Third-order finite element schemes for the solution of hyperbolic problems

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RESOLUTION D'EQUATIONS HYPERBOLIQUES PAR DES SCHEMAS DU TROISIEME ORDRE EN ELEMENTS FINIS

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Abstract

In this paper, we introduce a two-step version of the third-order accurate LW-TG scheme of J. Donea. The new scheme has properties similar to those of its one-step counterpart but is easier to be generalized to multidimensional equations and to nonlinear problems. The two schemes are presented and analyzed for a linear equation in one and two dimensions and the extension of the schemes to systems of nonlinear equations is shortly discussed.

Résumé

Dans ce rapport, on présente une version prédicteur-correcteur du schéma du troisième ordre LW-TG de J. Donea. Ce nouveau schéma à des propriétés analogues a ceux du schéma à un seul pas mais semble être plus facilement généralisable pour des systèmes d' équations et pour des problèmes non linéaires ou multidimensionnels. Les deux schémas sont présentés et analysés pour des équations linéaires à une ou deux dimensions spatiales et on discute brièvement la généralisation de ces schémas pour la résolution de systèmes d' équations non linéaires.

Contents

I. Introduction

In this report, we present a finite element third-order two-step scheme for the solution of hyperbolic problems. It belongs to the class of the Taylor-Galerkin methods introduced recently by J.Donea[1]. In these methods, the differential equation is first discretized in time by means of a Taylor series expansion in which the time derivatives of the unknown are expressed in terms of the governing equation and its derivatives. The generalized time-discretized equation thus obtained is subsequently discretized in space by means of the standard Galerkin finite element method[2]. In this way, we can obtain parameter-free second- or higher-order schemes which possess very low dissipation and dispersion errors.

The two-step version of the original third-order Taylor-Galerkin (named here LW-TG) scheme has similar properties (same phase response) than its one-step counterpart. Moreover, the two-step scheme seems easier to be generalized to systems of equations and to nonlinear or multidimensional problems. In particular, the two-dimensional scheme has a larger domain of numerical stability than the single step method.

In order to compare the numerical properties of the schemes, the modified equation method of Warming and Hyett[3] will be employed. The modified equation is the actual partial differential equation which is solved numerically, apart from round-off errors, when a given finite difference scheme is applied to solve an initial value problem. The terms appearing in this equation which are not in the original partial differential equation represent the truncation error of the numerical scheme. The main advantage of this approach is that the error terms provide immediate informations about the dissipation and dispersion properties of the numerical scheme. In fact, the even and odd derivative terms are found to be associated respectively to amplitude and phase errors. Also a necessary condition for numerical stability can be easily obtained by inspecting the sign of the coefficient of the lowest-order even derivative term appearing in the modified equation. The well-known von Neumann method[4] must, however, be employed to have a complete stability analysis.

An outline of the report follows. In section II, the LW-TG and LW-TG(2-s) schemes are presented and analyzed for the case of a scalar equation in one dimension. In section III, the previous analysis is extended to the case of the two-dimensional equation. Finally, in section IV, we shortly discuss the extension of the schemes to systems of nonlinear equations.

II. One-dimensional equation

As a simple example for illustrating the proposed schemes, we consider the scalar convection equation in one dimension

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \tag{1}$$

where a is a constant. This equation modelize the propagation phenomenon and its solution is the translation of an initial profile with the velocity a.

1. The LW-TG scheme

The time dicretization of Eq.(1) is obtained by considering the following Taylor series expansion in the time step Δt

$$u^{n+1} = u^n + \Delta t\frac{\partial u^n}{\partial t} + \frac{1}{2}\Delta t^2\frac{\partial^2 u^n}{\partial t^2} + \frac{1}{6}\Delta t^3\frac{\partial^3 u^n}{\partial t^3} + O(\Delta t^4) \tag{2}$$

where the superscript n is the time level, i.e., $t^n = n\Delta t$.
Now, successive differentiations of Eq.(1) indicate that

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} \quad ; \quad \frac{\partial^3 u}{\partial t^3} = a^2\frac{\partial^2}{\partial x^2}(\frac{\partial u}{\partial t}) \tag{3}$$

Using Eqs.(1) and (3), the semi-discrete equation (2) may be replaced by the following equation

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{a^2\Delta t^2}{6}\frac{\partial^2}{\partial x^2}(\frac{\partial u^n}{\partial t}) = -a\frac{\partial u^n}{\partial x} + a^2\frac{\Delta t}{2}\frac{\partial^2 u^n}{\partial x^2} \tag{4}$$

The third-order derivative term has been expressed in a mixed spatial-temporal form to allow the use of finite elements with C^0 continuity for the spatial discretization. This mixed form leads to a simple modification of the usual consistent mass matrix. In fact, by substituting

$$\frac{\partial u^n}{\partial t} = \frac{u^{n+1} - u^n}{\Delta t} + O(\Delta t)$$

into Eq.(4) a third-order accurate generalization of the Lax-Wendroff time differencing[5] is obtained

$$[1-\frac{a^2\Delta t^2}{6}\frac{\partial^2}{\partial x^2}]\frac{u^{n+1}-u^n}{\Delta t}=-a\frac{\partial u^n}{\partial x}+a^2\frac{\Delta t}{2}\frac{\partial^2 u^n}{\partial x^2} \tag{5}$$

To obtain a fully discrete equation we apply the Galerkin formulation to Eq.(5) with local approximation of the form

$$u(x,t)\simeq U(x,t)=\sum_j N_j(x)U_j(t)$$

where $N_j(x)$ belongs to a space of interpolation functions and $\{U_j\}$ is the vector of nodal values. Denoting by $< u,v >$ the L_2-inner product $\int uvdx$ over the domain of the problem, the Galerkin equations take the form

$$< [1-\frac{a^2\Delta t^2}{6}\frac{\partial^2}{\partial x^2}](U^{n+1}-U^n)+a\Delta t\frac{\partial U^n}{\partial x}-a^2\frac{\Delta t^2}{2}\frac{\partial^2 U^n}{\partial x^2}\ ,\ N_i >=0 \tag{6}$$

for all i, where $U^n = U(x,n\Delta t)$. In the case of piecewise linear functions on an uniform mesh of size h, Eq.(6), after integrating by parts the second-order terms, may be written in terms of the nodal parameters U_j^n as

$$[1+\frac{1}{6}(1-\nu^2)\partial^2](U_j^{n+1}-U_j^n)=-\nu\Delta_o U_j^n+\frac{\nu^2}{2}\partial^2 U_j^n \tag{7}$$

where $\nu=\dfrac{a\Delta t}{h}$ is the Courant number, $\Delta_o U_j=\dfrac{1}{2}(U_{j+1}-U_{j-1})$ and $\partial^2 U_j = U_{j+1}-2U_j+U_{j-1}$.

The modified equation associated to scheme (7) takes the form

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=-\frac{a^2h^2\Delta t}{24}(1-\nu^2)\frac{\partial^4 u}{\partial x^4}+\frac{ah^4}{180}(1-5\nu^2+4\nu^4)\frac{\partial^5 u}{\partial x^5} \tag{8}$$

It indicates that the scheme is third-order accurate in both Δt and h on an uniform mesh. The dispersion error is of the fourth order and a necessary condition for numerical stability is $|\nu|\leq 1$.

The analysis of the properties of the LW-TG can be completed using the von Neuman method. We sustitute a Fourier mode e^{ikx} into Eq.(7) and setting $\xi = kh$ a non dimensional wave number, we find an amplification in one time step of :

$$G_1(\nu,\xi)=1+[1-\frac{2}{3}(1-\nu^2)sin^2\frac{\xi}{2}]^{-1}[-i\nu sin\xi-2\nu^2 sin^2\frac{\xi}{2}] \tag{9}$$

to be compared with $e^{-i\nu\xi}$ for the differential equation. The stability condition ($|G|\leq 1$) reads $|\nu|\leq 1$ in accordance with the result obtained by the modified equation method. In fact, if $|\nu|>1$, $|G|>1$ for all ξ. Moreover, the scheme possesses the so called unit CFL property, i.e. signals are propagated without distorsion when the characteristics pass through the nodes. Finally, it should be noted that this scheme is identical to the Petrov-Galerkin scheme EPG(II) of Morton and Parrott[6].

2. The two-step LW-TG scheme (LW-TG(2-s))

Using a two-step strategy, we can obtain a form of the third-order scheme which does not involve the generalized mass matrix appearing in Eq.(7). In fact, the Taylor series expansion (2) can also be obtained combining the two following expressions

$$\begin{aligned} \tilde{u}^n &= u^n + \frac{\Delta t}{3}\frac{\partial u^n}{\partial t} + \frac{\Delta t^2}{9}\frac{\partial^2 u^n}{\partial t^2} \\ u^{n+1} &= u^n + \Delta t\frac{\partial u^n}{\partial t} + \frac{\Delta t^2}{2}\frac{\partial^2 \tilde{u}^n}{\partial t^2} \end{aligned} \tag{10}$$

where the term $\frac{\Delta t^2}{9}\frac{\partial^2 u^n}{\partial t^2}$ in the first step appears only to obtain a scheme with the same phase response as the LW-TG scheme.
The spatial discretization by means of the weak Galerkin formulation and of linear finite elements provides the following two-step scheme :

$$\begin{aligned} [1+\frac{1}{6}\partial^2](\tilde{U}_j^n - U_j^n) &= -\frac{\nu}{3}\Delta_o U_j^n + \frac{\nu^2}{9}\partial^2 U_j^n \\ [1+\frac{1}{6}\partial^2](U_j^{n+1} - U_j^n) &= -\nu\Delta_o U_j^n + \frac{\nu^2}{2}\partial^2 \tilde{U}_j^n \end{aligned} \tag{11}$$

The modified equation associated to Eq.(11) is

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{a^2h^2\Delta t}{24}(1-\frac{\nu^2}{3})\frac{\partial^4 u}{\partial x^4} + \frac{ah^4}{180}(1-5\nu^2+4\nu^4)\frac{\partial^5 u}{\partial x^5} \tag{12}$$

The third order accuracy is preserved by the two-step procedure. A necessary condition of numerical stability is then $|\nu| \leq \sqrt{3}$ and the dispersion error is of fourth-order accuracy. Moreover, the LW-TG and LW-TG(2-s) have the same phase response. In fact, the amplification factor of scheme (11) :

$$G_2(\nu,\xi) = 1 + [1 - \frac{2}{3}sin^2\frac{\xi}{2}]^{-1}[-i\nu sin\xi - 2\nu^2 sin^2\frac{\xi}{2}\ G_3(\nu,\xi)] \tag{13}$$

where

$$G_3(\nu,\xi) = 1 + [1 - \frac{2}{3}sin^2\frac{\xi}{2}]^{-1}[-i\frac{\nu}{3}sin\xi - \frac{2\nu^2}{9}sin^2\frac{\xi}{2}]$$

can be written as the product of the amplification factor (9) by the number

$$\frac{[1-\frac{2}{3}(1-\nu^2)sin^2\frac{\xi}{2}][1-\frac{2}{3}(1+\nu^2)sin^2\frac{\xi}{2}]}{[1-\frac{2}{3}sin^2\frac{\xi}{2}]}$$

which, since it is real, does not modify the phase of the amplification factor (9). The stability condition reads $|\nu| \leq \frac{\sqrt{3}}{2}$. This reduction of the interval of stability, with respect to the one predicted by the modified equation method, is due to the high frequencies and especially to the π mode. Finally, we note that the unit CFL property is lost.

3. Numerical results

To illustrate and compare the performances of the LW-TG schemes discussed so far, we consider the advection problem over the spatial interval $[0,1]$ defined by the initial and boundary conditions :

$$u(x,0) = \begin{cases} \frac{1}{2}\{1 + cos[\pi \frac{x - x_0}{\sigma}]\} & \text{if } |x - x_0| \leq \sigma \\ 0 & \text{if } |x - x_0| > \sigma \end{cases}$$

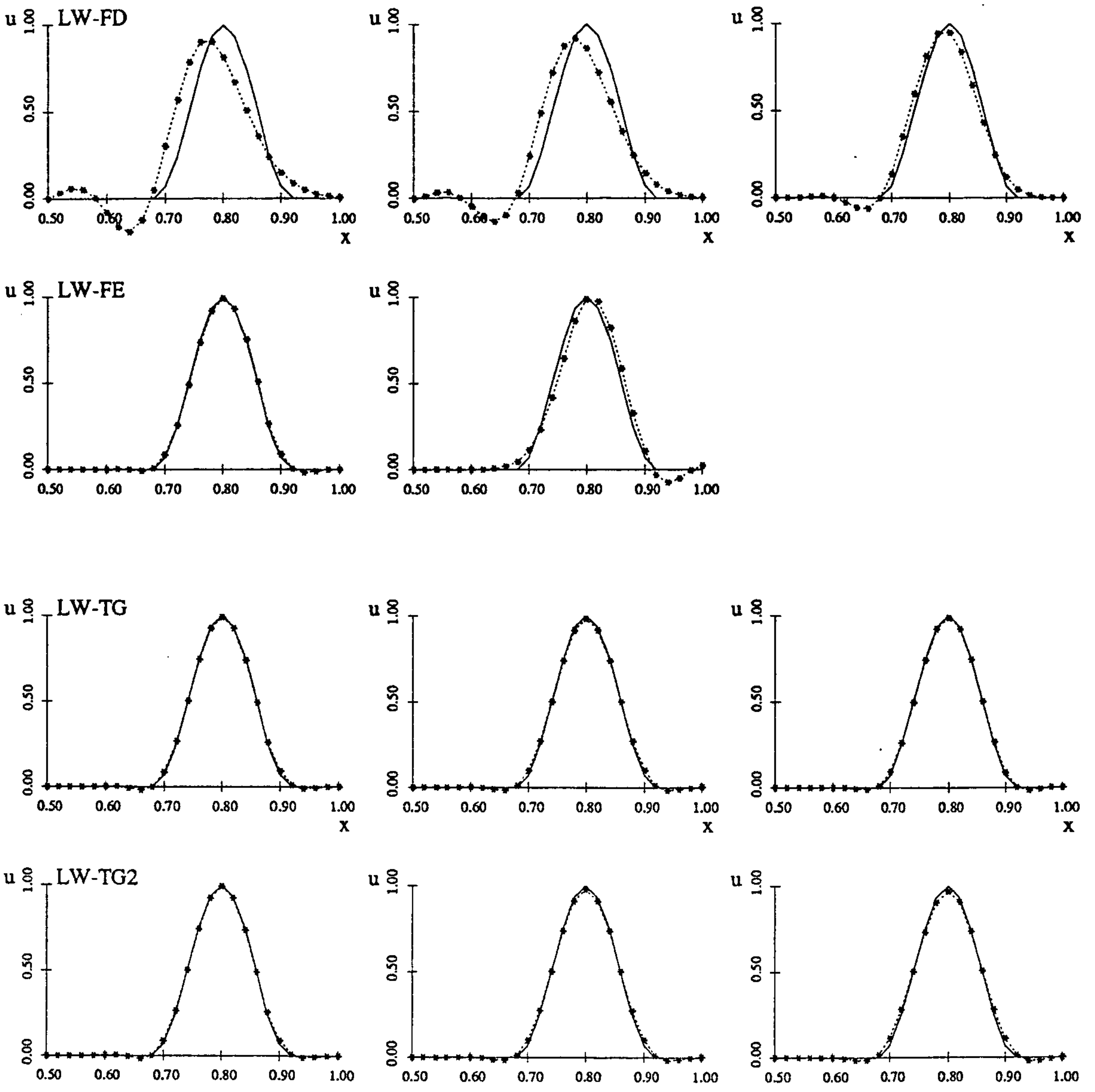
$$u(0,t) = 0 \quad ; \quad t \geq 0$$

with $x_0 = 0.2$ and $\sigma = 0.12$. The exact solution of Eq.(1) with $a = 1$ corresponds to the translation to the right of the initial profile with a unit velocity. In Fig. 1, we compare the numerical solutions, obtained using a uniform mesh of 50 elements and different values of the Courant number, with the exact solution at $t = 0.6$. The different schemes are the finite difference and finite element Lax-Wendroff schemes [7] which are second-order accurate and the LW-TG schemes. Furthermore, Table 1 provides the corresponding L^2-error defined by $err = \|U - u\|/\|u\|$, where $\|u\|^2 = \sum_k [u(x_k)]^2$.

The greater accuracy of the schemes of finite element type is clearly seen just as the superiority of the third-order schemes which have a rather uniform behaviour over the entire interval of numerical stability.

	LW-FD	LW-FE	LW-TG	LW-TG(2-s)
$\nu = 0.200$	0.1237	0.0005	0.0005	0.0005
$\nu = 0.500$	0.0733	0.0133	0.0011	0.0015
$\nu = 0.833$	0.0149	*unstable*	0.0015	0.0024

Table 1 : Propagation of a cosine profile. L^2-errors.



$\nu = 0.200$ $\nu = 0.500$ $\nu = 0.833$

Fig.1: Propagation of a cosine profile.

III. Two-dimensional equations

In order to study the behaviour of the LW-TG schemes for multidimensional problems, we consider the advection equation in two dimensions :

$$\frac{\partial u}{\partial t} + \vec{a} \cdot \vec{\nabla} u = 0 \tag{14}$$

where $\vec{a} = (a_x, a_y)$ is a constant velocity vector and $\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$. If one assumes an initial condition of the form :

$$u(\vec{x}, 0) = u_0 \; e^{i \, \vec{k} \cdot \vec{x}} \tag{15}$$

where $\vec{k} = (k_x, k_y)$ is the wave number vector, the exact solution to Eq.(14) is

$$u(\vec{x}, 0) = u_0 \; e^{i(\vec{k} \cdot \vec{x} + \omega t)} \tag{16}$$

where the exact phase velocity ω is defined simply by $\omega(\vec{a}, \vec{k}) = -\vec{a} \cdot \vec{k}$.

The spatial discretization of Eqs.(7) and (11) is obtained through the use of bilinear isoparametric elements on an uniform rectangular mesh with sizes h_x and h_y in the two directions. We can define for each scheme an amplification factor

$$G = G(\vec{p}, \vec{\nu})$$

which depends on the components of two non dimensional vectors :
the (non dimensional) wave number $\vec{p} = (p_x, p_y) = (k_x h_x, k_y h_y)$ and
the 'Courant number' vector $\vec{\nu} = (\nu_x, \nu_y) = (\frac{a_x \Delta t}{h_x}, \frac{a_y \Delta t}{h_y})$.
For more details about the expressions of this amplification factors see [7].

1. Numerical stability

In Fig. 2, the domain of numerical stability in the plane (ν_x, ν_y) is depicted for the Lax-Wendroff type schemes(LW-FD, LW-FE, LW-TG and LW-TG(2-s)).

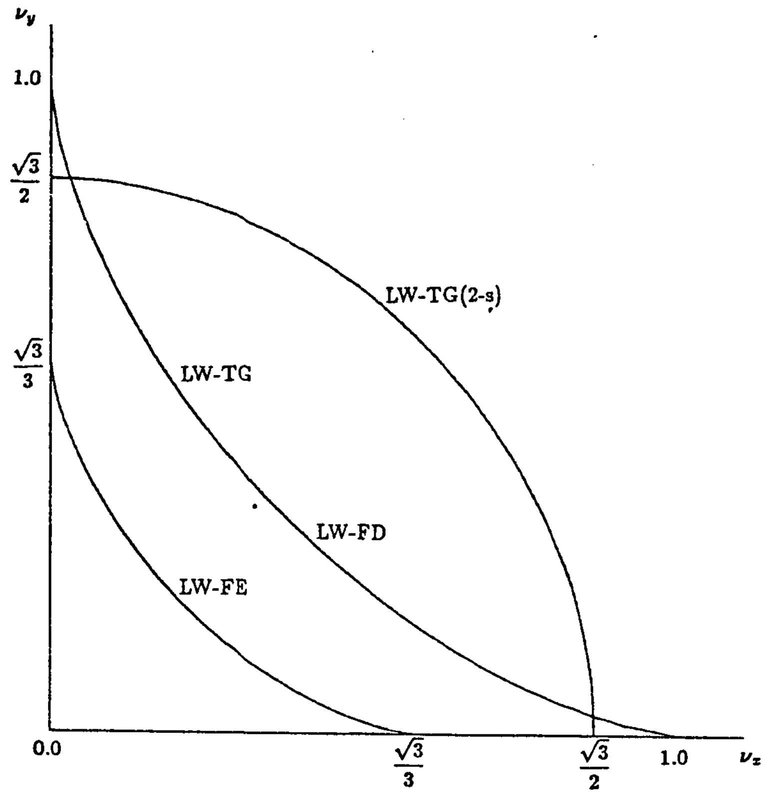


Fig.2: Domains of numerical stability of the Lax-Wendroff schemes.

Contrary to the one dimensional case, the LW-TG(2-s) scheme has a greater stability domain than the LW-TG scheme. We also note that the biggest admissible Δt barely depends on the orientation of the vector $\overrightarrow{\nu}$. In fact, we can approximate the equation of the limit curve by the following circle equation :

$$\nu_x^2 + \nu_y^2 = (\frac{\sqrt{3}}{2})^2$$

There is just a slight flattening in the neighbourhood of the direction $\nu_x = \nu_y$; for this direction, the value of the polar radius is approximatively 0.854 instead of 0.866. On the contrary, for the other schemes, the biggest admissible timestep depends strongly on the orientation of the vector $\overrightarrow{\nu}$. The limit curves for these schemes are very close to the curves of equation :

$$(\nu_x)^{\frac{2}{3}} + (\nu_y)^{\frac{2}{3}} = (\frac{1}{\sqrt{3}})^{\frac{2}{3}} \quad (LW - FE)$$
$$(\nu_x)^{\frac{2}{3}} + (\nu_y)^{\frac{2}{3}} = 1 \quad (LW - FD \ ; \ LW - TG)$$

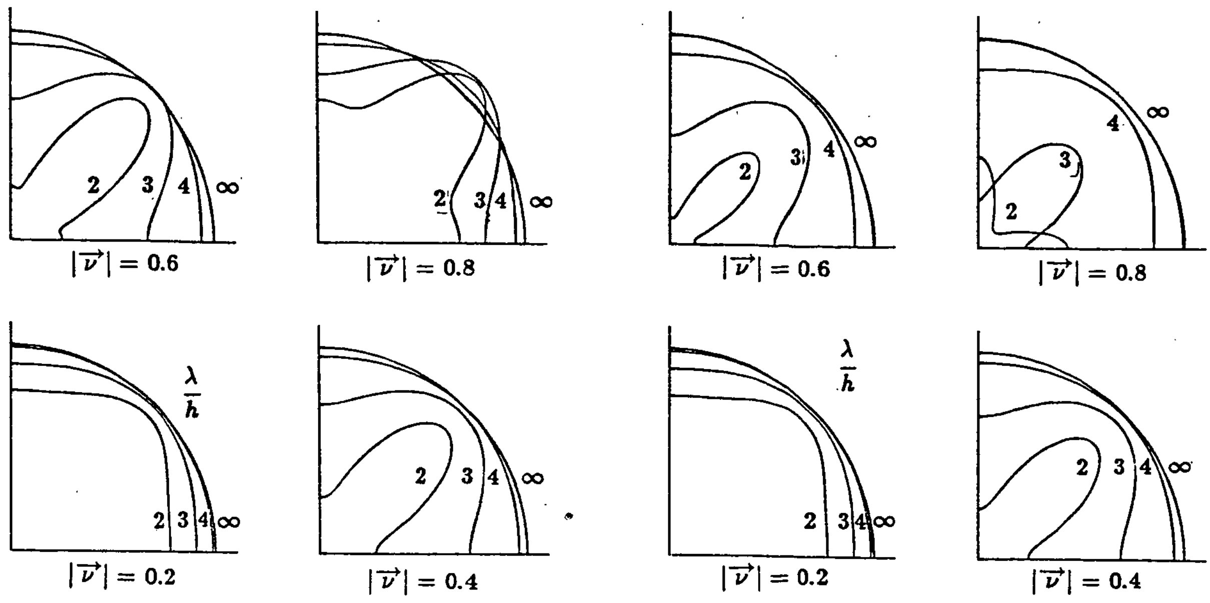


Fig.3: Amplification error of the LW-TG schemes(LW-TG left and LW-TG(2-s) right
The quoted numbers indicates the value of the dimensionless wavelength λ/h.

The value $\frac{1}{\sqrt{3}}$ of the similarity ratio between the two curves is verified numerically.

2. Amplification and phase responses

The numerical anisotropy that we have observed in the stability analysis is also present in the amplification and phase responses. To evaluate the properties of the schemes, the velocity vector $\vec{a}$ is first expressed as

$$\vec{a} = |\vec{a}|(cos\alpha, sin\alpha)$$

where α is the advection direction and the responses of the schemes is then analyzed by considering

$$\vec{p} = |\vec{p}|(cos\alpha, sin\alpha) \quad .$$

In the case of a square mesh $h_x = h_y = h$ and we can define the wavelength by

$$\lambda = \frac{2\pi}{|\vec{k}|} = \frac{2\pi h}{|\vec{p}|} \quad .$$

The properties of the LW-TG schemes are compared in Figs. 3 and 4 which provide a polar representation of the modulus of the amplification factor $\rho(|\vec{\nu}|, |\vec{p}|, \alpha)$ and of the relative phase error

$$\phi(|\vec{\nu}|, |\vec{p}|, \alpha) = \frac{\omega_{num}(|\vec{\nu}|, |\vec{p}|, \alpha)}{\omega(|\vec{\nu}|, |\vec{p}|, \alpha)}$$

for different values of λ/h.
We note that the schemes are very accurate for medium and long wavelengths and relatively anisotropic only for very short wavelengths.

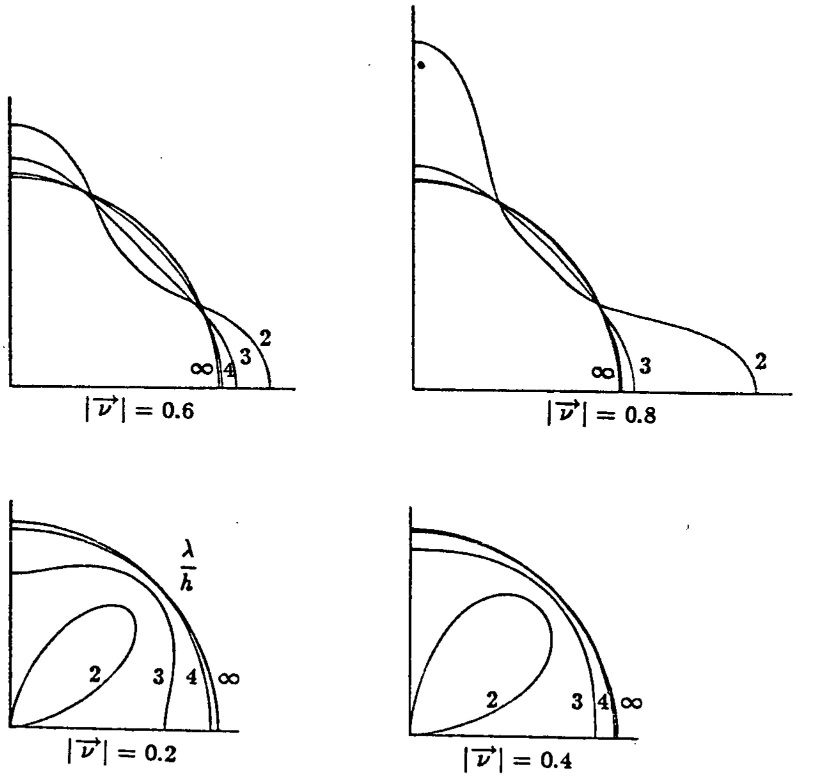


Fig.4: Phase velocity error of the LW-TG scheme

3. Numerical experiments

To compare the two-dimensional LW-TG schemes considered so far, the advection of a product-cosine hill in a pure rotation velocity field is considered. The initial condition is

$$u(x,y,0) = \begin{cases} \frac{1}{4}[1+cos\pi X][1+cos\pi Y] & \text{if } X^2+Y^2 \le 1 \\ 0 & \text{if } X^2+Y^2 > 1 \end{cases}$$

where $\overrightarrow{X} = (X,Y) = (\overrightarrow{x} - \overrightarrow{x}_0)/\sigma$, $\overrightarrow{x}_0$ and σ being the initial position of the center and the radius of the cosine hill, respectively. The advection field is a pure rotation with a unit angular velocity, namely $\overrightarrow{a} = (-y,x)$, so that a non constant coefficient linear equation is solved. A uniform mesh of 30×30 quadrilateral elements over the unit square $[-\frac{1}{2},\frac{1}{2}] \times [-\frac{1}{2},\frac{1}{2}]$ has been employed in the calculations. The numerical solutions for the case where $\overrightarrow{x}_0 = (\frac{1}{6},\frac{1}{6})$ and $\sigma = 0.2$ are shown in Figs. 6, 7 and 8 after a complete revolution. The maximum and minimum values of the computed solutions are also provided together with the corresponding L^2-error in Table 2.

$\Delta t = \frac{2\pi}{200}$	LW-FD	LW-FE	LW-TG	LW-TG2
u_{max}	0.852	0.987	0.988	0.988
u_{min}	−0.167	−0.016	−0.022	−0.022
err	0.2463	0.0021	0.0016	0.0018

$\Delta t = \frac{2\pi}{120}$	LW-FD	LW-FE	LW-TG	LW-TG2
u_{max}	0.826		0.978	0.975
u_{min}	−0.162	*unstable*	−0.020	−0.021
err	0.2628		0.0020	0.0024

$\Delta t = \frac{2\pi}{90}$	LW-FD	LW-FE	LW-TG	LW-TG2
u_{max}				0.966
u_{min}	*unstable*	*unstable*	*unstable*	−0.020
err				0.0032

Table 2

The greater accuracy of the schemes of finite element type is clearly seen. Admittedly, they are computationally more expensive than the explicit FD scheme

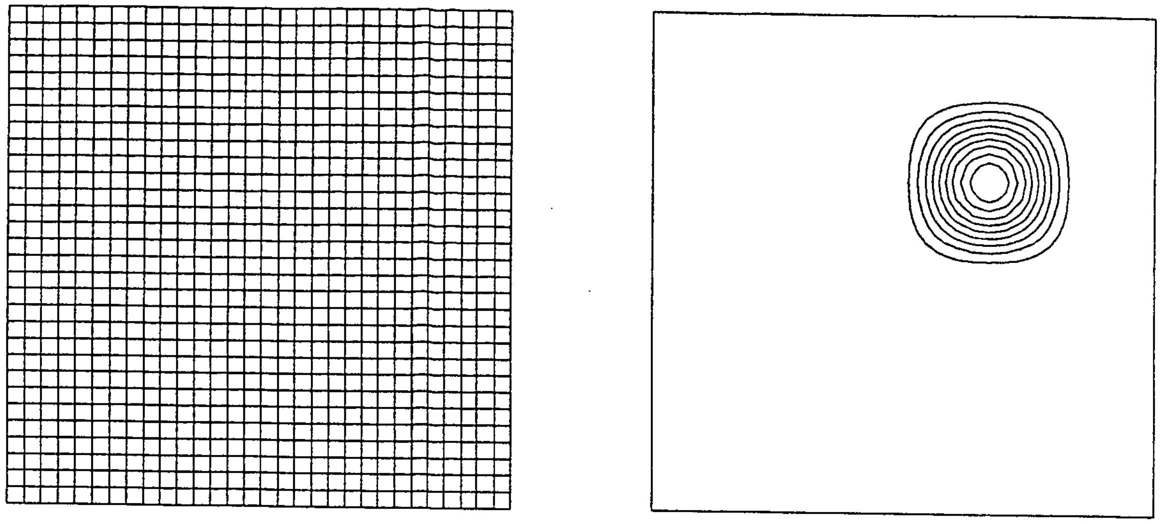


Fig.5: Advection of a cosine hill. Mesh and initial condition.

because at each time step the solution of a banded linear system is required. The inversion of the mass matrix can however be approximated by a purely explicit iterative technique of Jacobi type which takes a particulary simple form due to the incremental character of the equations[1].

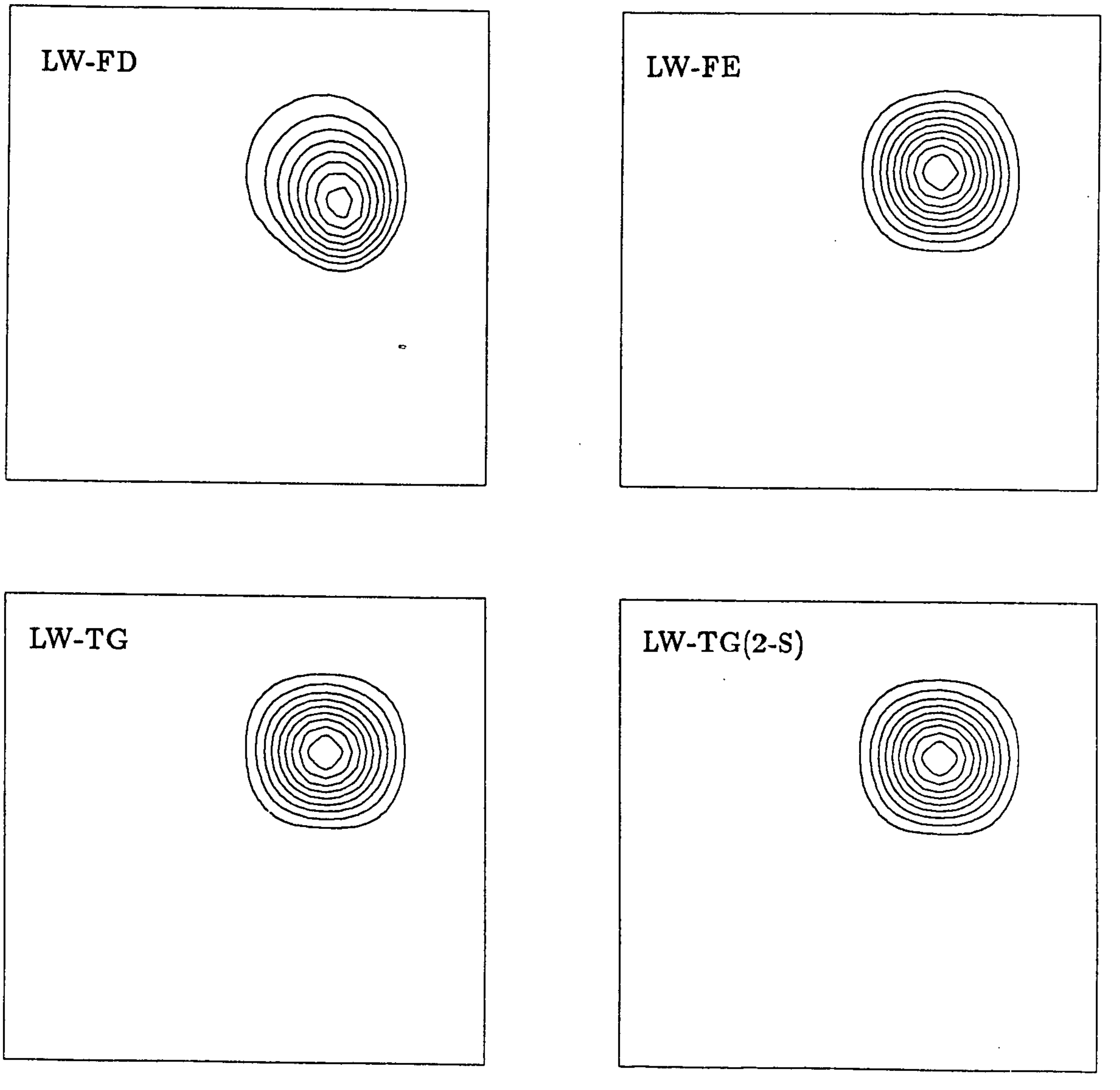


Fig.6: Advection of a cosine hill. Numerical solutions for $\Delta t = 2\pi/200$.

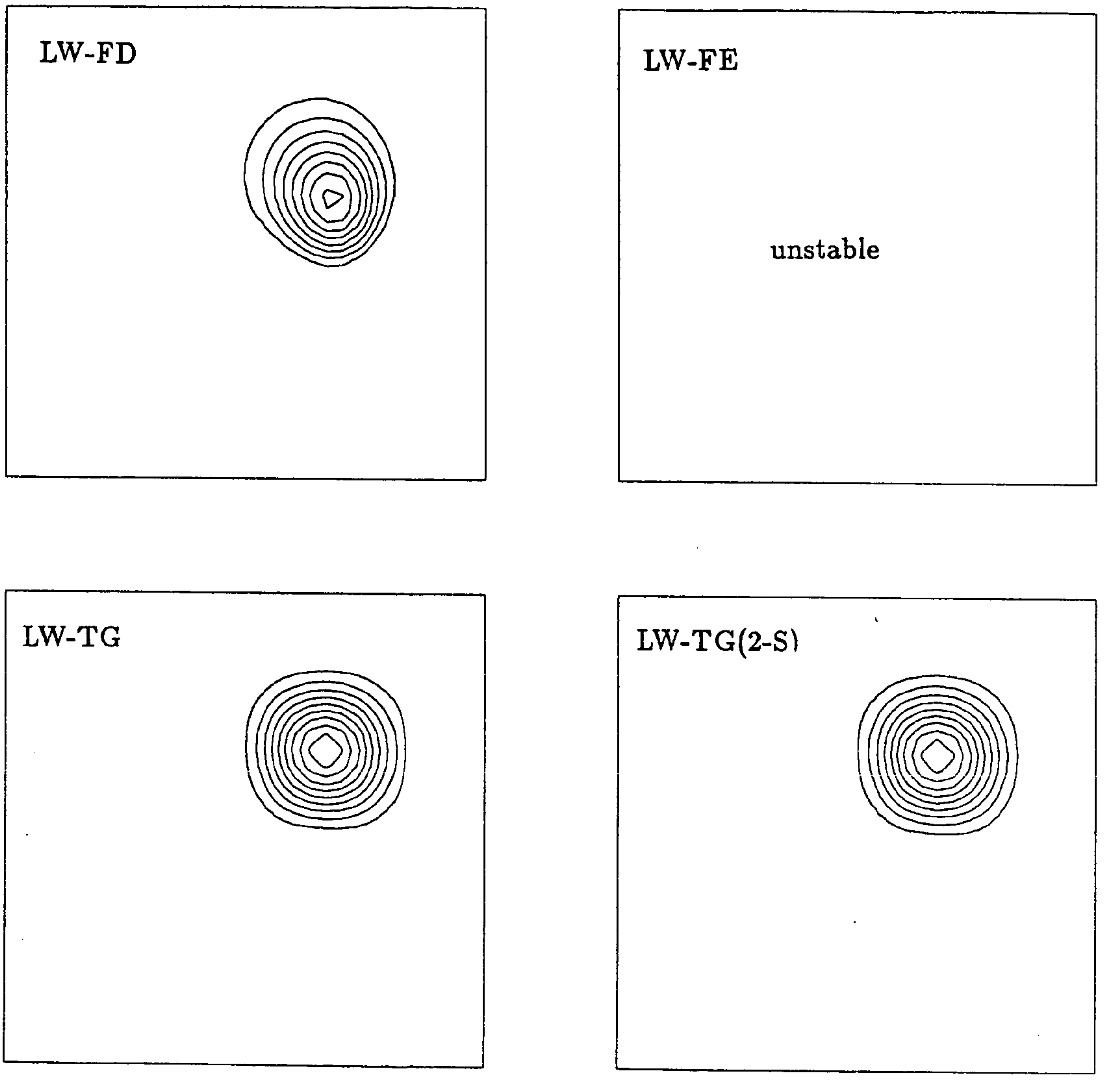


Fig.7: Advection of a cosine hill. Numerical solutions for $\Delta t = 2\pi/120$.

IV. Considerations about systems of nonlinear equations

We discuss rapidly here the generalization of the third-order schemes to the solution of nonlinear equations. For this purpose, we consider a hyperbolic system of conservation laws of the form :

$$\frac{\partial w}{\partial t} + \frac{\partial F(w)}{\partial x} = 0 \tag{17}$$

where w is an m-vector of unknowns and $F(w)$, the flux, is some vector-valued function of w, such that the Jacobian matrix $A = \partial F(w)/\partial w$ has only real eigenvalues. The generalization of LW-TG and LW-TG(2-s) schemes for this system is respectively :

- For the LW-TG scheme :

$$< \ [1 + L_x] \left(\frac{W^{n+1} - W^n}{\Delta t} \right) , \ N_i \ > = \ - < \ \frac{\partial}{\partial x} \left(F^n - \frac{\Delta t}{2} A^n \frac{\partial F^n}{\partial x} \right) , \ N_i \ > \tag{18}$$

where $F^n = F(W^n)$, $A^n = A(W^n)$ and in wich L_x is the following nonlinear operator :

$$L_x \, W \equiv 1 - \ \frac{\Delta t^2}{6} \, \frac{\partial}{\partial x} \left\{ \left[- \frac{\partial A^n}{\partial t} \Big|_{(x)} + A^n \frac{\partial A^n}{\partial x} \right] W \ + \ A^n A^n \frac{\partial W}{\partial x} \right\}$$

The notation $\dfrac{\partial A^n}{\partial t} \Big|_{(x)}$ is employ to emphasize that the derivative of the Jacobian matrix in time has to be expressed in terms of spatial derivatives using Eq.(17).

- For the LW-TG2 scheme :

$$\begin{aligned} &< \frac{\tilde{W}^n - W^n}{\Delta t} , \ N_i \ > \ = \ -\frac{1}{3} < \ \frac{\partial}{\partial x} \left(F^n - \frac{\Delta t}{3} A^n \frac{\partial F^n}{\partial x} \right) , \ N_i \ > \\ &< \frac{W^{n+1} - W^n}{\Delta t} , \ N_i \ > \ = \ - < \ \frac{\partial}{\partial x} \left(F^n - \frac{\Delta t}{2} \tilde{A}^n \frac{\partial \tilde{F}^n}{\partial x} \right) , \ N_i \ > \end{aligned} \tag{19}$$

In view of Eqs. (18) and (19), we can make the following remarks :
a) As the operator L_x is nonlinear, the terms of the generalized mass matrix must be calculated at each time step contrary to the case of linear problems. Thus, for

nonlinear problems, the amount of calculations is close for the two schemes.

b) In the case of systems of equations, the operator L_x couples the different equations what needs much more computer memory.

c) A Richtmyer two-step algorithm[4] can be used to calculate the right hand side of Eqs.(18) and (19). The advantage of this algorithm is that it does not require the evaluation and subsequent multiplication of the Jacobian matrix A which are very time-consuming. Moreover, it can be easily vectorized. This technique is not applicable to the left hand side of Eqs.(18). Thus, using this technique, the computational time of the two-step scheme can be much lower than the one-step scheme.

d) Finally, we note that the operator L_x is very complicated and it will be more and more difficult to obtain for multidimensional problems. In addition, remember that, in this case, the LW-TG(2-s) scheme have a greater domain of numerical stability than the LW-TG scheme.

In conclusion, the two-step version of the LW-TG scheme is easier to be generalized to nonlinear and multidimensional systems of equations.

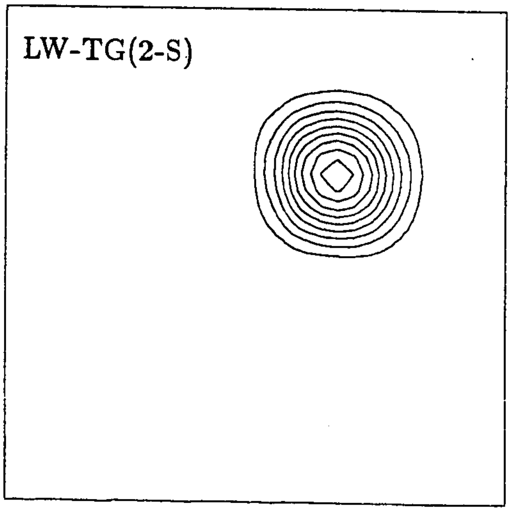


Fig.8: Advection of a cosine hill. Numerical solutions for $\Delta t = 2\pi/90$.

V. Conclusions

In this report, we have compared a one-step and a two-step version of a finite element third-order scheme applicable to the solution of hyperbolic equations and we have conducted a theoretical study of these schemes for linear equations. The two versions of the LW-TG scheme have similar properties (same phase response) and the numerical results obtained with these schemes are very close. Nevertheless, if the LW-TG scheme performs better for linear problems, the analysis has shown that the LW-TG(2-s) scheme in the two-dimensional case has a greater domain of numerical stability and appears also more convenient to generalize to systems of nonlinear equations.

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