Construction of the stationary regime of queues with locking

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CONSTRUCTION DU REGIME STATIONNAIRE DE FILES D'ATTENTE AVEC VERROUILLAGE

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RESUME

On considère un système de files d'attente avec N serveurs et deux types de clients. Les clients de type 1 demandent à être traités par l'un des N serveurs alors que ceux de type 2 doivent être traités simultanément par les N serveurs. On généralise le schéma de Loynes à ce type de files d'attente. On en déduit la condition de stabilité et diverses propriétés sur le régime stationnaire de ce système.

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ABSTRACT

We consider a queueing system with N servers and two types of customers: Simple customers which require a service from one of the N servers and Locking customers which have to be served simultaneously by all N servers. Loynes' increasing schema is generalized to this type of queueing system. Various properties such as the stability condition and the uniqueness of the stationary regime are then derived.

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1. Introduction

We consider a queueing system with N servers and two types of customers, Simple and Locking. One arrival stream of Simple customers is associated with each of the N servers. The Simple customers arriving to server k require a service from this server only, so that in the absence of Locking customers, each of the N servers gives rise to a classical First In First Out queue of Simple customers. On the other hand, the stream of Locking customers is unique, and each Locking customer requires service from all of the N servers. More precisely, a Locking customer requires these N services simultaneously. Hence, when one of the servers is ready to serve one of the Locking customers, it has to wait until all of the $N-1$ other servers are ready to serve the same customer in parallel. The interaction of the two types of customers is now fully specified when adding that a global First In First Out discipline is enforced everywhere.

Such locking mechanisms are commonly used in data bases to enforce consistency ([1]). In a system of this type "updates" (our Locking customers) have to acquire locks on all of the "data elements" on which they operate before beginning any real processing, while "queries" (the Simple ones) concerning one data element may be processed by reading this data element independently. The conflict resolution mechanism based on time stamps gives rise directly to the global First In First Out discipline mentioned above.

In a recent paper [2], the stability condition for this type of queueing system was given under certain assumptions on the input and service processes. In particular, it was assumed that Locking customers arrive in a renewal process, all Simple customers arrive in Poisson processes, and all service time sequences are iid. In addition it was assumed that all of the above processes are mutually independent. The proof was based on explicitly constructed pathwise upper and lower bounding systems of $G/G/1$ type.

The aim of the present paper is on the one hand to generalize this stability condition and on the other hand to build and analyze the stationary regime of this system while relaxing the various independence and exponentiality assumptions. We only assume stationarity and ergodicity of the input and service processes. An increasing schema showing both the existence and uniqueness of the stationary waiting times under the stability condition is analyzed in Section 3. The proofs are extensions of the argument of Loynes [3] for simple $G/G/1$ queues. The basic formalism to be used is outlined in Section 2.

This type of approach was already used to analyze the stability and stationary regime of other queueing systems with synchronization constraints [4] and seems to be of general applicability.

2. Preliminaries

We denote by $\mathbf{M}$ the space of integer valued measures on $\mathbf{R}$ which are finite on finite intervals and have no double points. Any ϕ in $\mathbf{M}$ has a unique representation of the form

$$\phi(\cdot) = \sum_{i \in \mathbb{Z}} \delta_{t_i}(\cdot)$$

where δ_a, $a \in \mathbf{R}$ represents the Dirac measure at point a, and the t_i's are real numbers satisfying the inequalities:

$$\cdots\ t_{-1} < t_0 \leqslant 0 < t_1 < \ \cdots\ .$$

The space $\mathbf{M}$ is endowed with the σ-field $\mathcal{M}$ generated by the events $\phi(x) = k$, where x varies over the Borel sets of $\mathbf{R}$ and k over the nonnegative integers. All of the random variables used in the present paper are defined on a common probability space $(\Omega, \mathcal{A}, \mathbf{P}, \theta_t)$, where $(\theta_t, t \in \mathbf{R})$ is a group of measurable automorphisms from Ω onto Ω leaving $\mathbf{P}$ invariant. It is also assumed that $\mathbf{P}$ is θ-ergodic. Within this framework, a stationary and ergodic point process K is a measurable function from $(\Omega, \mathcal{A}, \mathbf{P})$ onto $(\mathbf{M}, \mathcal{M})$:

$$\omega \rightarrow K(\omega,\cdot) = \sum_{i \in \mathbb{Z}} \delta_{t_i(\omega)}(\cdot),$$

which satisfies the condition

$$K(\theta_s(\omega),\cdot) = \sum_{i \in \mathbb{Z}} \delta_{t_i(\omega)-s}(\cdot),$$

for all s in $\mathbf{R}$.

A Palm space $(\Omega^0, \mathscr{A}^0)$, along with its Palm probability measure $\mathbf{P}^0$ on $(\Omega^0, \mathscr{A}^0)$, can be associated with the point process K, where

$$\Omega^0 = \Omega \cap \{t_0(\omega) = 0\}$$

and

$$\mathscr{A}^0 = \mathscr{A} \cap \{t_0(\omega) = 0\}.$$

We do not give the definition of $\mathbf{P}^0$ which can be found for instance in [5]. We should point out, however, that $\mathbf{P}^0$ can be understood as the law of K conditioned by the event that K has a point at zero (i.e., $t_0 = 0$).

Let $\theta = \theta_{t_1}$. The main results to be used in the paper are the following two properties:

$$\mathbf{P} \text{ is } \theta_t\text{-invariant iff } \mathbf{P}^0 \text{ is } \theta\text{-invariant}$$

and

$$\mathbf{P} \text{ is } \theta_t\text{-ergodic iff } \mathbf{P}^0 \text{ is } \theta\text{-ergodic}.$$

3. Stability Analysis

3.1 Assumptions and Notation

The assumptions on the arrival point processes and the sequences of service times are limited to stationarity and ergodicity. There is, in particular, no independence hypothesis between these processes. Within the mathematical framework outlined in Section 2, this gives rise to the following model. Let K and K_j, $j = 1, \ldots, N$ be stationary point processes on $(\Omega, \mathscr{A}, \mathbf{P}, \theta_t)$ representing respectively the arrival point processes of Locking and Simple customers destined for server j. These processes can be written as

$$K(\omega,\cdot) = \sum_{i \in \mathbb{Z}} \delta_{t_i(\omega)}(\cdot)$$

and

$$K_j(\omega,\cdot) = \sum_{i \in \mathbb{Z}} \delta_{s_i^j(\omega)}(\cdot).$$

The Palm spaces of K and K_j are denoted by $(\Omega^0, \mathcal{A}^0, \mathbf{P}^0)$ and $(\Omega_j^0, \mathcal{A}_j^0, \mathbf{P}_j^0)$ respectively.

The service requirements of the Locking customers (defined as the duration of their simultaneous residence time in all N servers once these are locked) are characterized by a real valued and nonnegative random variable σ defined on $(\Omega^0, \mathcal{A}^0, \mathbf{P}^0)$. The random variable σ represents the service requirement of the Locking customer arriving at time t_0 ($= 0$ on Ω^0). This specifies fully the sequence of service requirements when noticing that the service requirement of the k-th Locking customer (the one arriving at time t_k) is given by $\sigma \circ \theta_{t_k}$.

Similarly, the service requirements of the Simple customers of K_j are fully characterized by a nonnegative random variable σ_j defined on $(\Omega_j^0, \mathcal{A}_j^0, \mathbf{P}_j^0)$.

It is assumed that σ is $\mathbf{P}^0$ integrable, σ_j is $\mathbf{P}_j^0$ integrable, and the intensities of K and K_j are all finite, for $j = 1, \ldots, N$.

3.2 Evolution Equations for the Waiting Time of the Locking Customers

From now on, the basic probability space we work on will be $(\Omega^0, \mathcal{A}^0, \mathbf{P}^0)$, the Palm space of the Locking customers. For a given $j = 1, \ldots, N$ consider a sample path of K_j and $(\sigma_j \circ \theta_{s_i^j}, i \in \mathbb{Z})$ on this probability space. For t and $w \in \mathbf{R}^+$, let $F_j(t,w)$ be the residual workload of simple customers in front of the j-th server at time t, if the initial workload at time zero is zero, the server is blocked up to time w, and this server is fed with simple customers only. It is clear that $F_j(t,w)$ is a nonnegative real valued random variable on $(\Omega^0, \mathcal{A}^0, \mathbf{P}^0)$ since this residual workload can be obtained pathwise from the values of s_i^j and $\sigma_j \circ \theta_{s_i^j}$ for $i \geq 0$ defined on the same probability space. We assume that the function $t \to F_j(t,w)$ is left continuous. Notice that $F_j(t,w)$ is measurable in both arguments t and w. The behavior of the function $t \to F_j(t,w)$, w fixed, is exemplified in Figure 1.

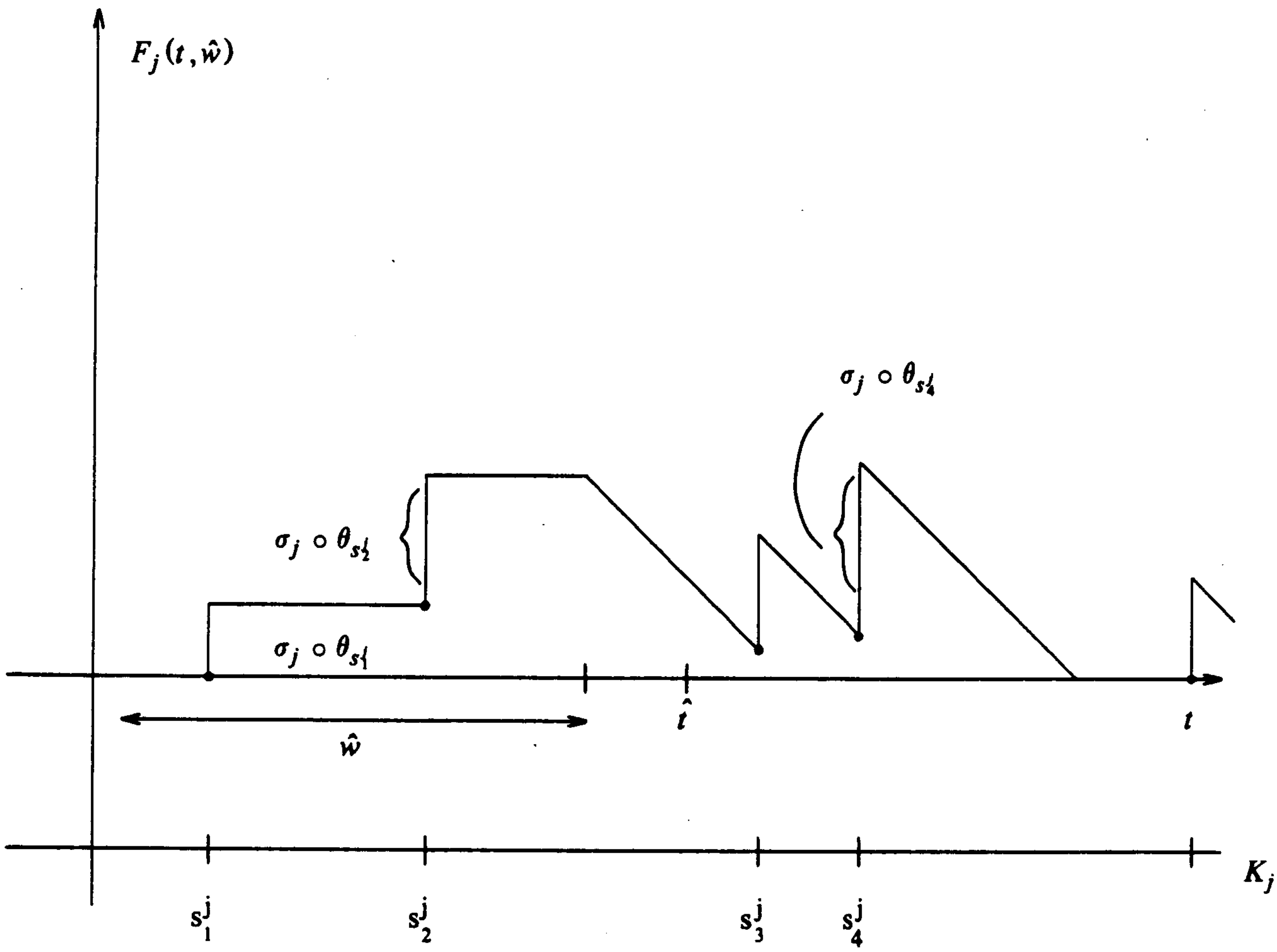

Figure 1

Let

$$a_j(t) = \sum_{\substack{i \\ 0 \leqslant s_i^j < t}} \sigma_j \circ \theta_{s_i^j}$$

and

$$b_j(t) = t - a_j(t) + F_j(t,0).$$

Considered as a function of w, with t held fixed, $F_j(t,w)$ is equal to $F_j(t,0)$ for $0 \leqslant w \leqslant b_j(t)$, increases linearly with slope 1 for $b_j(t) \leqslant w \leqslant t$, and is equal to $a_j(t)$ for $w \geqslant t$. This behavior is depicted in Figure 2.

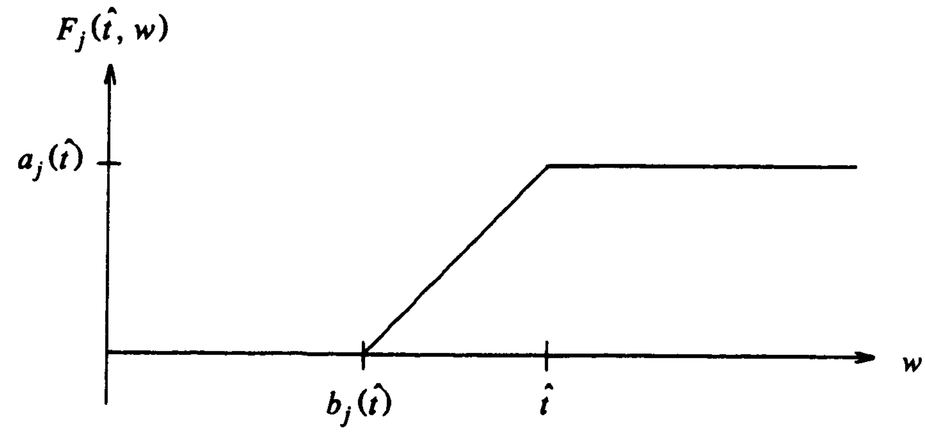

Figure 2

Let

$$h_j(t,w) = \max(w-t,0) + F_j(t,w).$$

Then $w \rightarrow h_j(t,w)$, t fixed, is constant and equal to $F_j(t,0)$ up to time $b_j(t)$ and linear with slope 1 after (Fig. 3).

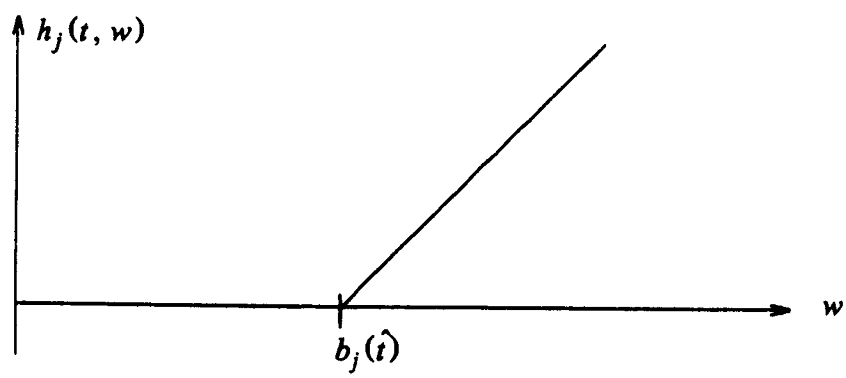

Figure 3

Assume one knows W, the waiting time of the Locking customer arriving at time $t_0 = 0$. (This waiting time is defined as the time between its arrival and the beginning of its simultaneous services). Let us then determine the waiting time incurred by the next Locking customer (arriving at time t_1). Due to the First In First Out assumption, one sees that the delay to exhaust the total workload (Simple plus Locking) in queue j at the arrival of customer t_1 is the random variable

$F_j(t_1, W + \sigma)$ if $t_1 > W + \sigma$ and the random variable $W + \sigma - t_1 + F_j(t_1, W + \sigma)$ if $t_1 \leqslant W + \sigma$. Thus, this delay is actually equal to

$$F_j(t_1, W + \sigma) + (W + \sigma - t_1)^+,$$

where $[a]^+ = \max(a, 0)$ for $a \in \mathbf{R}$. Our assumption on the left continuity of $t \rightarrow F_j(t, w)$ implies that if Locking and Simple customers arrive simultaneously, priority will be given to the Locking customer. The waiting time of the Locking customer arriving at t_1 is equal to

$$\max_{1 \leqslant j \leqslant N} \left[F_j(t_1, W+\sigma)\right] + [W+\sigma-t_1]^+.$$

The existence of a stationary regime means that the waiting time of the Locking customer arriving at t_1 is the waiting time of the one arriving at t_0, W, composed with θ_{t_1}. Thus, if one denotes θ_{t_1} by θ, the existence of a stationary regime is equivalent to the existence of a nonnegative real valued random variable W on $(\Omega^0, \mathscr{A}^0, \mathbf{P}^0)$ which satisfies the evolution equation

(3.1) $$W \circ \theta = [W+\sigma-t_1]^+ + \max_{1 \leqslant j \leqslant N} (F_j(t_1, W+\sigma)).$$

3.3 Stability Condition

Let E^0 denote the expectation with respect to $\mathbf{P}^0$.

Theorem 1. If the condition

(3.2) $$E^0[\sigma] + E^0\left[\max_{1 \leqslant j \leqslant N} a_j(t_1)\right] < E^0[t_1]$$

is fulfilled, there exists a finite random variable W on $(\Omega^0, \mathscr{A}^0, \mathbf{P}^0)$ which satisfies (3.1). ∎

Proof. Consider the following sequence of random variables on $(\Omega^0, \mathscr{A}^0, \mathbf{P}^0)$:

(3.3a) $$V_0 = 0$$

and

(3.3b) $$V_{n+1} = \left([V_n+\sigma-t_1]^+ + \max_{1 \leqslant j \leqslant N} F_j(t_1, V_n+\sigma)\right) \circ \theta^{-1}.$$

For any fixed $t \geqslant 0$, the function $w \rightarrow F_j(t, w)$ is pathwise continuous and nondecreasing. These properties imply that the function

$$g(w) = [w+\sigma-t_1]^+ + \max_{1 \leq j \leq N} F_j(t_1, w+\sigma)$$

is also pathwise continuous and nondecreasing. We can now show that the schema V_n is nondecreasing. Assume that $V_n \geqslant V_{n-1}$ (this is true for $n=1$). Then, one gets from (3.3) that

$$V_{n+1} \circ \theta = g(V_n) \geqslant g(V_{n-1}) = V_n \circ \theta,$$

so that the nondecreasingness of V_n is proven by induction. Let V be defined as the increasing limit of the V_n's. The continuity property of $g(w)$ implies that V satisfies the relation (3.1), so that

$$V \circ \theta = [V+\sigma-t_1]^+ + \max_{1 \leq j \leq N} (F_j(t_1, V+\sigma)). \tag{3.4}$$

Accordingly, the theorem will be established if we show that the condition (3.2) implies the finiteness of V. To show this, notice first that (3.4) implies that the event $(V=\infty)$ is θ invariant. (The random variables σ, t_1, and $F_j(t_1, V+\sigma)$ are a.s. finite, so that $[V+\sigma-t_1]^+$ is a.s. finite if and only if V is finite.) Thus, the ergodic theorem shows that either $V=\infty$ a.s. or $V<\infty$ a.s. To show that (3.2) implies that $V<\infty$ a.s., it is sufficient to prove that the assumption $V=\infty$ a.s. implies that (3.2) is not satisfied.

We need the following Lemma.

Lemma 1. For $n \geqslant 1$, V_n is $\mathbf{P}^0$ integrable.

Proof. The proof is by induction. Assume that V_n is integrable. So V_{n+1} will also be integrable if one proves that

$$[V_n+\sigma-t_1]^+ + \max_{1 \leq j \leq N} F_j(t_1, V_n+\sigma) \tag{3.5}$$

is integrable. Note that (3.5) is bounded above pathwise by

$$V_n + \sigma + \sum_{1 \leq j \leq N} a_j(t_1). \tag{3.6}$$

Using the Neveu exchange formula between two Palm measures [5], we obtain

$$\lambda E^0 a_j(t_1) = \lambda_j E^j[\sigma_j],$$

where E^j denotes the expectation with respect to $\mathbf{P}_j^0$ and where λ (resp. λ_j) denotes the intensity of K (resp. K_j). Thus, the expectation of (3.6) with respect to $\mathbf{P}^0$ is

(3.7) $$E^0[V_n] + E^0[\sigma] + \lambda^{-1} \sum_{1 \leqslant j \leqslant N} \lambda_j \, E^j[\sigma_j],$$

which is finite due to our integrability assumptions. This completes the proof of the Lemma.

Since $V_{n+1} \geqslant V_n$, using Lemma 1 we obtain

(3.8) $$E^0[V_{n+1} - V_n] \geqslant 0.$$

Since $\mathbf{P}^0$ is θ-invariant, $E^0[V_{n+1}] = E^0[V_{n+1} \circ \theta]$. Thus, (3.8) implies that

$$E^0[g(V_n) - V_n] \geqslant 0.$$

Assume now that $V = \infty$ a.s. or equivalently that V_n increases to ∞ a.s. Using (3.6), it is clear that $g(w) - w$ is bounded above by

(3.9) $$\sigma + \sum_{1 \leqslant j \leqslant N} a_j(t_1),$$

which is integrable as established in (3.7). In addition, $g(w) - w$ is bounded below by $\sigma - t_1$, so that Lebesgue's Dominated Convergence Theorem yields

(3.10) $$\begin{aligned} 0 &\leqslant \lim_{n \to \infty} E^0[g(V_n) - V_n] \\ &= E^0[\lim_{n \to \infty} g(V_n) - V_n]. \end{aligned}$$

Since the function $g(w) - w$ satisfies the limiting relation

$$\lim_{w \to \infty} g(w) - w = \sigma - t_1 + \max_{1 \leqslant j \leqslant N} a_j(t_1),$$

one sees that (3.10) implies that (3.2) is not satisfied which concludes the proof of the theorem. ■

Theorem 2. If

$$E^0[\sigma] + E^0\left[\max_{1 \leqslant j \leqslant N} a_j(t_1)\right] \neq E^0[t_1],$$

then equation (3.1) has a unique solution. If

(3.11) $$E^0[\sigma] + E^0\left[\max_{1 \leqslant j \leqslant N} a_j(t_1)\right] = E^0[t_1],$$

and if the variable V defined in Theorem 1 is finite, then any other solution of (3.1) will be of the form $V + c$ where c is a positive constant. ■

Proof. Let W be any solution of (3.1). Consider again the sequence V_n of Theorem 1. Since $W \geqslant V_0 = 0$, one gets by induction (when using the increasingness of $w \to g(w)$) that $W \geqslant V_n$ for all n so that

(3.12) $$W \geqslant V.$$

If $V = \infty$ a.s., one obtains from (3.12) that any other solution of (3.1) will be infinite as well, so that the uniqueness is established. Consider now the case $V < \infty$ a.s. Notice first that when it is differentiable, the slope of the function $g(w)$ is either zero or one. Since $g(w)$ is also continuous, one gets then

$$g(w) - g(v) \leqslant w - v \quad \text{for } w \geqslant v.$$

This implies the relation

(3.13) $$W \circ \theta - V \circ \theta = g(W) - g(V) \leqslant W - V.$$

For $t \in \mathbf{R}$, let $A_t = \min(W - V, t)$. Then (3.13) implies that $A_t \circ \theta \leqslant A_t$ so that the pointwise ergodic theorem yields

$$E[A_t] = \lim_{n \to \infty} \frac{\sum_{j=1}^{n} A_t \circ \theta^j}{n} \leqslant A_t.$$

Thus, A_t is a constant for all t so that $W - V$ is a constant as well. We have thus established that in the case $V < \infty$, W is necessarily of the form $W = V + c$, $c \geqslant 0$. Now, let us show that c can only be different from zero if (3.11) is satisfied. We have

(3.14) $$g(V+c) = g(W) = W \circ \theta = V \circ \theta + c = g(V) + c.$$

Since $g(w) = \max_{1 \leqslant j \leqslant N} h_j(t_1, w+\sigma)$, (3.14) implies that $g(w)$ increases linearly for $w > V$. The properties of the functions $w \to h_j(t_1, w)$ established in Section 3.2 imply that there exists a j_0 such that $V > b_{j_0}(t_1) - \sigma$ and

(3.15)
$$\begin{aligned} g(V) &= V + \sigma + a_{j_0}(t_1) - t_1 \\ &= V + \sigma + \max_{1 \leqslant j \leqslant N} a_j(t_1) - t_1 . \end{aligned}$$

Thus, since $g(V) = V \circ \theta$, (3.15) implies that $V \circ \theta - V$ is integrable and that

(3.16)
$$E^0[V \circ \theta - V] = E^0[\sigma] + E^0\left[\max_{1 \leqslant j \leqslant N} a_j(t_1)\right] - E^0[t_1].$$

Since $E^0[V_n \circ \theta - V_n] = 0$ for all n, and due to the bound (3.9) on $g(w) - w$, we conclude that the left hand term in (3.16) is zero which completes the proof of the theorem.

Theorem 3. If

(3.17)
$$E^0[\sigma] + E^0\left[\max_{1 \leqslant j \leqslant N} a_j(t_1)\right] > E^0[t_1]$$

then (3.1) has no finite solution. ■

Proof. Using the fact that $b_j(t) \geqslant [t - a_j(t)]^+$, we obtain

(3.18)
$$g(w) \geqslant \left[w + \sigma + \max_{1 \leqslant j \leqslant N} a_j(t_1) - t_1\right]^+ = k(w).$$

Hence, if we consider the sequence W_n on $(\Omega^0, \mathscr{A}^0, \mathbf{P}^0)$ defined by $W_0 = 0$ and $W_{n+1} \circ \theta = k(W_n)$, $n \geqslant 0$, we obtain immediately by induction from (3.18) that $V_n \geqslant W_n$, $n \geqslant 0$. Now, classical results on $G/G/1$ imply that if (3.17) is satisfied, then $W_n \to \infty$ a.s. ■

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