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On the equivalence of A-stability and G-stability

Equivalence entre A-stabilité et G-stabilité

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Abstract : We prove an algebraic result which gives a new characterization of the A-stability property for linear multistep methods. This enables us to obtain a new simple proof of the equivalence between A-stability and G-stability for one-leg methods. Using the algebraic result, we also give an alternative proof of the second Dahlquist barrier.

Résumé : On montre un résultat algébrique qui fournit une nouvelle caractérisation de la A-stabilité pour les méthodes linéaires à pas multiples. Nous en déduisons une démonstration simple de l'équivalence entre A- et G-stabilité ainsi que la seconde barrière de Dahlquist.

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" On the equivalence of A-stability and G-stability."

by Claudio Baiocchi and Michel Crouzeix.

Abstract : We prove an algebraic result which gives a new characterization of the A-stability property for linear multistep methods. This enables us to obtain a new simple proof of the equivalence between A-stability and G-stability for one-leg methods. Using the algebraic result, we also give an alternative proof of the second Dahlquist barrier.

1- The algebraic result.

Given two polynomials ρ and σ , of degree $r \geq 1$, with real coefficients, we consider the properties

(A) " $|z| \geq 1 \Rightarrow \operatorname{Re} \rho(z)\, \sigma(\bar{z}) \geq 0$ "

and

(B) there exist (r+1) real polynomials $p_1,\dots,p_r$ and q with $\deg(p_k) \leq r-1$ and $\deg(q) \leq r$, such that, for arbitrary $z, w \in \mathbf{C}$,

$$\rho(z)\, \sigma(w) + \rho(w)\, \sigma(z) + (1 - zw) \sum_{k=1}^{r} p_k(z)\, p_k(w) = q(z)\, q(w).$$

Theorem 1 . *The properties* (A) *and* (B) *are equivalent ; furthermore, if the polynomials* ρ *and* σ *are relatively prime, then the polynomials* p_k *are linearly independent .*

Proof . Implication (B) $\Rightarrow$ (A) is obtained by taking $w = \bar{z}$. For the converse, we can assume that the polynomials ρ and σ are relatively prime, otherwise we may divide them by their common factor.

As a first step, we show that there exists a polynomial q , with real coefficients and $\deg(q) \leq r$, such that

$$E(z) \equiv \rho(z)\, \sigma(1/z) + \rho(1/z)\, \sigma(z) = q(z)\, q(1/z) . \tag{1}$$

Indeed, if we exclude the trivial case $E \equiv 0$, the roots of E are different from zero and their number (counted with their multiplicities) cannot exceed 2r; since $E(z) = E(1/z)$, if z_j is a root of E with multiplicity n_j , $1/z_j$ is also a root of E with multiplicity n_j ; when $|z| = 1$, property (A) yields $E(z) = 2 \operatorname{Re} \rho(z)\, \sigma(\bar{z}) \geq 0$, therefore the multiplicities of the unimodular roots of E are even. Then E can be written as follows :

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$$E(z) = a_o \prod_{j:\,|z_j|\geq 1} (z - z_j)^{m_j} (1/z - z_j)^{m_j} ,$$

where $m_j = n_j$ if $|z_j| > 1$ and $m_j = n_j/2$ if $|z_j| = 1$. We have $a_o > 0$ since $E(z) \geq 0$ for $|z| = 1$. Then we can choose $q(z) = \sqrt{a_o} \prod (z - z_j)^{m_j}$; the coefficients of the polynomial q are real, since each complex root is associated with its conjugate.

Now the relation (1) is satisfied ; this means that the polynomial $q(z)\, q(w) - \rho(z)\, \sigma(w) - \rho(w)\, \sigma(z)$ is equal to zero when $zw = 1$. It can then be written as follows

$$q(z)\, q(w) - \rho(z)\, \sigma(w) - \rho(w)\, \sigma(z) = (1 - zw) \sum_{k,\ell=1}^{r} g_{k\ell}\, z^{\ell-1}\, w^{k-1} . \tag{2}$$

By lemma 2 below, there exists a real matrix $A = (a_{ij})$, such that $G = A^T A$; the theorem follows by taking

$$p_k(z) = \sum_{i=1}^{r} a_{ki}\, z^{i-1} .$$

Since the matrix A is regular, the polynomials p_k are linearly independent.

Lemma 2. *The symmetric matrix* $G = (g_{k\ell})$ *is positive definite.*

Proof . For a multiple root z of $\rho + \lambda\sigma$, we have $\rho(z) + \lambda\, \sigma(z) = \rho'(z) + \lambda\, \sigma'(z) = 0$, whence $\rho(z)\, \sigma'(z) - \rho'(z)\, \sigma(z) = 0$; so that the multiple roots have a finite number of possible values. Since the polynomials ρ and σ are supposed relatively prime, the number of values of λ such that the polynomial $\rho(z) + \lambda\, \sigma(z)$ has multiple roots is finite. It follows that there exist a real number λ and r distinct complex numbers z_i satisfying,

$$\lambda > 0 , \qquad \rho(z_i) + \lambda\, \sigma(z_i) = 0 \quad \text{and} \quad \sigma(z_i) \neq 0 , \qquad \text{for } i = 1,\dots, r . \tag{3}$$

Furthermore, property (A) yields

$$|z_i| < 1 , \qquad \text{for } i = 1,\dots,r . \tag{4}$$

Now we consider the Vandermonde, r x r , matrix

$$V = (v_{ij}) \quad \text{where} \quad v_{ij} = z_j^{i-1} ;$$

from (1) we deduce $V^* G\, V = B = (b_{ij})$, where

$$b_{ij} = [\, q(\overline{z}_i)\, q(z_j) - \rho(\overline{z}_i)\, \sigma(z_j) - \rho(z_j)\, \sigma(\overline{z}_i)\,] / (1 - \overline{z}_i\, z_j)$$

$$= [\, q(\overline{z}_i)\, q(z_j) + 2\,\lambda\, \sigma(\overline{z}_i)\, \sigma(z_j)\,] / (1 - \overline{z}_i z_j) .$$

The bound (4) allows the expansion

$$b_{ij} = \sum_{m=0}^{\infty} [\, \bar{z}_i^m \, q(\bar{z}_i) \, q(z_j) \, z_j^m + 2\,\lambda\, \bar{z}_i^m \, \sigma(\bar{z}_i) \, \sigma(z_j) \, z_j^m \,] .$$

Since $\sigma(z_i) \neq 0$, for $i = 1,..., r$, we deduce that the matrix B , and consequently the matrix G , is positive definite .

2. Application 1. The equivalence of A- and G-stability.

In order to approximate the solution of differential equations, we consider the linear multistep method asssociated with the pair of polynomials (ρ, σ) . For a sequence $y_0 , y_1 ,..., y_n ,...$ of complex numbers, we denote by E the unitary shift operator : $E\, y_n = y_{n+1}$. According to G. Dahlquist, the (ρ, σ) method is A-stable if, for all complex numbers λ such that $\mathrm{Re}\, \lambda < 0$ and all sequences y_n satisfying $\rho(E)\, y_n = \lambda\, \sigma(E)\, y_n$, we have $\lim_{n \to \infty} y_n = 0$; or equivalently

$$\forall\, \lambda \in \mathbf{C} \text{ , } \mathrm{Re}\, \lambda < 0 \text{ and } \rho(z) - \lambda\, \sigma(z) = 0 \;\Rightarrow\; |\, z \,| < 1 .$$

Therefore, A-stability implies property (A), (and is equivalent to it, when the polynomials ρ and σ are relatively prime).

Consider the column vector $Y_n = (y_n, y_{n+1}, ..., y_{n+r-1})^T$ in $\mathbf{C}^r$; according to G. Dahlquist, the (ρ, σ) method is G-stable if there exists a symmetric, positive definite matrix G such that

$$\mathrm{Re}\, \rho(E)\, y_n \,.\, \overline{\sigma(E)\, y_n} \leq 0 \quad \Rightarrow \quad Y_{n+1}^{\,*}\, G\, Y_{n+1} \leq Y_n^{\,*}\, G\, Y_n \quad . \tag{5}$$

We have seen that the property (B) implies (2) , therefore

$$q(E)\, y_n \,.\overline{q(E)\, y_n} - 2\, \mathrm{Re}\, \rho(E)\, y_n \,.\, \overline{\sigma(E)\, y_n} \;=\; Y_n^{\,*}\, G\, Y_n \,-\, Y_{n+1}^{\,*}\, G\, Y_{n+1} \;,$$

which gives (5) . Conversely, using the sequence $y_n = z^n$, we obtain the property (A) from (5) . We get in another way , the following result of G. Dahlquist .

Theorem 3. *If the polynomials* ρ *and* σ *are relatively prime , A-stability and G-stability are equivalent.*

We remark also that the proof of theorem 1 provides a way to construct the matrix G.

3. Application 2. The second barrier of Dahlquist.

Theorem 4. *The order of accuracy of a linear multistep A-stable method is limited to two. For the method of order two, the best possible error constant is* 1/12 .

Proof. The order of a (ρ , σ) method is 2 if

$$\rho(e^z) / \sigma(e^z) = z - c\, z^3 + O(z^4), \quad (\text{as } z \to 0) \ ; \tag{6}$$

it is greater than 2 when the error constant c is zero. Furthermore A-stability and (6) imply property (B) and $\sigma(1) \neq 0$. Setting

$$\pi_k(z) = p_k(e^z) / \sigma(e^z) \ , \ \varphi(z) = q(e^z) / \sigma(e^z) \ ,$$

we obtain

$$z + w - c\,(z^3 + w^3) + (1 - e^{z+w}) \sum_{k=1}^{r} \pi_k(z)\, \pi_k(w) = \varphi(z)\, \varphi(w) + O(z^4 + w^4).$$

If we identify the coefficients in the Taylor expansion, we deduce that

$$\varphi(0) = \varphi\,'(0) = 0 \ ,$$

$$\sum_{k=1}^{r} \pi_k(0)^2 = 1 \ , \ \sum_{k=1}^{r} \pi_k(0)\, \pi_k\,'(0) = -1/2 \ , \ \sum_{k=1}^{r} \pi_k\,'(0)^2 = c + 1/6 \ .$$

From the Cauchy-Schwarz inequality $1/4 \leq c + 1/6$, whence $c \geq 1/12$.

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