Optimal control of semilinear multistate systems with state constraints

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CONTROLE OPTIMAL DE SYSTEMES MULTI-ETATS SEMILINEAIRES AVEC DES CONTRAINTES SUR L'ETAT

Résumé. Cet article traite de problèmes de contrôle optimal avec contraintes sur l'état gouvernées par une équation sémilinéaire multi-états. Nous montrons l'existence d'une solution et obtenons les conditions d'optimalité.

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OPTIMAL CONTROL OF SEMILINEAR MULTISTATE SYSTEMS WITH STATE CONSTRAINTS

JOSEPH FREDERIC BONNANS[\$] AND EDUARDO CASAS[&]

Abstract. This paper deals with state constrained optimal control problems governed by a semilinear multistate equation. We prove the existence of solutions and derive optimality conditions.

Key words. Optimal control, subdifferential calculus, optimality conditions, elliptic operators, semilinear equations, multistate systems.

AMS(MOS) Subject classification. 49B22, 49A22.

1. Introduction. This paper is concerned with state constrained optimal control problems governed by a semilinear elliptic operator. As we make no monotonicity assumption, the state equation may be unsolvable or may have several solutions. Our aim is to obtain existence results and to derive the optimality system.

There exists a vast litterature on the control of well-posed state-constrained systems. The subdifferential calculus of convex analysis is a useful tool when dealing with linear state equations : see Mackenroth [16] [17], Bonnans and Casas [7], and Casas [8] [9]. In the nonlinear case, Bonnans and Casas [4] [5] [6] derived the optimality system using results of Clarke [10].

The control of non-monotone elliptic systems, but without state constraints, has been studied by Lions [15] (see also Komornik [14]). The optimality system is derived there by penalizing the state equation and passing to the limit in the optimality conditions of the penalized problem.

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The novelty of this paper lies in the simultaneous presence of state constraints and of an ill-posed system. Our method consists in approximating the problem by removing the nonlinearity from the state equation and penalizing a part of the state constraints. We formulate the problem and get an existence result in section 2, derive the optimality system in section 3 and study several examples in section 4.

2. Formulation of the control problem. Let Ω be an open bounded subset of $\mathbb{R}^n$ ($n \leq 3$) with C^2 boundary Γ. Let us consider the system :

$$(2.1)\qquad \begin{cases} Ay + \phi(y) = u \text{ in } \Omega, \\ y = 0 \text{ on } \Gamma, \end{cases}$$

where

$$Ay = - \sum_{i,j=1}^{n} \partial_{x_j} (a_{ij}(x)\partial_{x_i} y) + a_0(x)y,$$

and

$$a_0 \in L^\infty(\Omega),\ a_0(x) \geq 0 \text{ a.e. } x \in \Omega,$$

$$(2.2)\qquad \begin{cases} a_{ij} \text{ is Lipschitz on } \bar{\Omega}\ (1 \leq i,j \leq n), \\ \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i\, \xi_j \geq \alpha_0 \|\xi\|^2,\ \alpha_0>0,\ \forall \xi \in R^n,\ \forall x \in \Omega, \end{cases}$$

$$(2.3)\qquad \phi : R \to R \text{ is } C^1.$$

Let K be a non-empty, convex, closed subset of $L^2(\Omega)$, $\sigma \geq 2$ and y_d in $L^\sigma(\Omega)$ be given, and let $J : L^\sigma(\Omega) \times L^2(\Omega) \to \mathbb{R}$ be the functional

$$(2.4)\qquad J(y,u) = \frac{1}{\sigma}\int_\Omega |y(x) - y_d(x)|^\sigma\, dx + \frac{N}{2}\int_\Omega u^2(x)dx.$$

Let Z be a Banach space, B a closed convex subset of Z with non-empty interior, and let a be given in $\mathbb{R}^m$ ($m \geq 0$; we identify $\mathbb{R}^0$ with $\{0\}$). Define $Y = H^2(\Omega) \cap H^1_0(\Omega)$, where $H^s(\Omega)$ and $H^s_0(\Omega)$ are the usual Sobolev spaces (see Adams [1], Nečas [18]). Let $C_0(\Omega)$ be the space of real continuous functions on $\bar{\Omega}$ vanishing on Γ, endowed with the supremum norm $\|\ \|_\infty$. It is known that Y is compactly embedded in $C_0(\Omega)$ for $n \leq 3$. The dual of $C_0(\Omega)$ is the space $M(\Omega)$ of real and regular Borel measures on Ω, endowed with the norm

$$\|\mu\|_{M(\Omega)} = |\mu|(\Omega),$$

where $|\mu|$ is the total variation measure of μ (Rudin [19]). Finally, let $T : C_0(\Omega) \to \mathbb{R}^m$ and $L : C_0(\Omega) \to Z$ be linear continuous mappings. In order to derive the optimality conditions, we will suppose that

(2.5) $\qquad T(Y) = \mathbb{R}^m$ and $\overline{L(Y)} = Z$.

We consider the following control problem :

(P) $\qquad \min J(y,u)$ s.t. (2.1), $u \in K$, $y \in Y$, $Ty = a$, $Ly \in B$.

Remark 1 : The assumptions on Ω and A imply (Nečas, [18]) that for each f in $L^2(\Omega)$ there exists a unique solution $y \in Y$ of the Dirichlet problem

$$Ay = f \text{ in } \Omega,$$

$$y = 0 \text{ on } \Gamma,$$

and moreover there exists C_1 independant of f such that

(2.6) $\qquad \|y\|_{H^2(\Omega)} \leq C_1\|f\|_{L^2(\Omega)}.$

In fact all our results still hold if we just assume that Ω is bounded, Y is compactly embedded in $C_0(\Omega)$ and (2.6) holds. This is the case, for instance, if A is symmetric and satisfies (2.2) and Ω is bounded and convex (Grisvard, [13]).

Remark 2 : The existence of several states associated to the same control has been obtained e.g. with cubic nonlinearities [11]. Anyway, the inclusion of Y in $C_0(\Omega)$ for $n \leq 3$ (Adams [1]) implies that $A + \phi$ maps Y into $L^2(\Omega)$: hence all elements of Y are associated to a control. For parabolic systems the situation is essentially different (Bonnans [3]).

Let us now give some examples of control problems which fall in the previous formulation.

(P1) $\qquad \min J(y,u)$ s.t. (2.1), $u \in K$, $y \in Y$, $y(x_i) = a_i$, $1 \leq i \leq m$.

Here $\{x_i\}$ are given in Ω and we may take $B = Z = C_0(\Omega)$, L is the identity in $C_0(\Omega)$, and $Ty = \{y(x_i)\}$.

(P2) $\min J(y,u)$ s.t. (2.1), $u \in K$, $y \in Y$, $\int_\Omega |y(x)|\,dx \le \delta$,

with $\delta > 0$. Here $m = 0, T = 0$, $Z = L^1(\Omega)$, B is the closed ball with center 0 and radius δ, and L is the canonical injection from $C_0(\Omega)$ into $L^1(\Omega)$.

$$\text{(P3)} \quad \begin{array}{l} \min J(y,u) \text{ s.t. } (2.1),\ u \in K,\ y \in Y,\ \int_\Omega y(x)\,dx = a, \\ |y(x)| \le \delta,\ \forall x \in \Omega, \end{array}$$

with $\delta > 0$. Here $m = 1$ and $Ty = \int_\Omega y(x)dx$, $Z = C_0(\Omega)$, B is the closed ball with radius δ and center 0 and L is the identity. These three examples obviously satisfy (2.5).

We now give a result about the existence of a solution to problem (P). We need for this a relation between σ and the nonmonotone part of ϕ.

Theorem 1 : Suppose that (2.2) and (2.3) hold and that

(i) there exists (y,u) satisfying the constraints of (P) (i.e., (P) is feasible),

(ii) either $N > 0$ or K is bounded in $L^2(\Omega)$,

(iii) We may write $\phi(t) = \phi_1(t) + \phi_2(t)$, with ϕ_i continuous, $i = 1,2$, $\phi_1(t)$ non decreasing, and such that for some $C > 0$:

$$|\phi_2(t)| \le C\,(1 + |t|^{\sigma/2}).$$

Then problem (P) has (at least) one solution. □

Proof : As (P) is feasible, there exists a minimizing sequence $\{(y_n, u_n)\}$ in $Y \times K$. Because of (ii), $\{u_n\}$ is bounded in $L^2(\Omega)$. We are going to prove that $\{Ay_n\}$ is bounded in $L^2(\Omega)$ and for this we may assume that ϕ_1 is differentiable. Otherwise, we would approximate ϕ_1 by a standard convolution technique and then pass to the limit.

The form of J implies that $\{y_n\}$ is bounded in $L^\sigma(\Omega)$; hence with (iii), $\phi_2(y_n)$ is bounded in $L^2(\Omega)$ and so is $f_n = -\phi_2(y_n) + u_n = Ay_n + \phi_1(y_n)$. As $\phi_1(y_n)$ is in $C_0(\Omega)$, Ay_n belongs to $L^2(\Omega)$. Computing the scalar product of f_n with Ay_n in $L^2(\Omega)$, and integrating by parts the nonlinear term, we obtain

$$\|Ay_n\|^2_{L^2(\Omega)} + \int_\Omega \phi_1'(y_n) \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial y_n}{\partial x_i} \frac{\partial y_n}{\partial x_j} dx \leq \|f_n\|_{L^2(\Omega)} \|Ay_n\|_{L^2(\Omega)}.$$

The second term of the left-hand side is non-negative because of (2.2) and the monotonicity of ϕ_1. Hence $\|Ay_n\|$ is bounded in $L^2(\Omega)$; with (2.6), this implies that $\{y_n\}$ is bounded in Y. As Y is compactly embedded in $C_0(\Omega)$ for $n \leq 3$, selecting a subsequence if necessary, we may assume that

$$y_n \to \bar{y} \text{ weakly in } Y, \text{ strongly in } C_0(\Omega),$$

$$Ay_n \to A\bar{y} \text{ weakly in } L^2(\Omega),$$

$$u_n \to \bar{u} \text{ weakly in } L^2(\Omega).$$

This implies $T\bar{y} = a$, $L\bar{y} \in B$ and $\phi(y_n) \to \phi(\bar{y})$ in $C_0(\Omega)$; hence Ay_n weakly converges in $L^2(\Omega)$ towards $\bar{u} - \phi(\bar{y})$; hence $(\bar{y}, \bar{u})$ satisfies (2.1). As K is closed and convex, hence weakly closed, $\bar{u}$ is in K. Finally, the convexity and continuity of J implies its weak lower semicontinuity ; the result follows. □

3. The optimality system. For any set C, denote by I_C its indicatrix, defined by

$$I_C(x) = \begin{cases} 0 \text{ if } x \in C, \\ +\infty \text{ if not.} \end{cases}$$

We denote the subdifferential of a convex function f by ∂f (see Barbu and Precupanu [2], Ekeland and Temam [12]). The spaces $W_0^{1,s}(\Omega)$ and $W^{1,s}(\Omega)$ are the usual Sobolev spaces (Adams [1]). We denote by T^* the adjoint operator of T and by $R(T^*)$ its range. The aim of this section is to prove the following result :

Theorem 2 : Let $(\bar{y}, \bar{u})$ be a solution of (P). We assume that (2.2)-(2.5) hold and that

(3.1) $$\partial(I_B \circ L)\,(\bar{y}) \cap R(T^*) = \{0\}.$$

Then there exists $\bar{p}$ in $W_0^{1,s}(\Omega)$ for all $s < n/(n-1)$, $\bar{\lambda}$ in $\mathbb{R}^m$, $\bar{\mu}$ in Z' and $\bar{\alpha} \geq 0$ such that

(3.2) $\quad \bar{\alpha} + \|\bar{p}\|_{W_0^{1,s}(\Omega)} > 0,$

(3.3) $\quad A^*\bar{p} + \phi'(\bar{y})\bar{p} = \bar{\alpha}\,|\bar{y} - y_d|^{\sigma-2}(\bar{y} - y_d) + T^*\bar{\lambda} + L^*\bar{\mu},$

(3.4) $\quad \langle \bar{\mu}, z - L\bar{y} \rangle \leq 0, \ \forall z \in B,$

(3.5) $\quad \int_\Omega(\bar{p} + \bar{\alpha} N \bar{u})(v - \bar{u})dx \geq 0, \ \forall v \in K.$

Remark 3 : As B has a non-empty interior, we deduce from (2.5) that $R(L) \cap \overset{o}{B} \neq \emptyset$. This implies (see Barbu-Precupanu [2], Ekeland-Temam [12]) that $\partial(I_B oL)(\bar{y}) = L^*\partial I_B(L\bar{y})$. □

Remark 4 : We will verify that hypothesis (3.1) holds in our three examples. However, if (3.1) does not hold, then by Remark 3 there exists $(\bar{\lambda}, \bar{\mu})$ in $\mathbb{R}^m \times \partial I_B(L\bar{y})$ such that $\|\bar{\lambda}\| + \|\bar{\mu}\| > 0$ and $T^*\bar{\lambda} + L^*\bar{\mu} = 0$. In other words, if all hypothesis of Theorem 2 are satisfied except perhaps (3.1), there exists $\bar{p}$, $\bar{\lambda}$, $\bar{\mu}$, $\bar{\alpha}$ as in Theorem 1, not all null, satisfying (3.3)-(3.5). □

In order to prove Theorem 2, we need to establish some preliminary results.

Lemma 1 : Let W be a Banach space and D be a convex subset of W (not necessarily closed) with non-empty interior. Let $\{(w_n, \eta_n)\}$ be a sequence in $W \times W'$ such that $w_n \in D$, $w_n \to w$ and $\eta_n \in \partial I_D(w_n)$. If $\liminf \|\eta_n\| > 0$, then 0 is not a weak star limit-point of $\{\eta_n\}$. □

Proof : Assume that the conclusion does not hold. Let w_o be given in $\overset{o}{D}$. There exists $r > 0$ such that $\|w\| \leq r$ implies that $w_o + w$ is in D ; hence

$$\langle \eta_n, w_o + w - w_n \rangle \leq 0,$$

and this implies

$$r\|\eta_n\| = \sup_{\|w\| \leq r} \langle \eta_n, w \rangle \leq \langle \eta_n, w_n - w_o \rangle.$$

The strong convergence of w_n allows to pass to the limit and we get

$$r \liminf \|n_n\| \leq 0,$$

which gives a contradiction. □

Lemma 2 : Let W be a Banach space, and f (resp.g) be a Gâteaux-differentiable (resp. convex) mapping from W into $\mathbb{R}$ (resp.$]-\infty, +\infty]$). Let $\bar{x}$ be a solution of the following problem :

$$\min f(x) + g(x), \ x \in W.$$

Then

$$< \nabla f(\bar{x}), x - \bar{x} > + g(x) - g(\bar{x}) \geq 0, \ \forall x \in W,$$

or, equivalently :

$$\nabla f(\bar{x}) + \partial g(\bar{x}) \ni 0. \ □$$

Proof : A straightforward application of the definition of the subdifferential [12] allows to verify the equivalence of the two statements of the conclusion. Now consider $x^t = \bar{x} + t\ (x - \bar{x})$ for t in $]0,1[$. We have, using the convexity of g : $f(x^t) + g(x^t) \leq f(x^t) + (1 - t)\ g(\bar{x}) + tg(x)$; hence, as $\bar{x}$ is a solution of the above problem :

$$0 \leq f(x^t) + g(x^t) - (f(\bar{x}) + g(\bar{x})) \leq f(x^t) - f(\bar{x}) + t\ (g(x) - g(\bar{x})).$$

Dividing by t and passing to the limit, we obtain the result. □

We now consider the following approximate problem. Let the state equation be

$$(3.6) \quad \begin{aligned} Ay &= u + w \text{ in } \Omega, \\ y &= 0 \text{ on } \Gamma, \end{aligned}$$

The control being now (u,w) in $L^2(\Omega) \times L^2(\Omega)$. We define

$$J_\varepsilon(y,u,w) = J\ (y,u) + \frac{1}{2\varepsilon} \int_\Omega (w + \phi(y))^2 dx +$$

$$+ \frac{1}{2\varepsilon} \|Ty - a\|^2 + \frac{1}{2} \int_\Omega (u - \bar{u})^2 dx + \frac{1}{2} \int_\Omega (w + \phi(\bar{y}))^2 dx.$$

The approximate problem is

(P_ε) $\quad$ $\min J_\varepsilon(y,u,w)$ s.t. (3.6), $u \in K$, $w \in L^2(\Omega)$, $y \in Y$, $Ly \in B$.

Theorem 3 : Let $(\bar{y}, \bar{u})$ be a solution of (P). We assume that (2.2) - (2.5) hold. Then

(i) problem (P_ε) has at least one solution.

(ii) to each solution $(y_\varepsilon, u_\varepsilon, w_\varepsilon)$ of (P_ε) is associated p_ε in $W_0^{1,s}(\Omega)$ for all $s < n/(n-1)$, $\mu_\varepsilon \in Z'$ and λ_ε in $\mathbb{R}^m$ such that

$$A^* p_\varepsilon = |y_\varepsilon - y_d|^{\sigma-2}(y_\varepsilon - y_d) + T^*\lambda_\varepsilon + L^*\mu_\varepsilon +$$

$$+ \frac{1}{\varepsilon} \phi'(y_\varepsilon)(w_\varepsilon + \phi(y_\varepsilon)),$$

$$p_\varepsilon = 0 \text{ on } \Gamma,$$

$$\langle \mu_\varepsilon, z - Ly_\varepsilon \rangle \leq 0, \ \forall z \in B,$$

$$\int_\Omega (p_\varepsilon + Nu_\varepsilon + u_\varepsilon - \bar{u})(v - u_\varepsilon) dx \geq 0, \ \forall v \in K,$$

$$p_\varepsilon + \frac{1}{\varepsilon}[w_\varepsilon + \phi(y_\varepsilon)] + w_\varepsilon + \phi(\bar{y}) = 0. \ \square$$

Proof : (i) The triple $(\bar{y}, \bar{u}, -\phi(\bar{y}))$ is feasible for (P_ε). Any minimizing sequence is bounded in $L^\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)$; hence by (3.6) in $Y \times L^2(\Omega) \times L^2(\Omega)$. Taking a subsequence if necessary and using the compactness of $Y \subset C_0(\Omega)$ $(n \leq 3)$ to pass to the limit in the nonlinear terms, we get the result as in the proof of Theorem 1.

(ii) Denote by $y_{u,w}$ the solution of (3.6) and by $\theta(u,w)$ the mapping $(u,w) \to J_\varepsilon(y_{u,w}, u, w)$. It is easy to verify that θ is C^1 and that

$$\theta'_u(u,w) = q + Nu + u - \bar{u},$$

$$\theta'_w(u,w) = q + \frac{1}{\varepsilon}(w + \phi(y_{u,w})) + w + \phi(\bar{y}),$$

where q is the solution of (A^* being the formal transpose of A) :

$$A^*q = |y_{u,w} - y_d|^{\sigma-2}(y_{u,w} - y_d) + \frac{1}{\varepsilon}\,\phi'(y_{u,w})\,(w + \phi(y_{u,w})) +$$

$$+ \frac{1}{\varepsilon}\,T^*(Ty_{u,w} - a) \text{ in } \Omega,$$

$$q \quad = 0 \text{ on } \Gamma.$$

Let $(y_\varepsilon, u_\varepsilon, w_\varepsilon)$ be a solution of (P_ε) and q_ε the associated adjoint-state. Let us define :

$$\hat{L} : L^2(\Omega) \times L^2(\Omega) \to Z,$$

$$(u,w) \qquad \to Ly_{u,w},$$

$$\hat{K} = K \times L^2(\Omega),$$

$$g(u,w) = I_B\,(\hat{L}(u,w)) + I_{\hat{K}}(u,w).$$

Problem (P_ε) is equivalent to

$$\min \theta(u,w) + g(u,w),\ (u,w) \in L^2(\Omega) \times L^2(\Omega).$$

Applying now Lemma 2, we get

$$\nabla\theta(u_\varepsilon,w_\varepsilon) + \partial g(u_\varepsilon,w_\varepsilon) \ni 0.$$

The mapping $w \to y_{u,w}$(with u fixed) is an isomorphism from $L^2(\Omega)$ onto Y. Hence by (2.1) there exists (u,w) in $\hat{K}$ with $\hat{L}(u,w)$ in $\overset{\circ}{B}$. This allows us ([12]) to apply the rules of subdifferential calculus to the mapping g and we get the equality

$$\partial g(u_\varepsilon,w_\varepsilon) = \hat{L}^*\ \partial I_B(Ly_\varepsilon) + \partial I_{\hat{K}}(u_\varepsilon,w_\varepsilon).$$

Hence there exists μ_ε in $\partial I_B(Ly_\varepsilon)$ such that

$$\nabla\theta(u_\varepsilon,w_\varepsilon) + \hat{L}^*\mu_\varepsilon + \partial I_{\hat{K}}(u_\varepsilon,w_\varepsilon) \ni 0,$$

or equivalently

$$(\theta'_u(u_\varepsilon,w_\varepsilon),\ u-u_\varepsilon) + (\theta'_w(u_\varepsilon,w_\varepsilon),\ w-w_\varepsilon) +$$

$$+ \langle \mu_\varepsilon,\ Ly_{u,w} - Ly_\varepsilon \rangle \geq 0,\ \forall (u,w) \in K\times L^2(\Omega),$$

Let r_ε be the solution of

$$\begin{cases} A^* r_\varepsilon = L^* \mu_\varepsilon \text{ in } \Omega, \\ r_\varepsilon = 0 \text{ on } \Gamma, \end{cases}$$

we get

$$(\theta'_u(u_\varepsilon,w_\varepsilon) + r_\varepsilon,\ u-u_\varepsilon) \geq 0,\ \forall u \in K,$$

and

$$\theta'_w(u_\varepsilon,w_\varepsilon) + r_\varepsilon = 0.$$

We obtain the result with $p_\varepsilon = q_\varepsilon + r_\varepsilon$ and $\lambda_\varepsilon = \frac{1}{\varepsilon}(Ty_\varepsilon - a)$. □

Lemma 3 : Let $\{(y_\varepsilon, u_\varepsilon, w_\varepsilon)\}$ be a sequence of solutions of (P_ε). Then

$$0 = \lim_{\varepsilon\to 0} \|y_\varepsilon - \bar{y}\|_Y = \lim_{\varepsilon\to 0} \|u_\varepsilon - \bar{u}\|_{L^2(\Omega)} = \lim_{\varepsilon\to 0} \|w_\varepsilon + \phi(\bar{y})\|_{L^2(\Omega)}.$$

Proof : From the inequality $J_\varepsilon(y_\varepsilon, u_\varepsilon, w_\varepsilon) \leq J_\varepsilon(\bar{y},\bar{u},-\phi(\bar{y})) = J(\bar{y},\bar{u})$ and the form of J, we deduce that $\{(y_\varepsilon, u_\varepsilon, w_\varepsilon)$ is bounded in $L^\sigma(\Omega) \times L^2(\Omega) \times L^2(\Omega)$; hence $\{y_\varepsilon\}$ is bounded in Y by (3.6) and (2.6). This implies that for $\varepsilon \in D$, D being a subset of $]0,\infty[$ having 0 as limit-point, we have for some (y,u,w) in $Y \times L^2(\Omega) \times L^2(\Omega)$ when $\varepsilon \to 0$:

$y_\varepsilon \to y$ in Y weak, $C_0(\Omega)$ strong,

$u_\varepsilon \to u$ in $L^2(\Omega)$ weak,

$w_\varepsilon \to w$ in $L^2(\Omega)$ weak,

with (y,u,w) satisfying (3.6). As K and B are closed and convex in $L^2(\Omega)$ and Z we have $u \in K$ and $Ly \in B$. The form of J_ε implies that $\|w_\varepsilon + \phi(y_\varepsilon)\|_{L^2(\Omega)} \to 0$ and $\|Ty_\varepsilon - a\| \to 0$; hence $w + \phi(y) = 0$; with (3.6) this implies that (y,u) satisfies (2.1). We have, as J is l.s.c. :

$$J(\bar{y},\bar{u}) \geq \limsup J_\varepsilon(y_\varepsilon,u_\varepsilon, w_\varepsilon)$$

$$\geq \limsup \{J(y_\varepsilon,u_\varepsilon) + \frac{1}{2}\|u_\varepsilon - \bar{u}\|^2_{L^2(\Omega)} + \frac{1}{2}\|w_\varepsilon + \phi(\bar{y})\|^2_{L^2(\Omega)}\}$$

$$\geq J(y,u) + \frac{1}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} + \frac{1}{2}\|w + \phi(\bar{y})\|^2_{L^2(\Omega)}.$$

As (y,u) is feasible for (P), this implies that $u = \bar{u}$ and $w + \phi(\bar{y}) = 0$; hence $\phi(y) = \phi(\bar{y})$. With (2.1) this implies that $y = \bar{y}$. But the above inequality also implies $\|u_\varepsilon - \bar{u}\|_{L^2(\Omega)} \to 0$ and $\|w_\varepsilon + \phi(\bar{y})\|_{L^2(\Omega)} \to 0$; using (2.6), the result follows. □

We now are in position to prove Theorem 2, by passing to the limit in the optimality system of (P_ε).

Proof of Theorem 2 : Let $(y_\varepsilon, u_\varepsilon, w_\varepsilon)$ denote a solution of (P_ε) and $(p_\varepsilon, \mu_\varepsilon, \lambda_\varepsilon)$ be given by Theorem 3. If $\{(p_\varepsilon, \mu_\varepsilon, \lambda_\varepsilon)\}$ is bounded we obtain the result with $\bar{\alpha} = 1$ by passing to the limit in the optimality system of (P_ε) with the help of Lemma 3. Suppose now that $\alpha_\varepsilon = 1/(\|p_\varepsilon\|_{L^2(\Omega)} + \|\mu_\varepsilon\|_{Z'} + \|\lambda_\varepsilon\|)$ converges towards 0. Multiplying by α_ε the optimality system given by Theorem 3 and defining

$$\bar{p}_\varepsilon = \alpha_\varepsilon p_\varepsilon,\ \bar{\mu}_\varepsilon = \alpha_\varepsilon \mu_\varepsilon,\ \bar{\lambda}_\varepsilon = \alpha_\varepsilon \lambda_\varepsilon,$$

we obtain, eliminating $\frac{1}{\varepsilon}(w_\varepsilon + \phi(y_\varepsilon))$ from the last equality of Theorem 3 :

$$(3.7)\quad \begin{cases} A^*\bar{p}_\varepsilon + \phi'(y_\varepsilon)\bar{p}_\varepsilon = \alpha_\varepsilon |y_\varepsilon - y_d|^{\sigma-2}(y_\varepsilon - y_d) + T^*\bar{\lambda}_\varepsilon + L^*\bar{\mu}_\varepsilon - \\ \qquad\qquad - \alpha_\varepsilon \phi'(y_\varepsilon)(w_\varepsilon + \phi(\bar{y})) \text{ in } \Omega, \\ \bar{p}_\varepsilon = 0 \text{ on } \Gamma, \\ \langle \bar{\mu}_\varepsilon, z - Ly_\varepsilon \rangle \leq 0,\ \forall z \in B, \\ \int_\Omega [\bar{p}_\varepsilon + \alpha_\varepsilon(Nu_\varepsilon + u_\varepsilon - \bar{u})](v - u_\varepsilon) \geq 0,\ \forall v \in K. \end{cases}$$

As $\|\bar{p}_\varepsilon\|_{L^2(\Omega)} + \|\bar{\mu}_\varepsilon\|_{Z'} + \|\bar{\lambda}_\varepsilon\|$ is bounded, using Lemma 3, we may pass to the limit in the above systems ; then we obtain (3.3)-(3.5), with here $\bar{\alpha} = 0$. It remains to prove that $\bar{p} \neq 0$. If $\bar{p} = 0$, then $T^*\bar{\lambda} + L^*\bar{\mu} = 0$ by (3.3). But (3.1) and the injectivity of T^* and L^*(by(2.5)) imply then that $\bar{\mu} = 0$ and $\bar{\lambda} = 0$. As $\{\bar{\lambda}_\varepsilon\}$ is in $\mathbb{R}^m$ and because of Lemma 1, this implies that $\liminf \|\bar{\mu}_\varepsilon\|_{Z'} = 0$ and $\|\bar{\lambda}_\varepsilon\| \to 0$; hence $\|\bar{p}_\varepsilon\|_{L^2(\Omega)} \to 1$. From (3.7) and Lemma 3 we deduce that $A^* \bar{p}_\varepsilon$ is bounded in $M(\Omega)$; hence $\{\bar{p}_\varepsilon\}$ is bounded in $W_0^{1,s}(\Omega)$ for all $s < n/(n-1)$. The compact injection from $W_0^{1,s}(\Omega)$ into $L^2(\Omega)$ (for $n \leq 3$ and s close to $n/(n-1)$) implies that $\|\bar{p}_\varepsilon\|_{L^2(\Omega)} \to \|\bar{p}\|_{L^2(\Omega)} = 0$, which gives a contradiction. □

4. Applications : In this section we are going to consider the three examples stated in section 2 and we will derive the optimality system for each of them.

EXAMPLE 1

THEOREM 4 : Let $(\bar{y},\bar{u}) \in Y \times K$ be a solution of (P1). Then there exist a real number $\bar{\alpha} \geq 0$ and elements $\bar{\lambda} \in R^m$ and $\bar{p} \in W_0^{1,s}(\Omega)$ for all $s < n/(n-1)$ satisfying

(4.1) $$\bar{\alpha} + \|\bar{p}\|_{W_0^{1,s}(\Omega)} > 0,$$

(4.2) $$\begin{cases} A\bar{y} + \phi(\bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases}$$

(4.3) $$\begin{cases} A^*\bar{p} + \phi'(\bar{y})\bar{p} = \bar{\alpha}|\bar{y} - y_d|^{\sigma-2}(\bar{y} - y_d) + \sum_{i=1}^m \bar{\lambda}_i \delta_{[x_i]} \text{ in } \Omega, \\ \bar{p} = 0 \text{ on } \Gamma, \end{cases}$$

(4.4) $$\int_\Omega (\bar{p} + \bar{\alpha}N\bar{u})(v - \bar{u})dx \geq 0 \ \forall v \in K.$$

Proof : Hypothesis 3.1 is trivially satisfied as $B = C_0(\Omega)$. Hence we may apply Theorem 2 which gives the result. □

In some cases it is possible to prove that the previous Theorem lies true with $\bar{\alpha} = 1$. We are going to study two situations where it is so.

THEOREM 5 : Let $a_{ij} \in C^2(\bar{\Omega})$, $1 \le i \le j \le n$. Then the results of Theorem 2 are obtained with $\bar{\alpha} = 1$ if Ω is connected and one of the two hypothesis holds :

i) There exists an open subset Ω_0 of Ω such that $K = K + \widetilde{L^2(\Omega_0)}$ ($\widetilde{L^2(\Omega_0)}$ is the extension by zero from $L^2(\Omega_0)$ to $L^2(\Omega)$),

ii) $K = \{v \in L^2(\Omega) : v(x) \ge 0$ a.e. $x \in \Omega\}$, and $u=0$ is not optimal for (P1).

Proof : i) If $\bar{\alpha} = 0$, it follows from (4.3) that

$$(4.5) \qquad \begin{cases} A^*\bar{p} + \phi'(\bar{y})\bar{p} = \sum_{i=1}^{m} \bar{\lambda}_i \delta_{[x_i]} & \text{in } \Omega, \\ \bar{p} = 0 & \text{on } \Gamma. \end{cases}$$

Now from (4.4) and the property of K, we get that $\bar{p}=0$ in Ω_0. Taking $\Omega_1 = \Omega \backslash \{x_i\}_{i=1}^m$, we have

$$(4.6) \qquad \begin{cases} A^* \bar{p} + \phi'(\bar{y})\, \bar{p} = 0 & \text{in } \Omega_1, \\ \bar{p} = 0 & \text{in } \Omega_0 \backslash \{x_i\}_{i=1}^m. \end{cases}$$

Then we can use the prolongation unicity Theorem (Saut and Scheurer [20])and we deduce that $\bar{p}=0$ in Ω_1, hence in Ω, which contradicts (4.1).

ii) If $\bar{\alpha} = 0$, we deduce from (4.4) that $\bar{p} \ge 0$ in Ω. If $\bar{p}$ is null on an open subset Ω_0 of Ω, we can do as in i) and we obtain a contradiction. Otherwise for each open subset Ω_0 with $\bar{\Omega}_0$ included in Ω_1 we have :

$$(4.7) \qquad \max_{x \in \Omega_0} \bar{p}(x) > 0.$$

We remark that $\bar{p}$ satifies

$$A^*\bar{p} + \max(0, \phi'(\bar{y}))\, \bar{p} \ge 0 \text{ in } \Omega_1,$$

$$\bar{p} = 0 \text{ on } \Gamma.$$

Applying the Harnack inequality to $A^* + \max(0, \phi'(\bar{y}))$ (Stampacchia [21]) as in [5] we deduce that $\bar{p}(x) > 0$ everywhere in Ω_1, wich implies with (4.4) that $\bar{u} = 0$ a.e. □

EXAMPLE 2

THEOREM 6 : If $(\bar{y},\bar{u}) \in Y \times K$ is solution of (P2), then there exists a real number $\bar{\alpha} \geq 0$ and elements $\bar{\mu} \in L^{\infty}(\Omega)$ and $\bar{p} \in W_0^{1,s}(\Omega)$ such that

(4.8) $$\bar{\alpha} + \|\bar{p}\|_{W_0^{1,s}(\Omega)} > 0,$$

(4.9) $$\begin{cases} A\bar{y} + \phi(\bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases}$$

(4.10) $$\begin{cases} A^*\bar{p} + \phi'(\bar{y})\bar{p} = \bar{\alpha}\,|\bar{y} - y_d|^{\sigma-2}(\bar{y} - y_d) + \bar{\mu} \text{ in } \Omega, \\ \bar{p} = 0 \text{ on } \Gamma, \end{cases}$$

(4.11) $$\int_{\Omega}\bar{\mu}(z - \bar{y})dx \leq 0 \ \forall z \in B,$$

(4.12) $$\int_{\Omega}(\bar{p} + \bar{\alpha}\, N\bar{u})\,(v - \bar{u})\,dx \geq 0 \ \forall v \in K.$$

Proof : Here again, (3.1) is satisfied because T=0. Hence we may apply Theorem 2 and remark that $Z' = L^{\infty}(\Omega)$ and L^* is the canonical injection into $M(\Omega)$. Moreover the regularity of $\bar{p}$ follows from (2.6), (4.10) and the fact of that $\bar{\alpha}|\bar{y} - y_d|^{\sigma-2}(\bar{y} - y_d) + \bar{\mu} - \phi'(\bar{y})\bar{p}$ belongs to $L^2(\Omega)$. □

EXAMPLE 3

THEOREM 7 : If $(\bar{y},\bar{u}) \in Y \times K$ is solution of (P3) then there exist a real number $\bar{\alpha} \geq 0$ and elements $\bar{p} \in W_0^{1,s}(\Omega)$ for all $s < n/(n-1)$, $\bar{\lambda} \in \mathbb{R}$ and $\bar{\mu} \in M(\Omega)$ such that

(4.13) $$\bar{\alpha} + \|\bar{p}\|_{W_0^{1,s}(\Omega)} > 0,$$

(4.14) $$\begin{cases} A\bar{y} + \phi(\bar{y}) = \bar{u} \text{ in } \Omega, \\ \bar{y} = 0 \text{ on } \Gamma, \end{cases}$$

(4.15) $$\begin{cases} A^*\bar{p} + \phi'(\bar{y})\,\bar{p} = \bar{\alpha}|\bar{y} - y_d|^{\sigma-2}(\bar{y} - y_d) + \bar{\lambda} + \bar{\mu} \text{ in } \Omega, \\ \bar{p} = 0 \text{ on } \Gamma, \end{cases}$$

(4.16) $$\int_\Omega (z - \bar{y})\, d\bar{\mu} \leq 0 \;\; \forall z \in B,$$

(4.17) $$\int_\Omega (\bar{p} + \bar{\alpha} N\bar{u})\,(v - \bar{u})dx \geq 0 \;\; \forall v \in K.$$

Proof : We have to verify that (3.1) is satisfied. For it remember that in this case L is the identity in $C_0(\Omega)$ and $T \in C_0(\Omega)'$. Take $\mu \in \partial I_B(\bar{y})$ and $\lambda \in R$ such that

$$< \mu, z > = < T^*\lambda, z > = \lambda \int_\Omega z dx, \;\; \forall z \in C_0(\Omega) \;;$$

this implies that $\mu = \lambda m$, where m is the Lebesgue measure. If $\lambda \neq 0$ this implies that $y(x) = \pm\, \delta$ a.e., which contradicts the boundary condition. □

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