# On termination of the direct sum of term rewriting systems

Michaël Rusinowitch

UNITÉ DE RECHERCHE
INRIA-LORRAINE

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél. (1) 39 63 55 11

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Michael RUSINOWITCH

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PAR

MICHAEL RUSINOWITCH

**Résumé** : Pour que la somme directe de deux systèmes de réécriture termine, nous montrons qu'il suffit qu'aucun membre droit de règle ne soit réduit à une variable, ou bien qu'aucun membre droit ne contienne plus d'occurence d'une variable que le membre gauche correspondant.

**Abstract** : Some sufficient conditions are given for the termination of the direct sum of two term rewriting systems : either no right-hand-side of a rule is a variable, or no right-hand-side contains more occurrences of a variable than the corresponding left-hand-side.

# ON TERMINATION OF THE DIRECT SUM OF TERM REWRITING SYSTEMS.

Michael Rusinowitch.
CRIN Campus Scientifique BP 239
54506 Vandoeuvre-les-Nancy Cedex France.

ABSTRACT

Some sufficient conditions are given for the termination of the direct sum of two term rewriting systems: either no right-hand-side of a rule is a variable, or no right-hand side contains more occurrences of a variable than the corresponding left-hand-side.

Keywords: term-rewriting system, termination.

## INTRODUCTION.

It was recently proved in [Toyama 86a] that the direct sum of two term-rewriting systems $R_0$ and $R_1$ is confluent iff $R_0$ and $R_1$ are confluent. However, the same theorem is false if we replace the word confluent by terminating; this was shown in [Toyama 86b].

Let $F_0$ and $F_1$ be two disjoint sets of symbols and $X$ an infinite set of variables. We suppose that $R_0$ (resp. $R_1$) is a noetherian term-rewriting system on $T(F_0,X)$ (resp. $T(F_1,X)$). Then the direct sum of $R_0$ and $R_1$, which is denoted by $R_0 \cup R_1$, is noetherian on $T(F_0 \cup F_1,X)$ if one of the following conditions is satisfied:

**Condition A: for every rule $l \to r$ in $R_0 \cup R_1$ we have $V(r) \subseteq V(l)$, where $V(t)$ denotes the multiset of variables of $t$.**

**Condition B: for every rule $l \to r$ in $R_0 \cup R_1$ , $r$ is not a variable.**

EXAMPLES:

* $R_0$: f(0,1,x,x,x) -> f(x,x,x,0,0)
  $R_1$: or(x,y) -> x
  or(x,y) -> y
* The previous result can be trivially applied to two closed TRS.

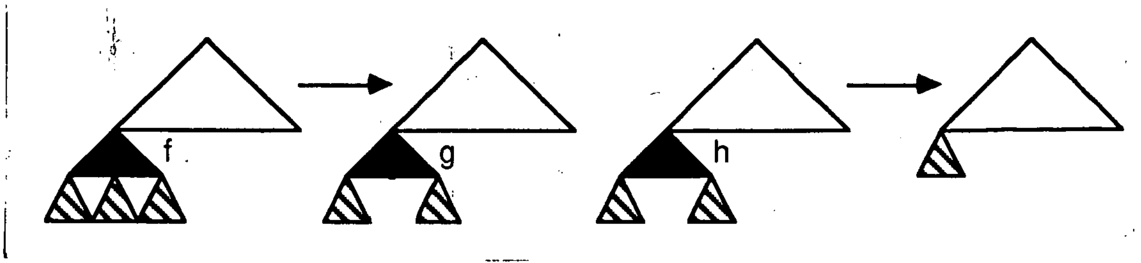
COUNTEREXAMPLES:

* [Toyama 86b] $R_0$: f(0,1,x) -> f(x,x,x)
  $R_1$: or(x,y) -> x
  or(x,y) -> y
* If we replace f(x,x,x) above by f(f(x,x,x),f(x,x,x),f(x,x,x)), we get a a system that is not even quasi-terminating.
* There is in [Barendregt & Klop 86] a direct sum of canonical term-rewriting systems which is not canonical.

DEFINITION: a rewrite rule is **collapsing** when its right-hand side is a variable, and **duplicant** when its right-hand side has strictly more occurrences of one variable than its left-hand side.

A technique to prove termination is to embed the rewriting relation in a well-founded ordering on terms [Dershowitz 85] such as simplification orderings [Dershowitz 82][Plaisted 78][Jouannaud etal. 82] which are usually recursively defined. Another idea is to associate with each term a quantity which decreases when the term is rewritten, but cannot decrease for ever. The quantity may be a polynomial interpretation of the terms (see,for instance, [Huet-Oppen 80]).

Our goal now is to find such a quantity. In order to prove termination under hypothesis A, we divide terms into connected domains, such that all the symbols within a domain belong to the same $F_i$(i=0 or 1). During a rewriting sequence, the number of frontier occurrences between two domains cannot increase, since there is no duplicant rule in the system (figure a). It may decrease, due to the presence of collapsing rules (figure b). When it is not decreasing, then the multiset of connected domains of the term is rewritten by $R_0 \cup R_1$, which is known to terminate on homogeneous terms.




using the rule f(x,x,y)->g(x,y)

**figure a**

using the rule h(x,x)->x

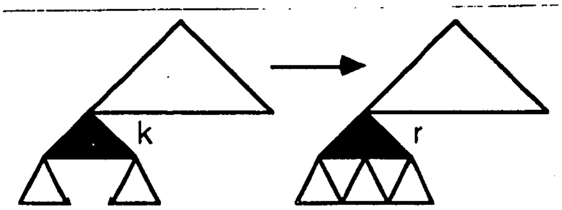
**figure b**

To deal with hypothesis B, we first note that the maximal number of connected domains traversed by a branch of a term is constant on every derivation starting from this term, due to the absence of collapsing rules. Then, we can show that the multiset of connected domains figuring at some level is decreasing, without the lower levels increasing (figure c). The termination is achieved as in the proof that nested multiset orderings [Dershowitz Manna 79] are well-founded.

level 1 is unchanged

level 2 decreases

level 3 increases




with the rule k(x,x)->r(x,x,x)

figure c

The main difficulty is that a rewriting step by $R_0$ at an occurrence may generate new rewriting opportunities by $R_1$ above that occurrence as in the following example:

$R_0$: a->b

$R_1$: f(x,x)->g

f(a,b)->f(b,b)->g.

Let us prove now more formally our statements.

## 1.DEFINITIONS

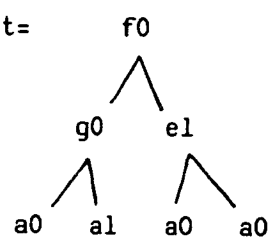
Terms are considered as trees. Positions in trees are defined by occurrences which are sequences of integers (see [Huet 80] for rigorous definitions). We call occ(t) (resp. $\mathcal{J}(t)$) the set of non-variable (resp. variable) occurrences of a term t. Given an occurrence i of a term t, p(i) denotes the maximal proper prefix of i. If $\varepsilon$ is the root occurrence of t, $p(\varepsilon)$ is undefined.

Let Z(t) be the set of occurrences i of t, such that the symbol at i comes from a different theory than the symbol at p(i). To put it more formally $Z(t)=\{i \in occ(t);\ t(i) \in F_j \text{ and } t(p(i)) \in F_{1-j}\ ,\ \text{for } j=0 \text{ or } 1\}$.

Let I(t) the subset of elements of Z(t) such that none of their strict left factors is in Z(t). Hence, every symbol occurring at $i \in I(t)$ (and before i) is in the same theory than the root symbol. In the following, function symbols whose index is j (j=0 or 1) are implicitly supposed to belong to $F_j$.

example:

t=




$f0, g0, a0 \in F_0 \quad e1, a1 \in F_1$

$Z(t)=\{12,2,21,22\}$

$I(t)=\{12,2\}$

Let SK(t) be the pure part of t. To be more precise, let $\Omega$ be a new variable, then we have:

$$SK(t)= t[i \leftarrow \Omega]_{i\in I(t) \cup \mathcal{J}(t)}.$$

It is easy to check that $\mathcal{J}(SK(t)) = I(t)\cup\mathcal{J}(t)$. In particular, when x is a variable, $SK(x)=\Omega$.

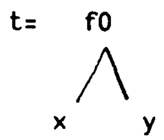
For any substitution $\theta$: $X \dashrightarrow T(F,X)$, and for i=1,0 we define $SK_i(\theta)$ to be the following substitution: for all $x\in X$,

$$SK_i(\theta)(x) = \begin{cases} \Omega \text{ if } \theta(x)(\varepsilon) \in F_{1-i} \\ SK(\theta(x)) \text{ otherwise} \end{cases}$$

We just state the next easy proposition:

**Proposition 1: if $t \in T(F_i,X)$ then $SK(\theta(t)) = SK_i(\theta)(t)$.**

example :

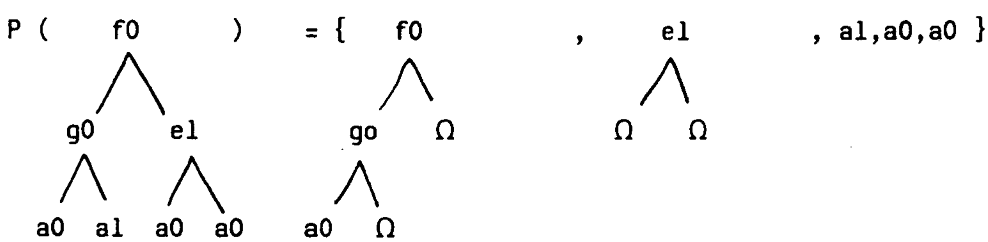



$\theta(x)=a1$ $\qquad$ $SK_0(\theta)(x)=\Omega$

$\theta(y)=f0(a1,a1)$ $\qquad$ $SK_0(\theta)(y)=f0(\Omega,\Omega)$

Let P(t) be the multiset of the maximal homogeneous terms we can find in t. To be more precise, P is the unique solution of the recursive equation: $P(t) = \{SK(t)\} \cup ( \cup_{i\in I(t)} P(t/i) )$. We shall call SP(t) the sub-multiset $(\cup_{i\in I(t)} SK(t/i))$ of P(t).

example :




The term-rewriting system $R_0 \cup R_1$ induces a rewriting relation "-->" on $T(F_0,\Omega) \cup T(F_1,\Omega)$. Since every "--> chain" is either a "$\to R_0$ chain" or a "$\to R_1$ chain", it is a finite chain:

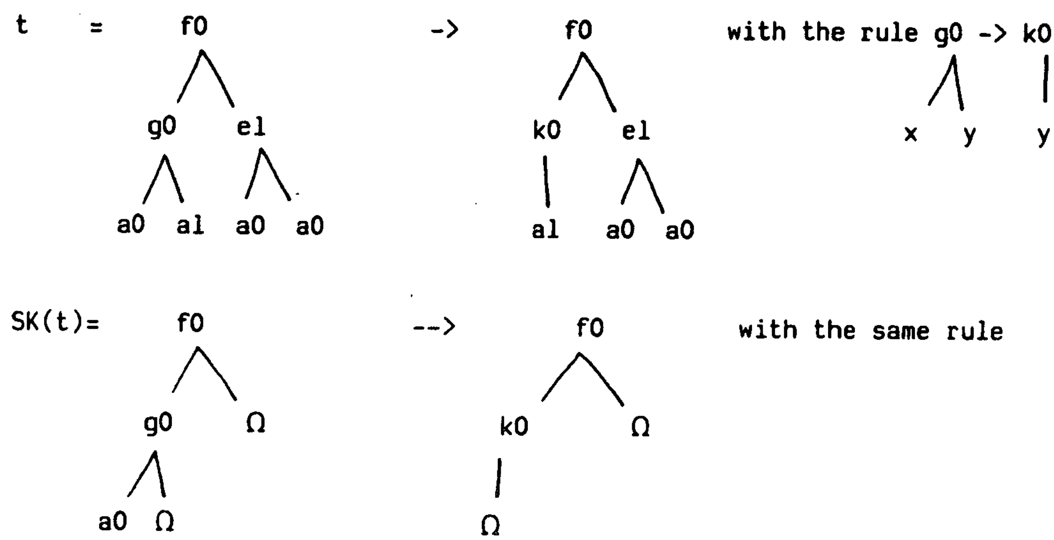
**Proposition 2: --> is noetherian.**

Every rewriting on t which is performed at an occurrence of the domain of SK(t) can be transposed into a rewriting of SK(t) (at the same occurrence), as it is showed by the next lemma.

**Lifting Lemma: suppose that t-> t' by application of the rule 1->r at occurrence i of t with the substitution $\theta$. We suppose that i is also an occurrence of occ(SK(t)) and $t(\varepsilon) \in F_0$ (the case $F_1$ is similar). Then**

$$SK(t) \dashrightarrow \begin{cases} \Omega \text{ if } i=\varepsilon \text{ and } r\in X \text{ and } \theta(r)(\varepsilon)\in F_1 \\ \\ SK(t') \text{ otherwise} \end{cases}$$

example :




proof: we have $t=t[i\leftarrow\theta(l)]$ and $t'=t[i\leftarrow\theta(r)]$. Since $i\in occ(SK(t))$ and $t(\varepsilon)\in F_0$, we also have $t(i)\in F_0$. Therefore $l(\varepsilon)=t(i) \in F_0$, $SK(t)=SK(t)[i\leftarrow SK(\theta(l))]$ . The proposition above can be applied to $\theta(l)$: $SK(\theta(l))=SK_0(\theta)(l)$. Now, if $\theta(r)(\varepsilon)\in F_0$ we also have $SK(t')=SK(t)[i\leftarrow SK(\theta(r))]$. If $\theta(r)(\varepsilon)\in F_1$ and $i\neq\varepsilon$ then $SK(t')=SK(t)[i\leftarrow\Omega]$. In any of these two previous cases: $SK(t')=SK(t)[i\leftarrow SK_0(\theta)(r))]$ and we proved that $SK(t)\to SK(t')$ by the same rule. Let us suppose now that $i=\varepsilon$ and $\theta(r)(\varepsilon)\in F_1$. This understands that r is a variable which obviously occurs in l and such that $t=\theta(l)$ and $t'=\theta(r)$. The definition of $SK(t)$ shows that $SK(t)=\theta'(l)$ for the substitution $\theta'$ defined by: $\theta'(x)= SK_0(\theta(x))$ for every $x\in X$. In particular, $\theta'(r)=SK_0(\theta(r))=\Omega$. This last result takes care of the first case in the lemma.

Let MP be the set of finite multisets of elements in $T(F_0,\Omega) \cup T(F_1,\Omega)$. We extend --> to MP in a standard way [Jouannaud Lescanne 82]: we still get a noetherian relation on MP. This relation will be denoted by $\Rrightarrow$. Let $\Rrightarrow+$ be the transitive closure of $\Rrightarrow$.

## 2. Duplication-free systems

Let us prove now that condition A is enough to ensure the termination property for the term-rewriting system $R_0 \cup R_1$, denoted by R. Our goal will be reached by defining a noetherian ordering on $T(F,X)$ which is compatible with R [Manna & Ness theorem]:

Let us define the ordering $>$ on $T(F,X)$ by:

$t>t'$ if $|Z(t)| > |Z(t')|$ or

$|Z(t)| = |Z(t')|$ and $P(t) \Rightarrow\!\!+ P(t')$

Since $>$ is the lexicographical composition of the noetherian orderings $(>,N)$ and $(\Rightarrow\!\!+, MP)$, it is also noetherian. Now let us show that: $t \to t'$ implies $t>t'$.

Suppose $t \to t'$ by applying the rule $l \to r$ at occurrence i with the substitution $\theta$. Let $\iota$ be the maximal prefix occurrence of i such that $\iota \in Z(t)$; otherwise $\iota$ is $\varepsilon$. We also suppose that $t(\iota) \in F_0$ (the case $F_1$ will be similar).

**CASE 1:** We suppose that i is equal to $\iota$ and $r \in X$. We also suppose that $\theta(r)(\varepsilon) \in F_1$ (the root symbol of $\theta(r)$ is in $F_1$).

We can get t' by the replacement of t/i at occurrence i within t, by $\theta(r)$. Let h be an occurrence of t/i such that $(t/i)/h = \theta(r)$. (it suffices to take h such that $l/h=r$). We can notice that $Z((t/i)/h) \subseteq Z(t/i)$. Then, we have $(t/i)/h = t/i.h$. The hypothesis implies that i does not belong any more to $Z(t')$. Hence we have:

$Z(t') = ((Z(t)\setminus \{i.k;\ k \in Z(t/i)\}) \setminus\{i\}) \cup \{i.k;\ k \in Z(t/i.h)\}$.

Since $|Z(t/i.h)| \leq |Z(t/i)|$, we have $|Z(t')|<|(Z(t)|$.

**CASE 2:** Otherwise.

In this case $|Z(t')| \leq |Z(t)|$, but one can no more ensure that the inequality is strict. However, we have:

$P(t') = (P(t)\setminus P(t/\iota)\ ) \cup P(t'/\iota)$.

$P(t/\iota) = \{SK(t/\iota)\} \cup SP(t/\iota)$

$P(t'/\iota) = \{SK(t'/\iota)\} \cup SP(t'/\iota)$

Firstly, the Lifting lemma implies that $SK(t/\iota) \dashrightarrow SK(t'/\iota)$. Now, if we can prove $SP(t'/\iota) \subseteq SP(t/\iota)$ we are done, since it is easy to derive from this inclusion : $P(t'/\iota) \Rightarrow^{+} P(t/\iota)$ and then $P(t) \Rightarrow^{+} P(t')$.

We have $t'/\iota = t/\iota[i \leftarrow \theta(r)]$. According to condition A, $V(r) \subseteq V(1)$. As a consequency, we have (*) $\{\theta(x);\ x \in V(r)\} \subseteq \{\theta(x);\ x \in V(1)\}$ (multiset inclusion). As $r \in T(F_0,X)$ and $SK(t/\iota) \in T(F_0,\Omega)$ we can remark that every element of $SP(t'/\iota)$ appears within a $\theta(x)$ for some $x \in V(r)$. Therefore (*) implies $SP(t'/\iota) \subseteq SP(t/\iota)$.

## 3. Collapse-free systems.

Let us prove now that Condition B ensures that $R_0 \cup R_1$ is noetherian. The technique used above fails, since, in the actual case, we may have $t' \to t$ with $Z(t) > Z(t')$. Let us introduce the notion of **level** of a term, which is the maximal number of homogeneous layers of symbols that can be encountered on a branch of the term.

The level of any term in $T(F_0,X) \cup T(F_1,X)$ is 1. The level of a term in $T(F_0 \cup F_1,X)$ is recursively defined: $\text{level}(t) = 1 + \max\{\text{level}(t/i)\ ;\ i \in I(t)\}$ (with $\max\{\emptyset\}=0$).

example:

```
level (     f0       )    = 3
           /  \
         g0    h1
        /  \   /  \
       x   a1 a0   a0
```

Let us notice that Condition B ensures that the level of terms is preserved under rewriting. As usual, we start defining an ordering > on $T(F_0 \cup F_1,X)$:

$t > t'$ if t and t' have the same level and (recursively)

$SK(t') \dashrightarrow SK(t)$ or

$SK(t') = SK(t)$ and $\{t'/i\}_{i \in I(t')} \gg \{t/i\}_{i \in I(t)}$

where $\gg$ is the multiset extension of >.

This ordering presents some similarity with the "nested multiset ordering" in [Dershowitz Manna 79]. Let us recall that --> is noetherian. By induction on the level of terms, we obtain :

**Proposition 4: > is noetherian.**

We are now in position to apply Manna & Ness method, contingent on proving:

**Proposition 5: $t \to_R t'$ implies $t > t'$.**

proof: if t is reduced to t' at occurrence i and i belongs to occ(SK(t)) then the Lifting Lemma yields: SK(t)-->SK(t'). Otherwise, since there is no collapsing rule, we have level(t)=level(t') , SK(t)=SK(t') and the rewriting takes place in a subterm of t, say t/i with $i \in I(t)$. Now, by induction on the level of terms, $\{t/i\}_{i \in I(t)} \gg \{t'/i\}_{i \in I(t')}$. From this we can derive t>t'.

## CONCLUDING REMARKS.

1. We conjecture that a system $R_0 \cup R_1$ is non-terminating only if $R_0$ contains a collapsing rule and $R_1$ contains a duplicant rule or vice-versa. However, we did not succeed to build a noetherian extension of the rewriting relation, in that case.

2. Attention has been recently focused on unification algorithms in combination of theories (see for instance [Yellick 1985]). We think our results may be useful in a term-rewriting approach of these problems, as it was developped in [Kirchner 1985], especially in the presence of collapsing axioms, which are difficult to handle in unification theory.

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