi_h^2, \phi_h^3) \cdot \tau) \\ & - \frac{1}{Re} \int_{\Gamma_h} (n \cdot D(u_h) \cdot n) ((\phi_h^2, \phi_h^3) \cdot \tau) + \frac{1}{Re} \int_{\Gamma_{\infty}} n \cdot D(u_h) \cdot (\phi_h^2, \phi_h^3) \end{aligned} \quad (73)$$

$$N_4(w_h, \phi_h) = -\frac{1}{Re} \int_{\Omega_h} u_h \cdot D(u_h) \cdot \nabla \phi_h^4 - \frac{\gamma}{Pr Re} \int_{\Omega_h} \nabla T_h \cdot \nabla \phi_h^4 \quad (74)$$

$$\text{where } \phi_h = (\phi_h^i)_{i=1,4}$$

u_h and T_h are P^1 approximations of $(\rho u)_h / \rho_h$ and $E_h / \rho_h - 1/2 ((\rho u)_h)^2 / \rho_h^2$ respectively (one can check that u_h and T_h are still first order approximations of u and T).

One is usually interested in stationary solutions, so classical time discretizations are delta schemes :

$$\int_{\Omega} M(w_h^{n+1} - w_h^n), \phi_h = \Delta t [E(w_h^n, \phi_h) + N(w_h^n, \phi_h)] \quad (75)$$

with M the identity operator for an explicit scheme, a linearization of E in most implicit schemes.

Slip boundary conditions introduce a new problem : A is often a large constant, so the system (75) may be ill-conditioned. That is why, while (75) with $M = I - \Delta t \partial E$, ∂E being a linearization of E , is quite efficient with no slip boundary conditions, it isn't here : the Courant number is drastically limited for A greater than unity.

So we introduce an implicit factored scheme :

$$M = (I - \Delta t \partial N)(I - \Delta t \partial E) \quad (76)$$

where ∂N is a complete linearization of N , including boundary integrals:

$$\begin{aligned} \int \partial N(\delta w_h, \phi_h) = & \frac{1}{Re} \int_{\Omega_h} D(\delta u_h) \cdot \nabla [\phi_h^2, \phi_h^3] + \frac{A}{Re} \int_{\Gamma_h} \delta \rho_h(u_h^n \cdot \tau) ([\phi_h^2, \phi_h^3] \cdot \tau) \\ & + \frac{A}{Re} \int_{\Gamma_h} \rho_h(\delta u_h \cdot \tau) ([\phi_h^2, \phi_h^3] \cdot \tau) - \frac{1}{Re} \int_{\Gamma_h} (n \cdot D(\delta u_h) \cdot n) [\phi_h^2, \phi_h^3] \cdot n \\ & + \frac{1}{Re} \int_{\Omega_h} \delta u_h \cdot D(u_h^n) \cdot \nabla \phi_h^4 + \frac{1}{Re} \int_{\Omega_h} u_h \cdot D(\delta u_h^n) \cdot \nabla \phi_h^4 + \frac{\gamma}{Re Pr} \int_{\Omega_h} \nabla(\delta T_h) \cdot \nabla \phi_h^4 \end{aligned} \quad (77)$$

$$\text{where } \delta w_h = w_h^{n+1} - w_h^n$$

(76) is solved at each time step in two successive operations :

$$(I - \Delta t \partial E)(\delta W_h) = \Delta t [E(w_h^n) + N(w_h^n)] \quad (78)$$

is solved with respect to the conservative variables, and

$$(I - \Delta t \partial N)(\delta w_h) = \delta W_h \quad (79)$$

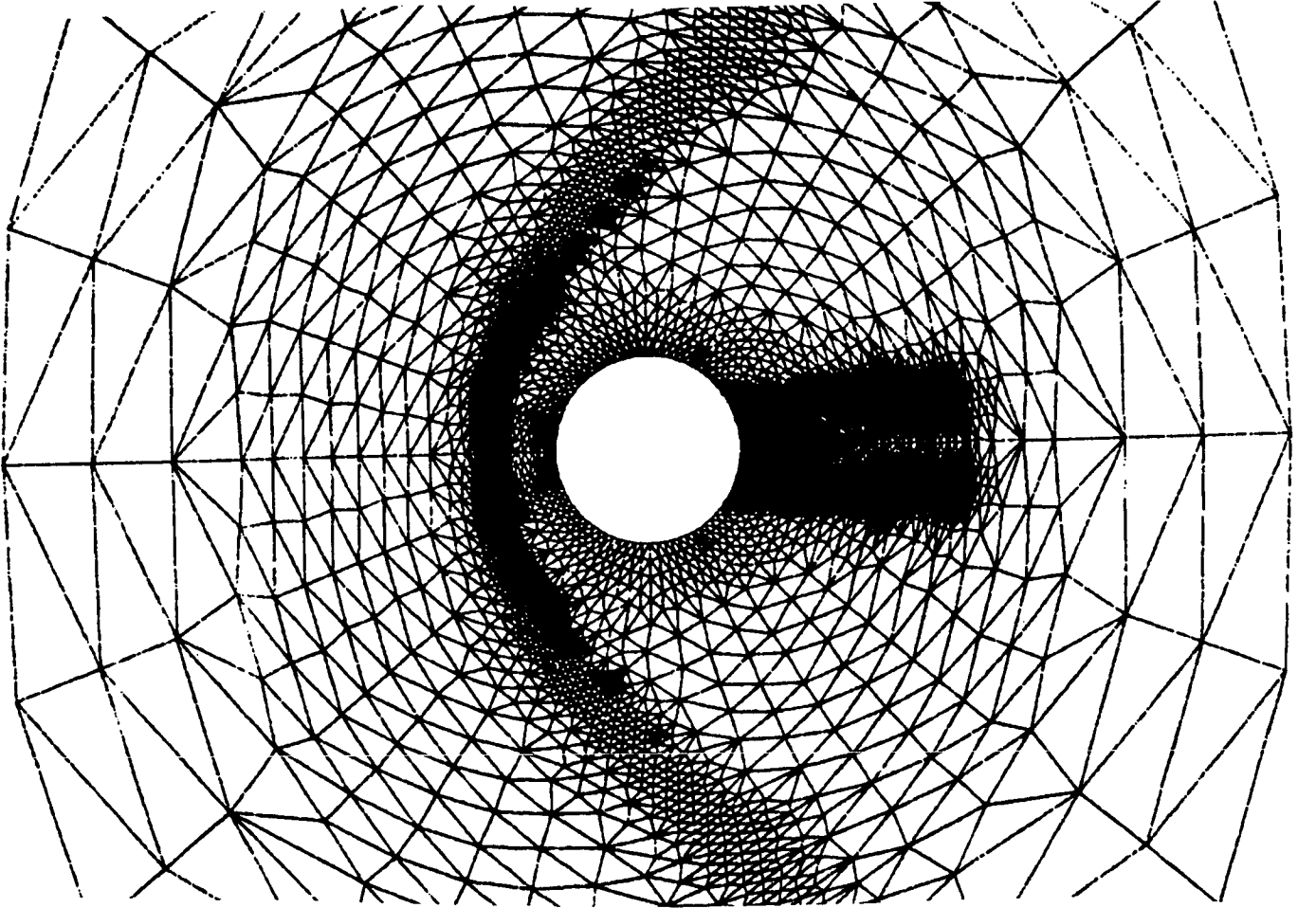
with respect to the non conservative variables.

Although an improvement over existing methods, (76) is not stable enough to allow very high mach and low Reynolds flow calculation, which are our goal for the study of a shuttle reentry. A totally implicit scheme with viscous terms calculated from the conservative variables is beeing developped.

V. Flow simulations :

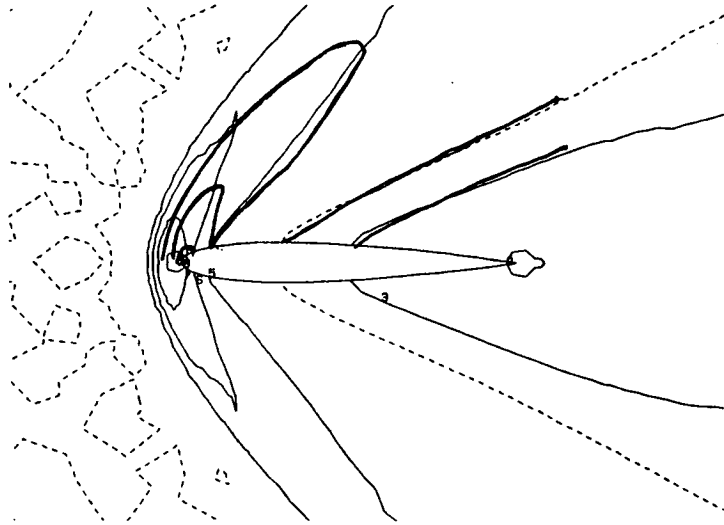
These computations where performed on the Cray of the C.C.V.R. and on the I.B.M. 3090 of AMD-BA.

Transonic, supersonic and hypersonic calculations were performed with slip and no slip boundary conditions, at different densities, around a NACA0012 airfoil and a cylinder ;comparison with experimental data taken from [1] is made.

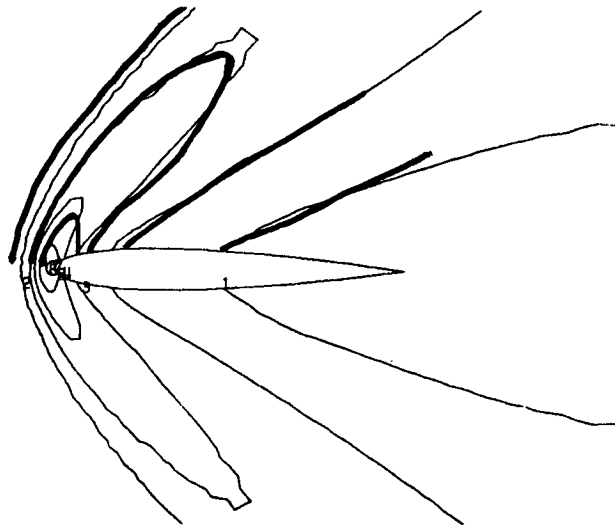


example of the unstructured meshes used

A) Flow over a NACA 0012 airfoil at a Mach number of 2 and a Reynolds number of 10^6 ; two results are presented: the no slip flow and the flow at a density of 10^{-5} kg/m^3 . The experimental flowfield, at a density of 10^{-7} kg/m^3 , taken from (1), are superimposed.



No-slip iso density lines

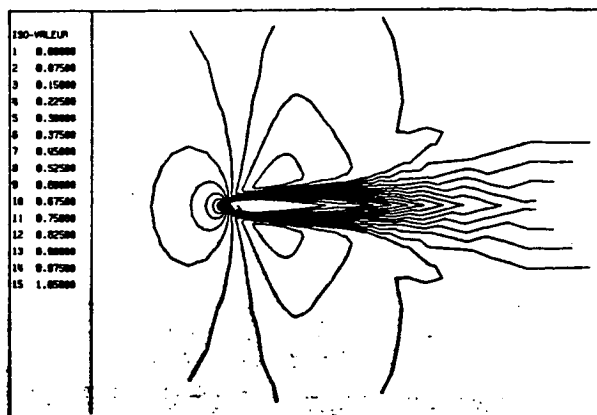


Slip iso density lines

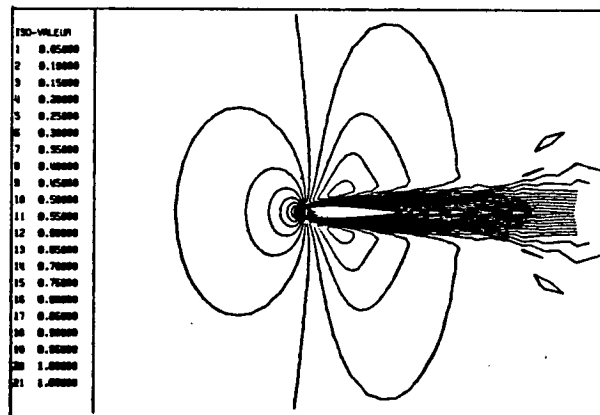
It appears that slip boundary conditions improves the precision of numerical results, although the closest to experiment result is obtained for $\rho_\infty = 10^{-5} \text{ kg/m}^3$, which is not the experimental density. But one must remember that the boundary layer model is very coarse, we only intend here to show that our approach is legitimate.

B) Flow over a NACA 0012 airfoil at a Mach number of 0.85 and a Reynolds number of 500 ;four results are presented :the no slip flow and three rarefied flows at densities of 10^{-5} kg/m^3 , 10^{-6} kg/m^3 and 10^{-7} kg/m^3 .

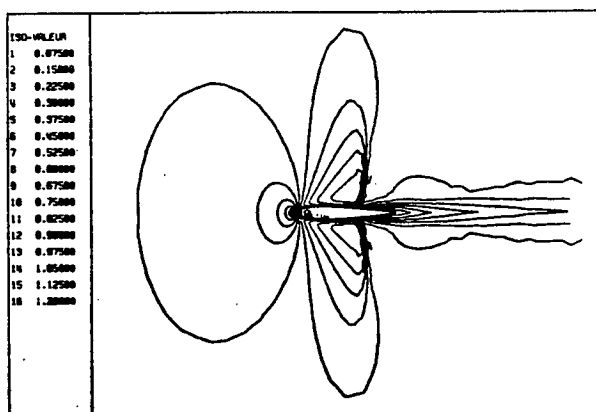
Iso-Mach lines



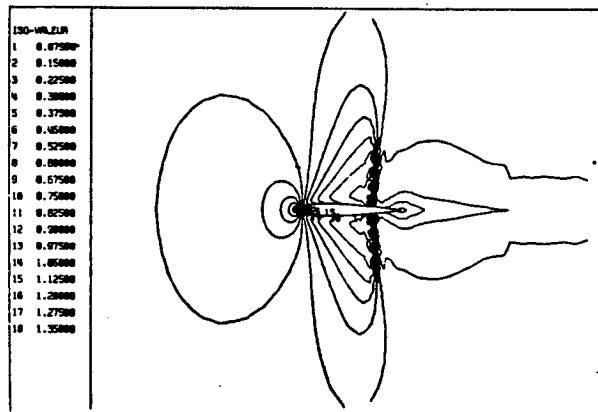
No slip



$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$

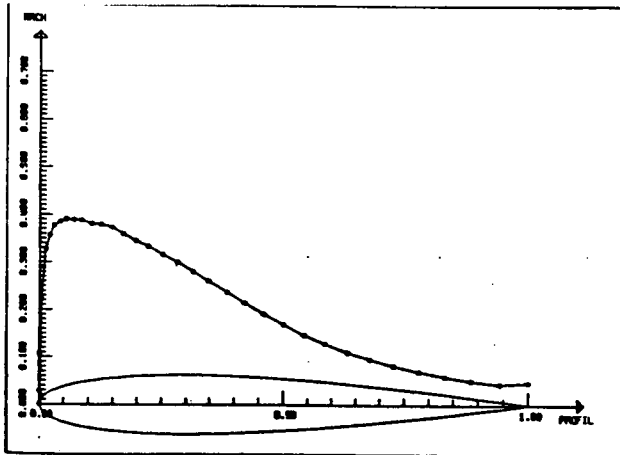


$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$

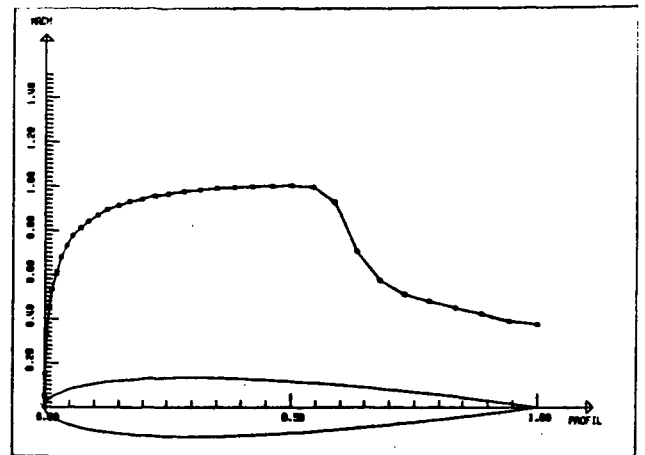


$\rho_{\infty} = 10^{-7} \text{ kg/m}^3$

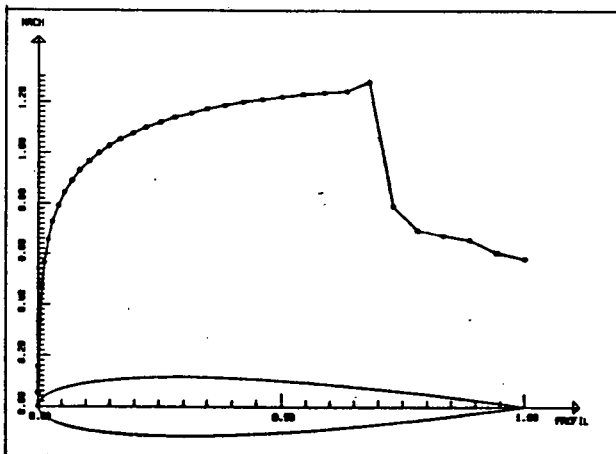
Mach number on the body



$$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$$



$$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$$

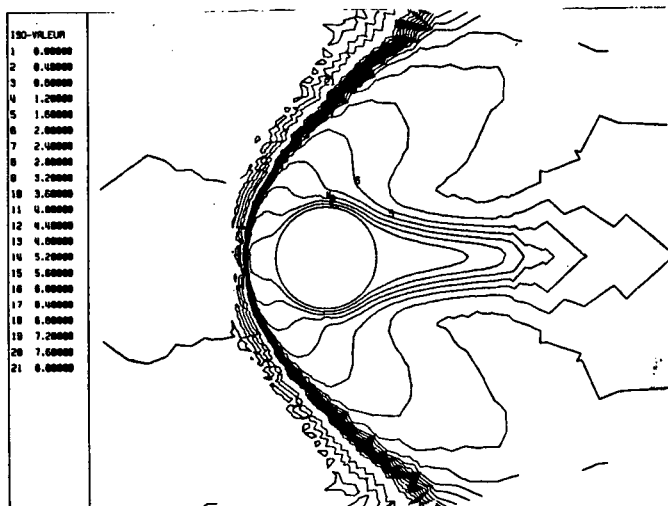


$$\rho_{\infty} = 10^{-7} \text{ kg/m}^3$$

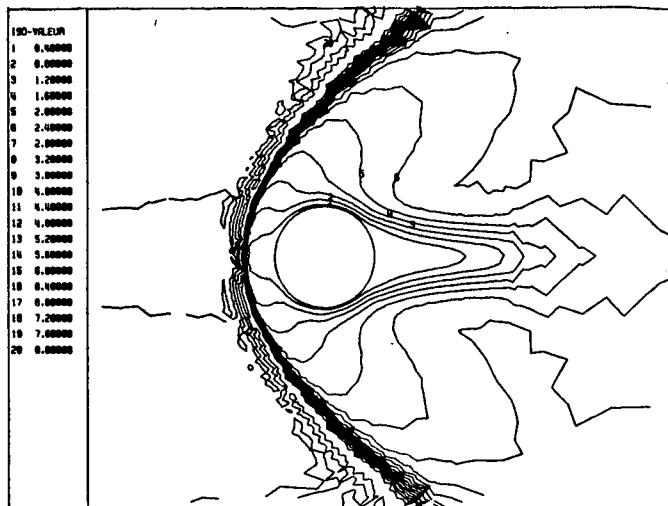
It is obvious that a transsonic flow depends very strongly on the boundary condition, so the boundary layer model must be carefully tuned in order to obtain reliable results.

C) Flow over a cylinder at a Mach number of 8 and a Reynolds number of 1000 ; four results are presented : the no slip flow and three rarefied flows at densities of 10^{-4} kg/m^3 , 10^{-5} kg/m^3 and 10^{-6} kg/m^3 .

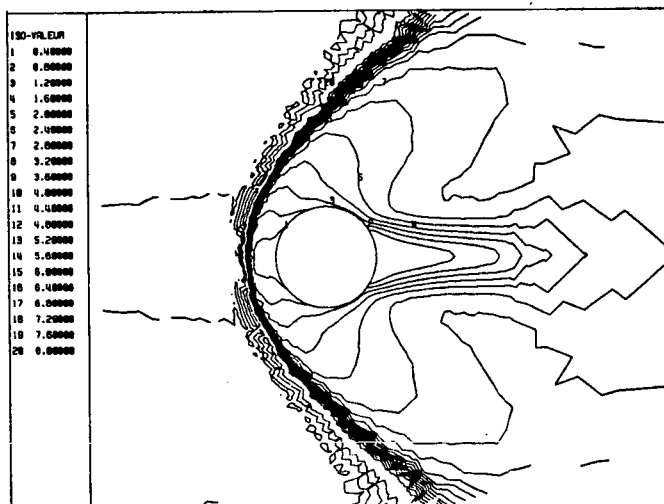
Iso-Mach lines



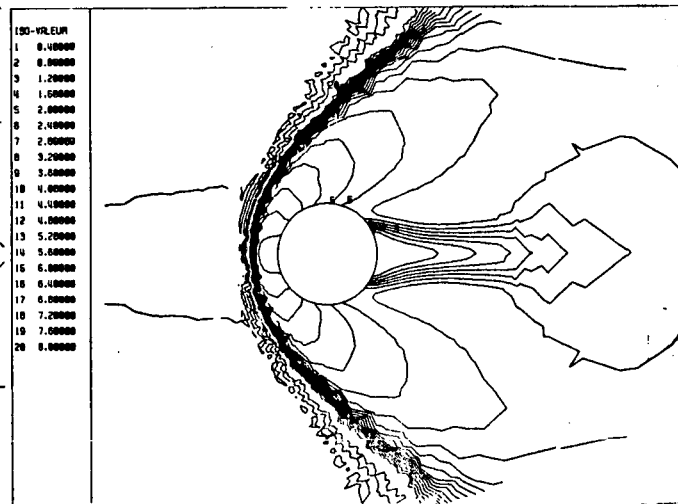
No slip



$\rho_{\infty} = 10^{-4} \text{ kg/m}^3$



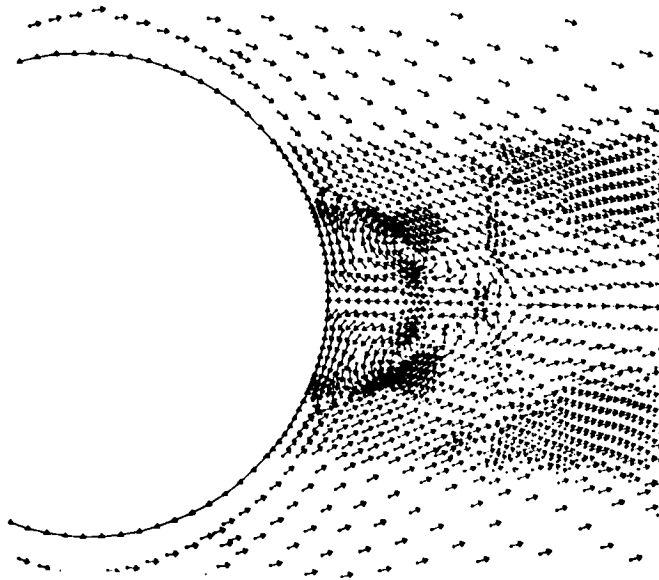
$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$



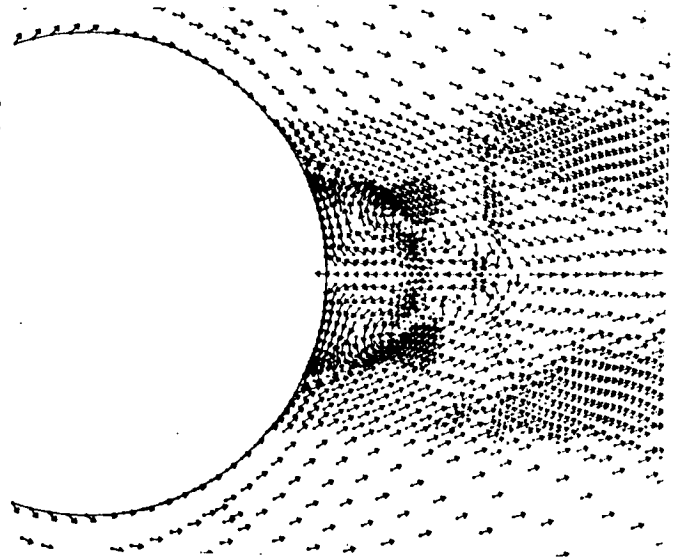
$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$

The bow shock is closer to the wall when the gas is rarefied ,while the viscous boundary layer shrinks up to almost disappearing ,except around the detachment point.

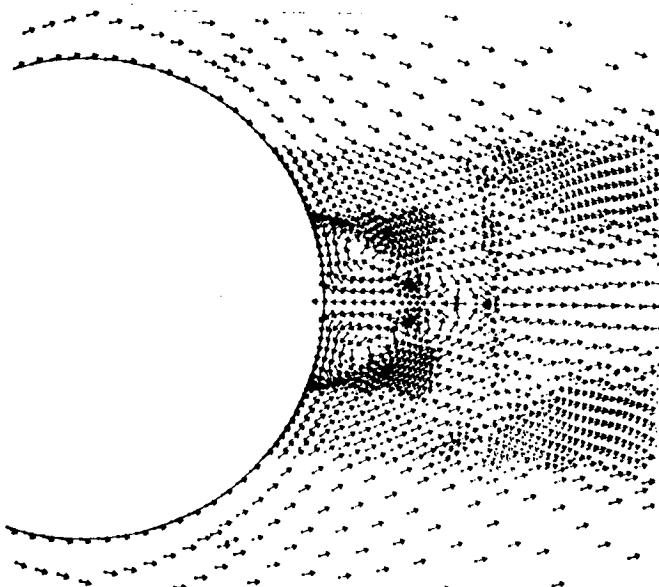
Speed vectors directions behind the cylinder



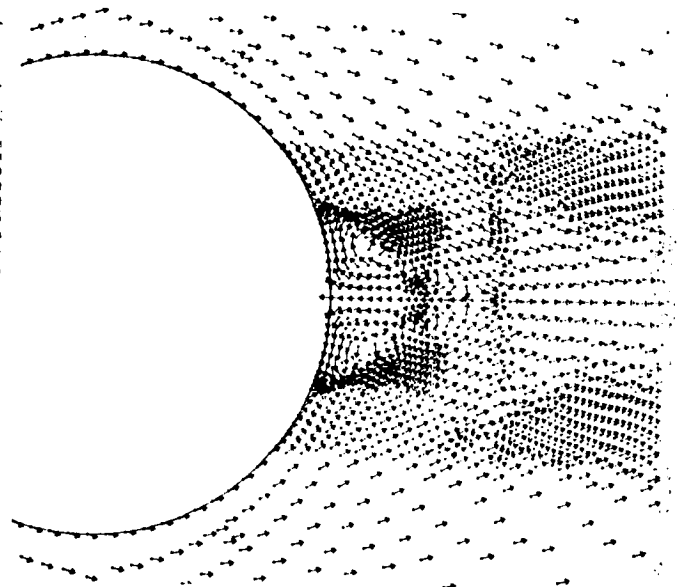
No slip



$\rho_{\infty} = 10^{-4} \text{ kg/m}^3$



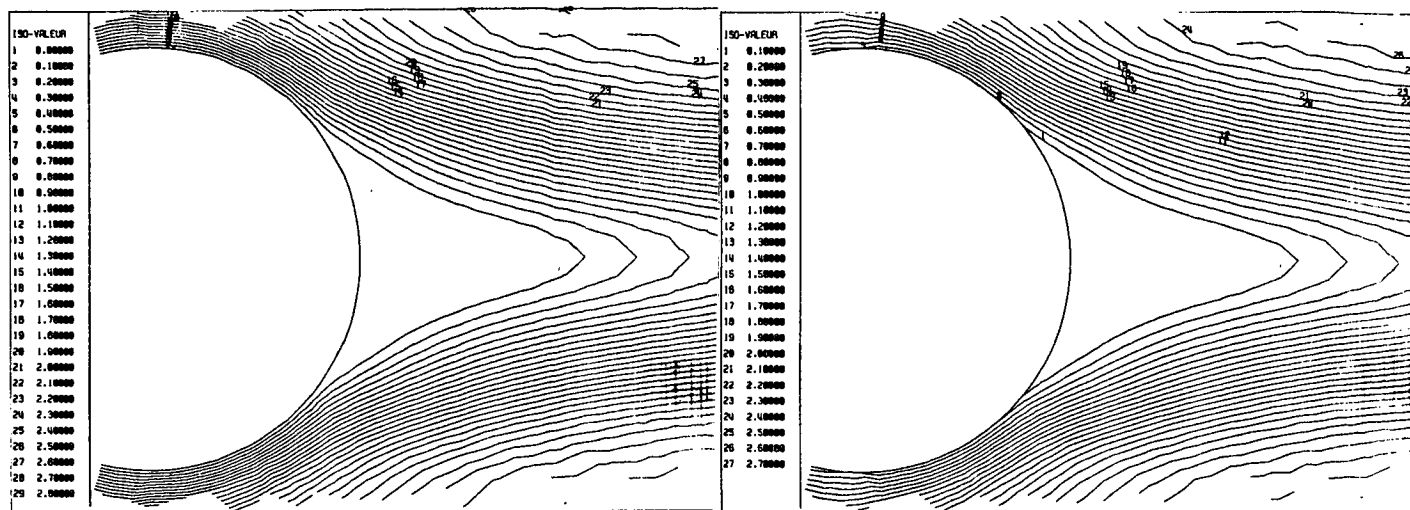
$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$



$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$

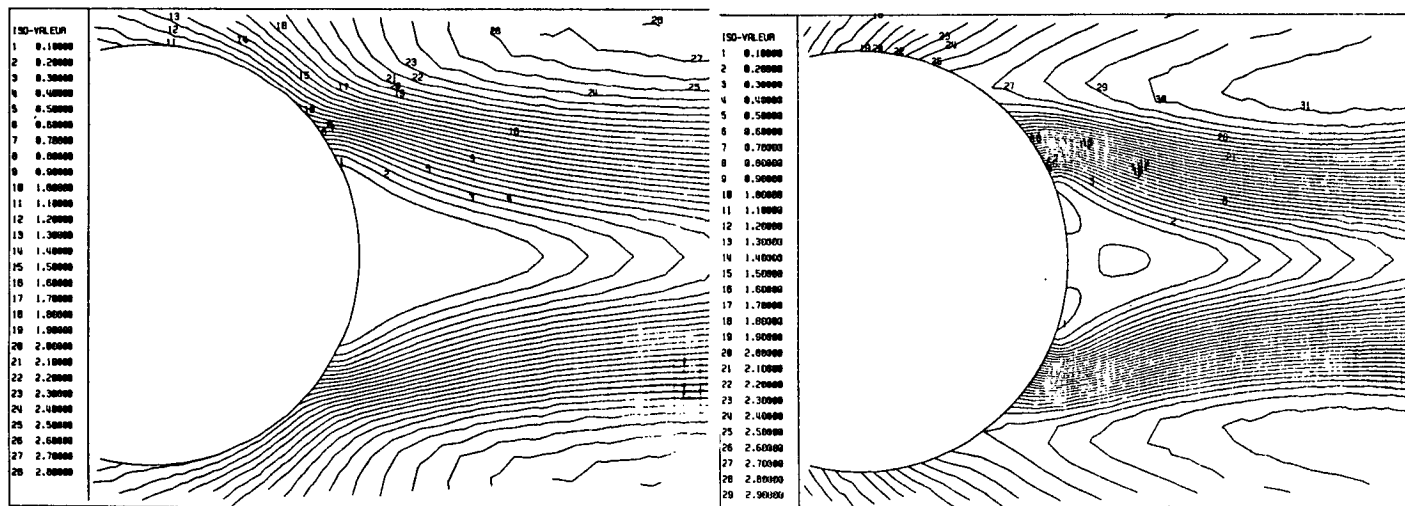
The recirculation zone between the wall and the stagnation point is smaller for rarefied flows ,moving with decreasing densities to the perfect gas solution.

Enlargement of the iso-Mach lines



No slip

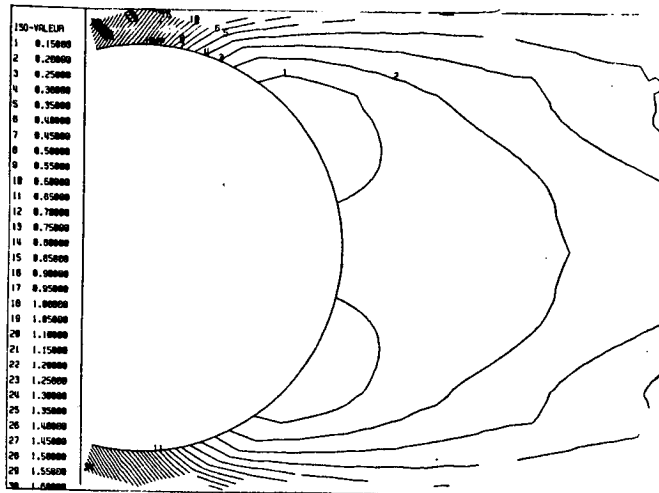
$$\rho_{\infty} = 10^{-4} \text{ kg/m}^3$$



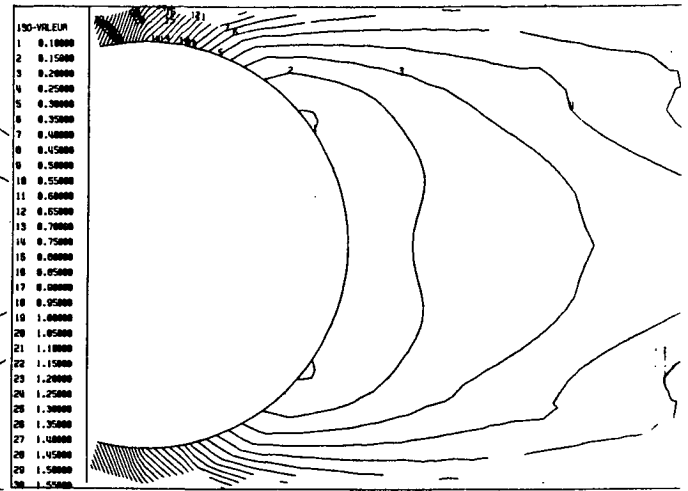
$$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$$

$$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$$

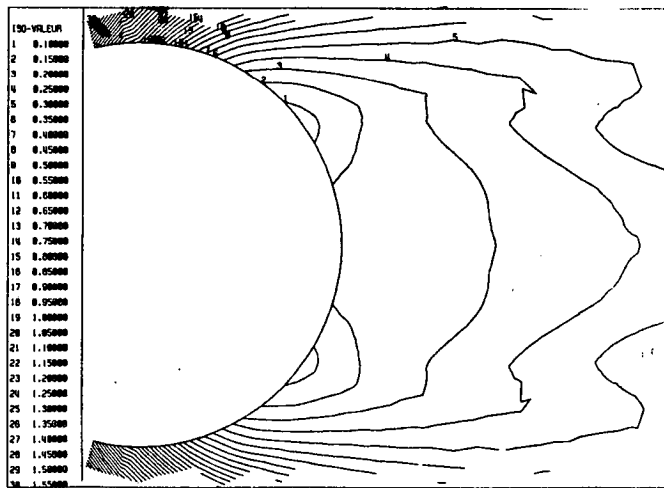
Enlargement of iso-density lines



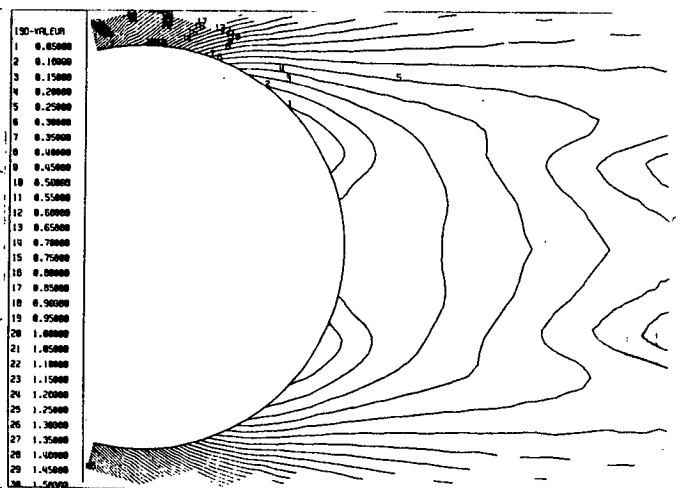
No slip



$\rho_{\infty} = 10^{-4} \text{ kg/m}^3$

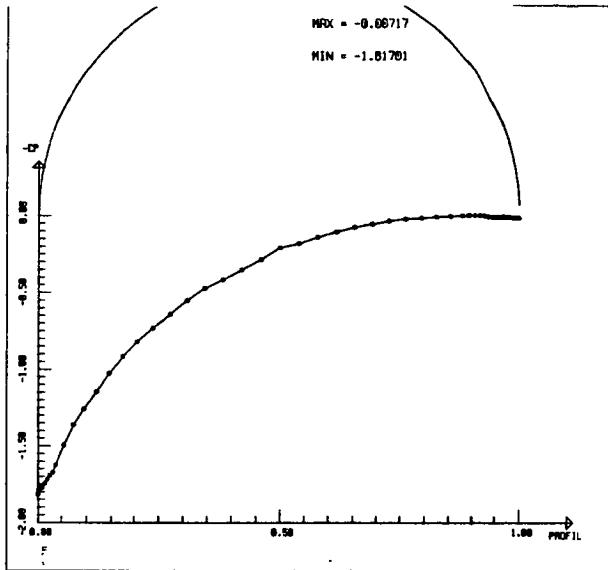


$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$

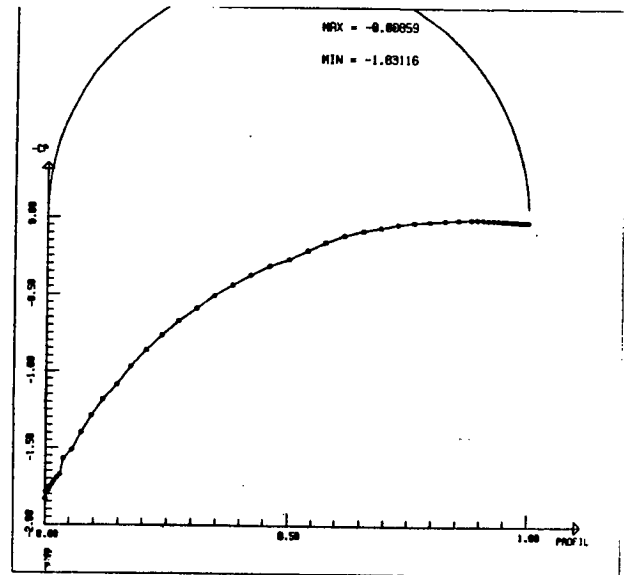


$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$

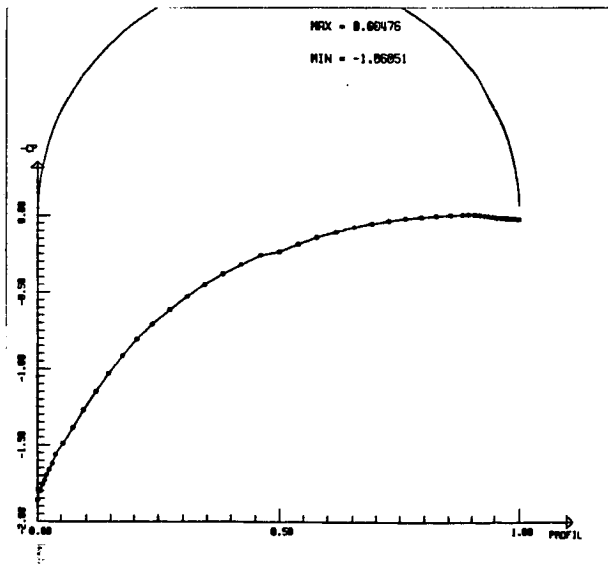
Pressure coefficient on the body



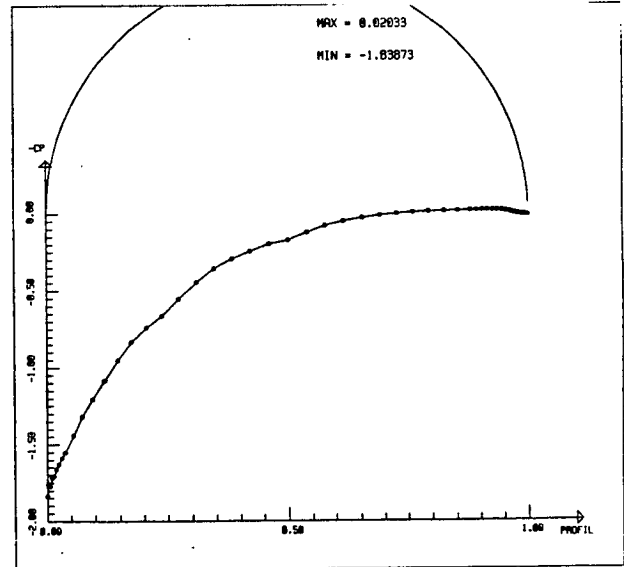
No slip



$\rho_{\infty} = 10^{-4} \text{ kg/m}^3$



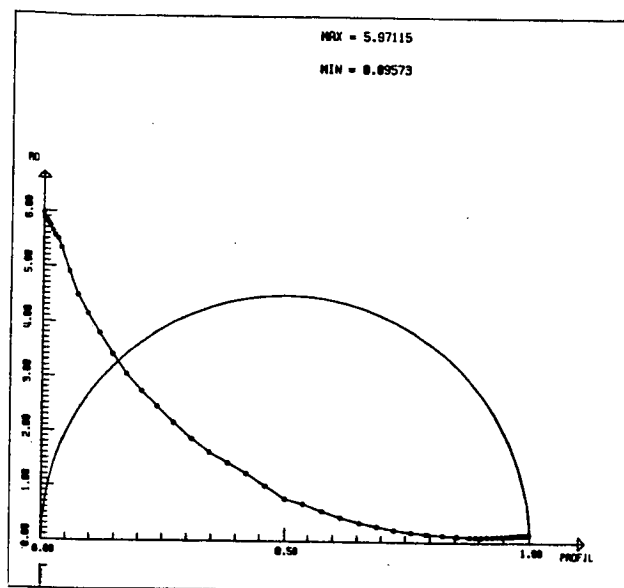
$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$



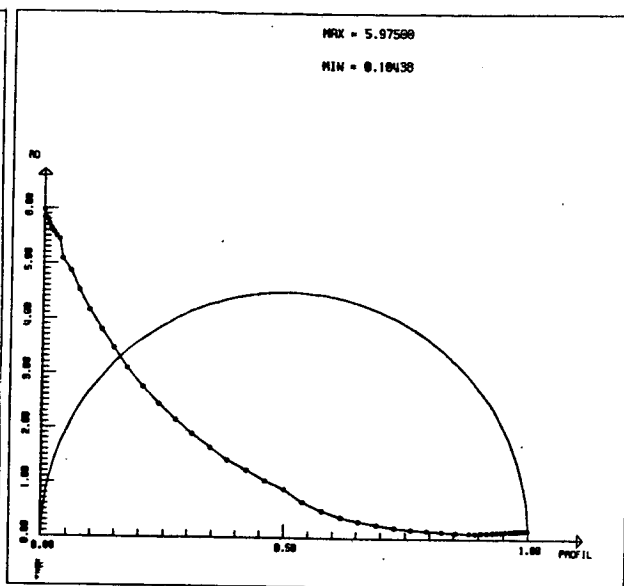
$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$

The pressure on the body doesn't much depend on the boundary condition, it is also very close to the perfect gas pressure.

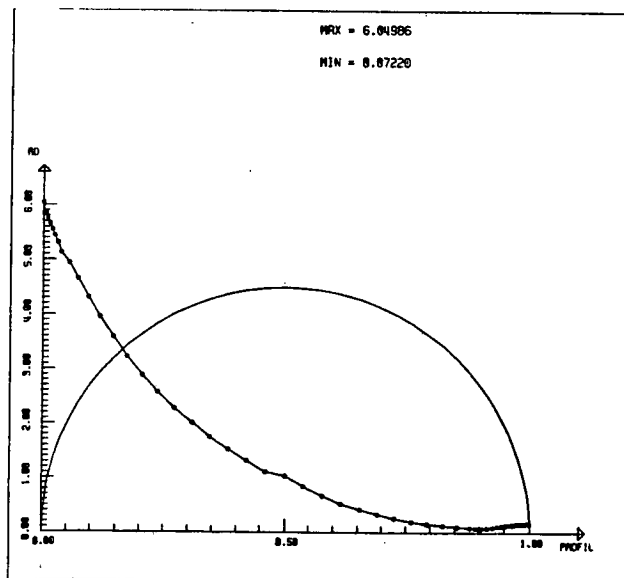
Density on the body ($\frac{\rho}{\rho_\infty}$)



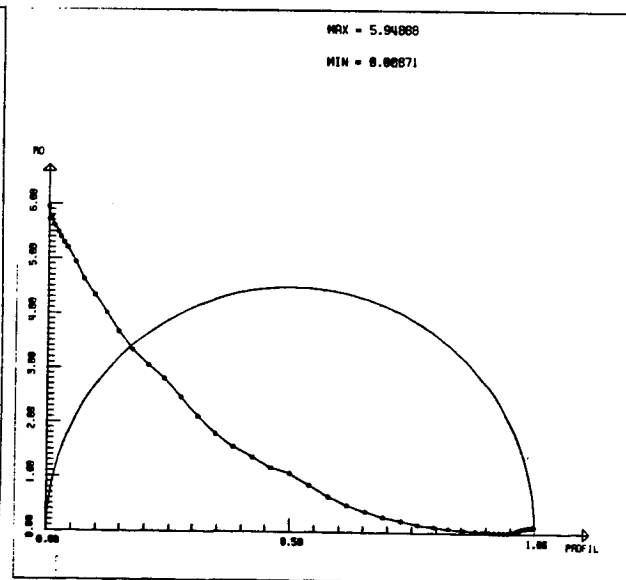
No slip



$\rho_\infty = 10^{-4} \text{ kg/m}^3$

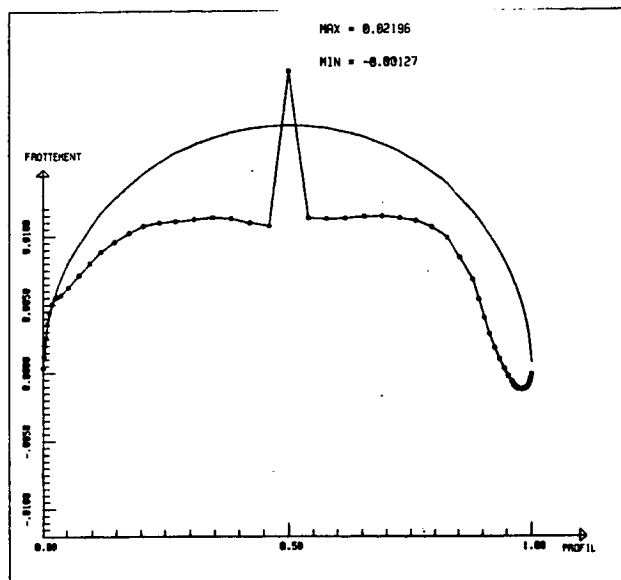


$\rho_\infty = 10^{-5} \text{ kg/m}^3$

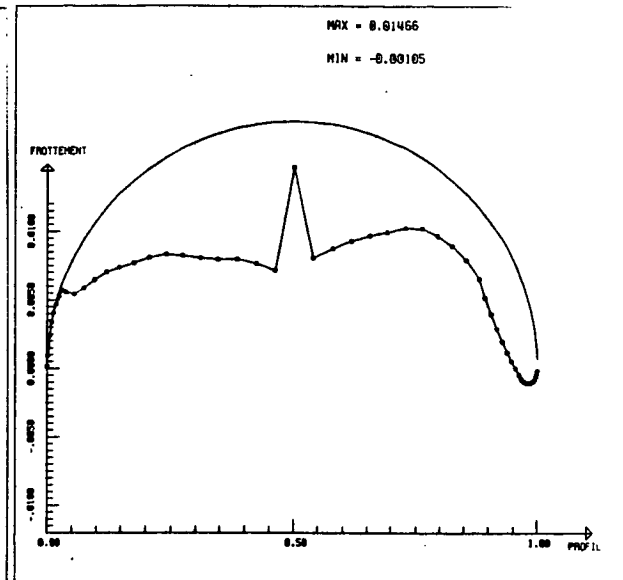


$\rho_\infty = 10^{-6} \text{ kg/m}^3$

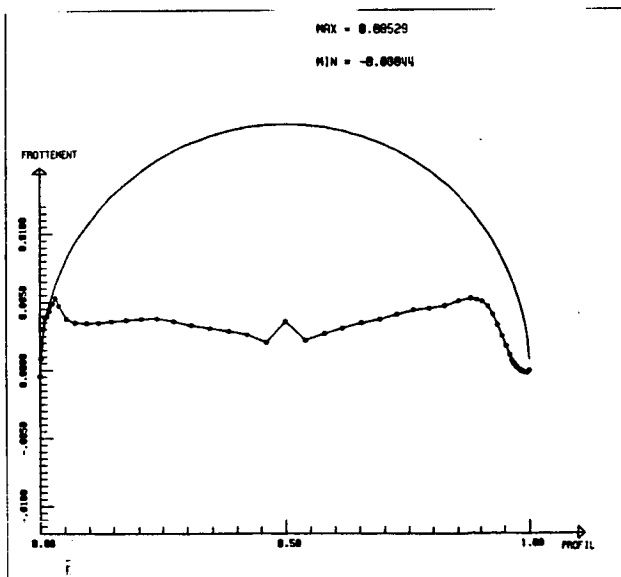
Skin friction coefficient



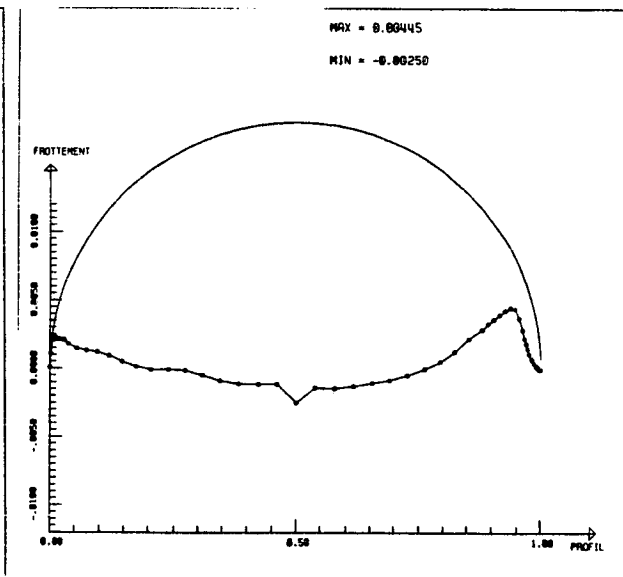
No slip



$\rho_{\infty} = 10^{-4} \text{ kg/m}^3$



$\rho_{\infty} = 10^{-5} \text{ kg/m}^3$



$\rho_{\infty} = 10^{-6} \text{ kg/m}^3$

CONCLUSION :

Significative differences appear between slip and no-slip simulations, the former seeming to be closer to experimental results. Nevertheless, more realistic boundary layer models, together with temperature dependant viscosities, will be necessary to compute accurately low Reynolds hypersonic flows.

ACKNOWLEDMENT : This work owes much to F.Angrand,M.O.Bristeau,O.Pironneau of INRIA, and to B.Mantel,J.Periaux and B.Stoufflet of AMD-BA ,whom I want to thank here.

REFERENCES :

- [1] J. Allègre, M. Raffin, L. Gottesdiener : Slip effects around Naca 0012 airfoils, SESSIA, Meudon, France, 1985.
- [2] R. Gupta, C. Scott, J. Moss : Slip boundary equations for multicomponent nonequilibrium airflow, NASA Technical paper 2452, (1985).
- [3] F. Coron, F. Golse, C. Sulem : A classification of well-posed kinetic layer problems, part I and II, preprint.
- [4] R. Brun : Transport et relaxation dans les écoulements gazeux, Masson (Paris), 1986.
- [5] S. Chapman, T. Cowling : The mathematical theory of non uniform gases, Cambridge University Press, 1952.
- [6] J.O.Hirschfelder,C.F.Curtiss,R.B.Bird : Molecular theory of gases and liquids ,John Wiley son inc. ,1967.
- [7] G.A. Bird : Molecular gas dynamics, Oxford University Press, 1976.
- [8] C. Bardos : Equations cinétiques et approximations fluides, Cours de DEA, Université Paris VI, 1986-87.
- [9] S. Mas-Gallic : A deterministic particle method for the Boltzman equation, to appear in "Transport theory and Statistical Physics".
- [10] C. Cercignani : Theory and applications of the Boltzman equation, Scottish Academic Press, (1975).
- [11] A. Valli : An existence theorem for compressible viscous fluids ,Ann. Math. pura ed applicata (IV),vol. 130,197-213 (1982).
- [12] R. Verfurth, Thesis, Ruhr Universitat, Bochum, June 1986.
- [13] J.A. Nitsche : On Korn's second inequality, RAIRO, Anal. Numer. 15, 237-248 (1981).
- [14] M.O. Bristeau, J. Périaux : Finite element methods for the calculation of compressible viscous flows using self adaptative mesh refinements, Lecture Notes on Computational Fluid Dynamics, March 3-7, 1986, Von Karman Institute for Fluid Dynamics.
- [15] F. Angrand : Viscous perturbation for the compressible Euler equations ,application to the numerical simulation of compressible viscous flows, GAMM, Nice, 1984.

[16] F. Angrand, A. Dervieux : Some explicit finite element schemes for the Euler equations, International Journal for Numerical Methods in Fluids, Vol. 4, 749-764, (1984).

[17] B. Stoufflet : Résolution numérique des équations d'Euler des fluides parfaits compressibles par des schémas implicites en éléments finis , Thesis, Université Paris VI, 1984.

[18] B. Stoufflet ,J.Periaux ,L.Fezzoui ,A.Dervieux : Numerical Simulations of 3-D Hypersonic Euler Flows Around Space Vehicles Using Adapted Finite Elements ,AIAA paper 87 0560 Reno 1987.

Imprimé en France
par
l'Institut National de Recherche en Informatique et en Automatique

