

# On multistep approximation of semigroups in Banach spaces

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**ON MULTISTEP APPROXIMATION  
OF SEMIGROUPS  
IN BANACH SPACES**

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**ON MULTISTEP APPROXIMATION OF SEMIGROUPS IN BANACH SPACES**  
**APPROXIMATION DE SEMI-GROUPES DANS DES ESPACES DE BANACH PAR DES**  
**METHODES MULTIPAS**

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**Abstract :** We consider a multistep rational approximation of a bounded, strongly continuous semigroup on a Banach space. We study the convergence when the time step converges to zero under a weak form of A-stability assumption.

**Résumé :** Etant donné un semi-groupe fortement continu dans un espace de Banach et son approximation par une méthode rationnelle multipas, nous étudions la convergence sous une hypothèse de A-stabilité affaiblie.

ON MULTISTEP APPROXIMATION OF SEMIGROUPS  
IN BANACH SPACES

by Michel Crouzeix

*Abstract* : we consider a multistep rational approximation of a bounded, strongly continuous semigroup on a Banach space. We study the convergence when the time step converges to zero under a weak form of  $A$ -stability assumption.

### 1. Introduction.

Given a uniformly bounded, strongly continuous semigroup  $e^{tA}$  on a Banach space  $X$ :

$$\|e^{tA}\| \leq C_0, \quad (1)$$

and  $u_0$  in  $X$ , we approximate the value  $u(t) = e^{tA} u_0$  at the time  $t_n = n \Delta t$  by the solution  $u_n$  of the  $q$ -step method

$$u_{n+1} = \sum_{j=0}^{q-1} r_j(\Delta t A) u_{n-j}, \quad n \geq q-1, \quad (2)$$

starting from the procedure

$$u_j = d_j(\Delta t A) u_0, \quad j=1, \dots, q-1, \quad (3)$$

where  $r_j(z), d_j(z)$  are rational functions uniformly bounded for  $\operatorname{Re} z \leq 0$  and  $\Delta t > 0$  denotes the time step.

We first consider the scalar case where  $X = \mathbb{C}$ ,  $u_0 = 1$  and  $A = \lambda \in \mathbb{C}$  with  $\operatorname{Re} \lambda \leq 0$ . In order to obtain a global error  $u(t_n) - u_n = O(\Delta t^p)$ , it is natural to assume a local error in (2) and in (3) of order  $p+1$ , that is to say, setting  $z = \lambda \Delta t$ ,

$$e^{qz} - \sum_{j=0}^{q-1} r_j(z) e^{(q-j-1)z} = O(z^{p+1}) \quad (4)$$

and

$$e^{jz} - d_j(z) = O(z^{p+1}), \quad j=1, \dots, q-1. \quad (5)$$

We introduce the polynomial

$$P(x; z) = x^q - \sum_{j=0}^{q-1} r_j(z) x^{q-j-1}, \quad (6)$$

associated with the linear recursion formula (2); then Condition (4) may be rewritten as

$$P(e^z; z) = O(z^{p+1}). \quad (7)$$

We shall say that Scheme (2) is A-stable with defect  $k \geq 0$  if :  
 for all  $z$  with  $\operatorname{Re} z \leq 0$ , the roots of  $P(\cdot; z)$  lie in the unit disk and the multiplicities of the unimodular roots are less than  $k+1$ .

The integer  $k$  is chosen as small as possible. We remark that, from (7), 1 is a root of  $P(\cdot; 0)$ . The case  $k = 0$  corresponds to the classical notion of A-stability.

**Remark :** With each monic polynomial  $P(\cdot; z)$ , whose coefficients are rational functions of  $z$ , we can associate a method of Form (2); so, we can construct high order schemes in a simple way :

- if the method associated with the polynomial  $P(\cdot; z)$  is of order  $p$  and A-stable with defect  $k$ , the method associated with  $P(\cdot; z)^m$  is of order  $mp+m-1$  and A-stable with defect  $km+m-1$ . Similarly, if another method, associated with  $Q(\cdot; z)$ , is of order  $q$  and A-stable with defect  $r$ , the method associated with  $P(\cdot; z)Q(\cdot; z)$  is of order  $p+q+1$  and A-stable with defect  $k+r+1$ .

The idea to consider this kind of construction was born during some discussions with G.A. Baker, see also [1].

In this note we prove the following theorem :

**Theorem 1.** *If Scheme (2) is A-stable with defect  $k$ , if Conditions (5) and (7) are satisfied and if  $u_0$  belongs to  $D(A^{p+1})$ , then the following inequality holds*

$$\|u(t_n) - u_n\|_X \leq C_1 C_0 t_n^{k+1} \Delta t^{p-k} \|A^{p+1} u_0\|_X,$$

where the constant  $C_1$  depends only on the rational functions  $r_j$  and  $d_j$ .

The proof of this theorem uses the mathematical framework described in the paper of Brenner and Thomée [4] and some technical lemmas; the case of analytical semigroups is considered in Section 3 and the hilbertian case in Section 4. For related works, see [2], [3], [6], [7].

## 2. Proof of Theorem 1.

We introduce the Frobenius matrix  $R(z)$  associated with  $P(\cdot; z)$  and the corresponding linear operator  $R(\Delta t A)$  in  $\mathcal{L}(X^q, X^q)$  defined by

$$R(z) = \begin{pmatrix} r_0 & r_1 & \dots & r_{q-1} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \quad R(\Delta t A) = \begin{pmatrix} R_0 & R_1 & \dots & R_{q-1} \\ I & 0 & \dots & 0 \\ 0 & I & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & I & 0 \end{pmatrix}$$

where  $r_j$  stands for  $r_j(z)$  and  $R_j$  for  $r_j(\Delta t A)$ ; we also consider the vectors in  $X^q$  of approximate and exact solutions :

$$U_n = \begin{pmatrix} u_{n+q-1} \\ u_{n+q-2} \\ \vdots \\ u_n \end{pmatrix}, V_n = \begin{pmatrix} u(t_{n+q-1}) \\ u(t_{n+q-2}) \\ \vdots \\ u(t_n) \end{pmatrix}, U_0 = \begin{pmatrix} d_{q-1}(\Delta t A) u_0 \\ \vdots \\ d_1(\Delta t A) u_0 \\ u_0 \end{pmatrix}, V_0 = \begin{pmatrix} e^{(q-1)\Delta t A} u_0 \\ \vdots \\ e^{\Delta t A} u_0 \\ u_0 \end{pmatrix}$$

Clearly (2) implies

$$U_{n+1} = R(\Delta t A) U_n, V_{n+1} = e^{\Delta t A} V_n;$$

therefore

$$U_n = R(\Delta t A)^n U_0, V_n = e^{n\Delta t A} V_0$$

and we have the error representation formula

$$U_n - V_n = H_n(\Delta t A) u_0 + G_n(\Delta t A) u_0, \quad (8)$$

with

$$H_n(z) = (R(z)^n - e^{nz} I) \begin{pmatrix} e^{(q-1)z} \\ \vdots \\ e^z \\ 1 \end{pmatrix}, G_n(z) = R(z)^n \begin{pmatrix} d_{q-1}(z) - e^{(q-1)z} \\ \vdots \\ d_1(z) - e^z \\ 0 \end{pmatrix} \quad (9)$$

**Lemma 2.** a) If  $z_0$  is not a pole of  $R(z)$ , there exist a neighborhood  $V_0$  of  $z_0$  in  $\bar{C}$  and a constant  $C$  such that

$$\forall n \geq k_0, \forall z \in V_0, \|R(z)^n\| \leq C [ \rho(R(z))^n + n^{k_0} \rho(R(z))^{n-k_0} ],$$

where  $\rho(R(z))$  denotes the spectral radius of  $R(z)$  and  $(k_0+1)$  the highest multiplicity of the eigenvalues of  $R(z_0)$  having modulus  $\rho(R(z_0))$ .

b) If furthermore the eigenvalues of  $R(z_0)$  which have a modulus equal to  $\rho(R(z_0))$  are differentiable over  $V_0$ , there exists a constant  $C$  such that

$$\forall n \geq k_0, \forall z \in V_0, \left\| \frac{d}{dz} (R(z)^n) \right\| \leq C [ n \rho(R(z))^{n-1} + n^{k_0+1} \rho(R(z))^{n-k_0-1} ].$$

**Sketch of proof.** a) In a neighborhood of  $z_0$ , it is possible to find an analytic and invertible matrix  $H(z)$  such that

$$H^{-1}(z) R(z) H(z) = \text{diag}(R_S(z)) \quad (10)$$

is a block diagonal matrix where each square matrix  $R_S(z)$  corresponds to one of the distinct eigenvalues of  $R(z_0)$ . Clearly  $\|R(z)^n\| \leq \max_S \|R_S(z)^n\|$ ;

since the matrix  $R(z)$  is not derogatory and the eigenvalues of  $R(z_0)$  are continuous at the point  $z_0$ , we can find invertible matrices  $P_S(z)$ , continuous at the point  $z_0$ , such that

$$P_S(z)^{-1} R_S(z) P_S(z) = \begin{pmatrix} \lambda_{s_0} & & & & \\ 1 & \lambda_{s_1} & & & \\ & 1 & \ddots & & \\ & & & \ddots & \\ 0 & & & & 1 & \lambda_{s_{k_s}} \end{pmatrix} = J_S(z),$$

where  $\lambda_{s_j}$  stands for  $\lambda_{s_j}(z)$ . Part a) follows from inequalities

$$\|J_S(z)^n\| \leq \rho(J_S(z))^n + n \rho(J_S(z))^{n-1} + \dots + \binom{n}{k_s} \rho(J_S(z))^{n-k_s}$$

and

$$\|R_S(z)^n\| \leq C \|J_S(z)^n\|.$$

b) From (10), we deduce

$$\left\| \frac{d}{dz} (R_S(z))^n \right\| \leq C \max_s (\|R_S(z)^n\| + \left\| \frac{d}{dz} (R_S(z))^n \right\|);$$

the only difficulty is to obtain an estimate of  $\left\| \frac{d}{dz} (R_S(z))^n \right\|$  when

$\rho(R_S(z_0)) = \rho(R(z_0))$ . But, in this case the matrix  $P_S(z)$  may be chosen differentiable and therefore

$$\left\| \frac{d}{dz} (R_S(z))^n \right\| \leq C (\|J_S(z)^n\| + \left\| \frac{d}{dz} (J_S(z))^n \right\|);$$

for the matrix  $J_S(z)^n$  we have the following upper bound

$$\left\| \frac{d}{dz} (J_S(z))^n \right\| \leq C [n \rho(J_S(z))^{n-1} + \dots + \binom{n}{k_s} (n - k_s - 1) \rho(J_S(z))^{n-k_s-1}].$$

**Lemma 3.** *If Scheme (2) is A-stable with defect  $k$ , the unimodular eigenvalues of  $R(iy)$ , when  $y$  is real, are twice differentiable near  $y$ .*

**Proof:** First, we remark that the eigenvalues of  $R(z)$  are the roots of the polynomial  $P(\cdot, z)$ . Without loss of generality, we give the proof only for the case  $y = 0$  and for the eigenvalues which converge to 1 as  $z \rightarrow 0$ . These eigenvalues can be written as Puiseux series

$$\lambda_j(z) = 1 + a_j z^{r_j} + b_j z^{s_j} + o(z^{s_j}),$$

where  $0 < r_j < s_j$  are rational and  $a_j \neq 0$ . Since  $|\lambda_j(z)| \leq 1$  when  $\operatorname{Re} z \leq 0$ , we cannot have  $r_j > 1$ ; neither can  $r_j = p_j / q_j$  be  $< 1$ , since each determination of  $z^{1/q_j}$  has to be considered. Therefore we have  $r_j = 1$  and  $a_j > 0$ . Considering the case  $z = iy$ , we must have  $0 \leq 1 - \operatorname{Re} \lambda_j(iy) \approx -\operatorname{Re}(b_j (iy)^{s_j})$ , which implies  $s_j \geq 2$ .

Using the compactness of  $\bar{\mathbf{R}}$  and the previous lemma, we obtain the following corollary.

**Corollary 4.** *If the scheme is A-stable with defect  $k$ , there exists a constant  $C$  such that*

$$\begin{aligned} \|R(iy)^n\| &\leq C n^k, \\ \forall n \geq 1, \forall y \in \mathbf{R}, \\ \left\| \frac{d}{dy} R(iy)^n \right\| &\leq C n^{k-1}. \end{aligned}$$

Now we resume the study of the convergence estimates. Since  $H_1(z) = (R(z) - e^z I)(e^{(q-1)z}, \dots, e^z, 1)^T = (P(e^z; z), 0, \dots, 0)^T = O(z^{p+1})$ , we obtain from (9) and (7)

$$H_n(z) = \sum_{j=0}^{n-1} e^j z R(z)^{n-j-1} H_1(z) = O(z^{p+1}).$$

Similarly, using (5), we have  $G_n(z) = O(z^{p+1})$ . Therefore the functions

$$\tilde{H}_n(z) = H_n(z)/z^{p+1}, \quad \tilde{G}_n(z) = G_n(z)/z^{p+1} \quad (11)$$

are analytic and uniformly bounded in the halfplane  $\text{Re } z \leq 0$ . Using the background described in Brenner-Thomée, there exist two functions  $\mathfrak{H}_n$  and  $\mathfrak{G}_n$  in  $L^1(\mathbb{R})^q$  such that, for  $\text{Re } z \leq 0$ ,

$$\tilde{H}_n(z) = \int_0^{\infty} e^{tz} \mathfrak{H}_n(t) dt, \quad \tilde{G}_n(z) = \int_0^{\infty} e^{tz} \mathfrak{G}_n(t) dt, \quad (12)$$

and

$$\mathfrak{H}_n(t) = \mathfrak{G}_n(t) = 0 \text{ for } t < 0. \quad (13)$$

We define the two operators in  $L(X^q, X^q)$

$$\tilde{H}_n(\Delta t A) = \int_0^{\infty} e^{t \Delta t A} \mathfrak{H}_n(t) dt, \quad \tilde{G}_n(\Delta t A) = \int_0^{\infty} e^{t \Delta t A} \mathfrak{G}_n(t) dt, \quad (14)$$

and we have from (8) and (11), if  $u_0$  is in  $D(A^{p+1})$ ,

$$U_n - V_n = \Delta t^{p+1} \tilde{H}_n(\Delta t A) A^{p+1} u_0 + \Delta t^{p+1} \tilde{G}_n(\Delta t A) A^{p+1} u_0.$$

Now, Theorem 1 is a simple consequence of the following lemma:

**Lemma 5.** *Under the hypotheses of Theorem 1, there exists a constant  $C$  such that*

$$\|\tilde{H}_n(\Delta t A)\| \leq C C_0 n^{k+1} \text{ and } \|\tilde{G}_n(\Delta t A)\| \leq C C_0 n^{k+1/2}.$$

**Proof.** We have, from (14) and (1),

$$\|\tilde{H}_n(\Delta t A)\| \leq C_0 \|\mathfrak{H}_n\|_{L^1(\mathbb{R})^q} \text{ and } \|\tilde{G}_n(\Delta t A)\| \leq C_0 \|\mathfrak{G}_n\|_{L^1(\mathbb{R})^q},$$

therefore it is sufficient to prove that

$$\|\mathfrak{H}_n\|_{L^1(\mathbb{R})^q} \leq C n^{k+1} \text{ and } \|\mathfrak{G}_n\|_{L^1(\mathbb{R})^q} \leq C n^{k+1/2}. \quad (15)$$



Let us introduce the functions :  $\hat{\mathfrak{H}}_n(y) = H_n(iy)$ ,  $\hat{\mathfrak{G}}_n(y) = G_n(iy)$  ;  
 from (12) and (13),  $\hat{\mathfrak{H}}_n$  and  $\hat{\mathfrak{G}}_n$  are the Fourier transforms of  $\mathfrak{H}_n$  and  $\mathfrak{G}_n$ .  
 A Carlson's inequality gives

$$\|\hat{\mathfrak{G}}_n\|_{L^1(\mathbb{R})^q} \leq C \|\hat{\mathfrak{G}}_n\|_{L^2(\mathbb{R})^q}^{1/2} \|\hat{\mathfrak{G}}_n'\|_{L^2(\mathbb{R})^q}^{1/2} ;$$

from Corollary 4, we have

$$\|\hat{\mathfrak{G}}_n(y)\| \leq C n^k \min\left(\frac{1}{|y|^{p+1}}, 1\right)$$

and

$$\|\hat{\mathfrak{G}}_n'(y)\| \leq C n^{k+1} \min\left(\frac{1}{|y|^{p+1}}, 1\right)$$

therefore

$$\|\hat{\mathfrak{G}}_n\|_{L^2(\mathbb{R})^q} \leq C n^k, \quad \|\hat{\mathfrak{G}}_n'\|_{L^2(\mathbb{R})^q} \leq C n^{k+1},$$

which shows the second inequality in (15).

In order to prove the first inequality, we introduce the notations

$$S(z) = \begin{pmatrix} s_0 & s_1 & \dots & s_{q-1} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \end{pmatrix} \quad D(z) = \begin{pmatrix} e^{(q-1)z} & & & \\ & \ddots & & \\ & & 0 & \\ & & & e^z \\ & & & & 1 \end{pmatrix}$$

where  $s_j$  stands for  $s_j(z) = r_j(z) e^{-(j-1)z}$ . We have

$R(z) = e^z D(z) S(z) D(z)^{-1}$ , therefore

$$\hat{\mathfrak{H}}_n(y) = \frac{1}{(iy)^{p+1}} e^{niy} D(iy) (S(iy)^{n-1}) E = e^{niy} D(iy) \hat{T}_n(y) \quad (16)$$

where  $E = (1, 1, \dots, 1)^T$  and

$$\hat{T}_n(y) = \frac{1}{(iy)^{p+1}} (S(iy)^n - I) E. \quad (17)$$

We remark that  $\hat{T}_n$  is the Fourier transform of  $T_n$  which, from (16), is related to  $\mathfrak{H}_n$  by the relation, (on the  $j^{\text{th}}$  component),

$$\mathfrak{H}_n(t)_j = T_n(t - (n+q-j))_j$$

therefore, using a Carlson's inequality, we obtain

$$\|\mathfrak{H}_n\|_{L^1(\mathbb{R})^q} \leq C \|T_n\|_{L^1(\mathbb{R})^q} \leq C \|\hat{T}_n\|_{L^2(\mathbb{R})^q}^{1/2} \|\hat{T}_n'\|_{L^2(\mathbb{R})^q}^{1/2} .$$

The following lemma completes the proof of the first inequality in (15).

**LEMMA 6** . If the multiplicity of the root 1 of  $P(\cdot; 0)$  is  $k+1$ , then we have

$$\|\hat{T}_n\|_{L^2(\mathbb{R})} \leq C n^{k+1-1/(2r+2)} \quad \text{and} \quad \|\hat{T}_n'\|_{L^2(\mathbb{R})} \leq C n^{k+1+1/(2r+2)},$$

where  $r$  is a rational (which will be defined in the proof).

If the multiplicity is less than or equal to  $k$ , then

$$\|\hat{T}_n\|_{L^2(\mathbb{R})} \leq C n^k \quad \text{and} \quad \|\hat{T}_n'\|_{L^2(\mathbb{R})} \leq C n^{k+1}.$$

**Proof** . We consider only the case where the multiplicity of the root 1 is  $k+1$ , the other case is easier. Relation (17) and Corollary 4 imply

$$\|\hat{T}_n(y)\| \leq C n^k / |y|^{p+1} \quad \text{and} \quad \|\hat{T}_n'(y)\| \leq C(n^k / |y|^{p+2} + n^{k+1} / |y|^{p+1}). \quad (18)$$

In the definition of  $S(z)$ , we remark that  $\mu(z)$  is an eigenvalue of  $S(z)$  if and only if  $\lambda(z) = e^z \mu(z)$  is an eigenvalue of  $R(z)$ , and that the algebraic multiplicity is preserved. From Lemma 2, we can find, in a neighborhood  $\mathcal{V}$  of zero on the imaginary axis, an invertible and twice differentiable matrix  $H(z)$  such that

$$\forall z \in \mathcal{V}, \quad S(z) = H(z)^{-1} \begin{pmatrix} A(z) & 0 \\ 0 & B(z) \end{pmatrix} H(z), \quad (19)$$

where the matrix  $B(z) - I$  is invertible and the  $(k+1) \times (k+1)$  matrix  $A(z)$  has the form

$$A(z) = \begin{pmatrix} \mu_0(z) & & & \\ 1 & \mu_1(z) & & 0 \\ & 1 & \ddots & \\ 0 & & \ddots & 1 & \mu_k(z) \end{pmatrix}, \quad \text{with} \quad \mu_j(0) = 1.$$

Now we introduce the twice differentiable vectors  $A_n(z)$ ,  $B_n(z)$ ,  $a(z)$ ,  $b(z)$ ,  $\beta(z)$  by

$$\begin{pmatrix} A_n(z) \\ B_n(z) \end{pmatrix} = H(z) \hat{T}_n(z), \quad \begin{pmatrix} a(z) \\ b(z) \end{pmatrix} = H(z) E, \quad \beta(z) = \frac{1}{z^{p+1}} b(z); \quad (20)$$

then

$$A_n(z) = \frac{1}{z^{p+1}} (A(z)^n - I) a(z), \quad B_n(z) = (B(z)^n - I) \beta(z). \quad (21)$$

Since  $\beta(z) = (B(z) - I)^{-1} B_1(z)$  is twice differentiable, we obtain

$$\|B_n(iy)\| \leq C n^k, \quad \|B_n'(iy)\| \leq C n^{k+1}, \quad \text{for } iy \in \mathcal{V}. \quad (22)$$

We assume that the order of the eigenvalues  $\mu_j(z)$  of  $A(z)$  has been chosen in order to ensure

$$0 \leq r = r_0 \leq r_1 \leq \dots \leq r_k,$$

for the first exponent of the Puiseux expansion

$$\mu_j(z) = 1 + a_j z^{r_j+1} + \dots, \quad r_j \in \mathbb{Q}, \quad a_j \neq 0.$$

Noticing that

$$A_n(z) = \sum_{j=0}^{n-1} A(z)^j A_1(z) \quad \text{and} \quad \|A'(z)\| \leq C |z|^\Gamma,$$

we get

$$\|A_n(z)\| \leq C n^{k+1}, \quad \|A_n'(z)\| \leq C n^{k+1} (1 + n |z|^\Gamma). \quad (23)$$

With the notations

$$a(z) = (a_0(z), \dots, a_k(z))^T, \quad e_0 = (1, 0, \dots, 0)^T$$

$$c(z) = (A(z) - I)(a(z) - a_0(z) e_0) = (0, c_1(z), \dots, c_k(z))^T, \quad (24)$$

(  $c(z) = 0$  when  $k = 0$  ),

we can also write

$$A_n(z) = \frac{1}{z^{p+1}} \left[ a_0(z) (A(z)^n - I) e_0 + \sum_{j=0}^{n-1} A(z)^j c(z) \right]. \quad (25)$$

Since the first row and the first column of  $A(z)$  are not used in  $A(z)^j c(z)$ , this term is bounded by  $C j^{k-1} \|c(z)\|$ . Then

$$\|A_n(z)\| \leq C n^k (|a_0(z)| + \|c(z)\|) / |z|^{p+1}. \quad (26)$$

Similarly, by derivation of (25), we obtain

$$\|A_n'(z)\| \leq (p+1) \|A_n(z)\| / |z| + C n^k (|a_0'(z)| + \|c'(z)\|) / |z|^{p+1} + C n^{k+1} |z|^\Gamma (|a_0(z)| + \|c(z)\|) / |z|^{p+1}. \quad (27)$$

From the relations

$$(A(z) - I) a(z) = z^{p+1} A_1(z),$$

$$(\mu_0(z) - 1) a_0(z) = z^{p+1} (A_1(z))_0,$$

we get

$$a_0(z) = O(z^{p-r}), \quad a_0'(z) = O(z^{p-r-1}),$$

$$c(z) = z^{p+1} A_1(z) - a_0(z) (A(z) - I) e_0 = O(z^{p-r}),$$

and

$$c'(z) = O(z^{p-r-1}).$$

Together with (26), (27) and (23), it yields

$$\|A_n(z)\| \leq C n^k / |z|^{r+1}, \quad \|A_n'(z)\| \leq C n^{k+1} / |z| \text{ for } z \in \mathcal{V}.$$

Considering (18), (22) and (23), we have proved that, for all  $y \in \mathbb{R}$ ,

$$\|\hat{T}_n(y)\| \leq C n^k \min(n, 1/|y|^{r+1}),$$

$$\|\hat{T}_n'(y)\| \leq C n^{k+1} \min(1+n|y|^r, 1/|y|).$$

The lemma follows from a simple calculation.

### 3. The case of holomorphic semigroups.

In this section, we make the stronger assumption that  $A$  generates a holomorphic semigroup on  $X$ ; more precisely we also assume that the spectrum of  $A$  is included in the sector  $S_\theta$  and

$$\forall z \in \mathbb{C} \setminus S_\theta, \quad \|(zI - A)^{-1}\| \leq C / |z|, \quad (28)$$

where  $\theta \in ]0, \pi/2[$ ,  $C$  is a constant and

$$S_\theta = \{z \in \mathbb{C}; \pi - \theta \leq |\text{Arg } z| \leq \pi \text{ or } z = 0\}.$$

We can make weaker assumptions on the scheme; we assume that the rational functions  $r_j$  and  $d_j$  are uniformly bounded in  $S_\theta$  and satisfy

$$P(e^z; z) = O(z^{p+1}) \quad (7)$$

and

$$e^z - d_j(z) = O(z^p), \quad j=1, \dots, q-1; \quad (29)$$

the last requirement is weaker than (5). We suppose also that the method is  $A(\theta)$ -stable with defect  $k$ , that is to say  
*for all  $z$  belonging to  $S_\theta$ , the roots of  $P(\cdot; z)$  lie in the unit disk and the multiplicities of the unimodular roots are less than  $k+1$ .*

We need also the following hypothesis: there exist  $\eta > 0$  and  $\mu$ ,  $0 < \mu < \cos \theta$ , such that

$$\begin{aligned} & \text{for all root } \lambda_j(z) \text{ of } P(\cdot; z) \text{ such that } \lambda_j(0) \text{ is} \\ & \text{unimodular and of multiplicity } k+1, \text{ we have} \\ & \forall z \in S_\theta \text{ with } |z| \leq \eta, \quad |\lambda_j(z)| \leq e^{-\mu|z|}. \end{aligned} \quad (30)$$

We have the following theorems:

**Theorem 7.** *If the scheme is  $A(\theta)$ -stable with defect  $k$ , if conditions (7), (28), (29) and (30) are satisfied and if  $u_0$  belongs to  $D(A^p)$ , then the following inequality holds*

$$\|u(t_n) - u_n\|_X \leq C t_n^k \Delta t^{p-k} \|A^p u_0\|_X$$

**Theorem 8.** *If the same assumptions are satisfied but  $u_0$  belongs to  $X$ , if furthermore, for all  $z \in S_\theta - \{0\}$  and for  $z = \infty$ , the roots of  $P(\cdot; z)$  lie in the open unit disk, then the following inequality holds*

$$\|u(t_n) - u_n\|_X \leq C \Delta t^{p-k} / t_n^{p-k} \|u_0\|_X$$

**Proof of Theorem 7.** We use the same representation formula (8)

$$U_n - V_n = H_n(\Delta t A) u_0 + G_n(\Delta t A) u_0, \quad (8)$$

as for Theorem 1, but instead of (11), we use the functions

$$\tilde{H}_n(z) = H_n(z)/z^p, \quad \tilde{G}_n(z) = G_n(z)/z^p. \quad (31)$$

Then

$$U_n - V_n = \Delta t^p (\tilde{H}_n(\Delta t A) + \tilde{G}_n(\Delta t A)) A^p u_0$$

and

$$\|U_n - V_n\|_X \leq C \Delta t^p (\|\tilde{H}_n(\Delta t A)\| + \|\tilde{G}_n(\Delta t A)\|) \|A^p u_0\|_X \quad (32)$$

Since  $\|\tilde{H}_n(z)\| \leq C_n \min(|z|, |z|^{-p})$ , we can use the Dunford-Taylor spectral representation

$$\tilde{H}_n(\Delta t A) = \frac{1}{2\pi i} \int_{\Gamma_\theta} (zI - \Delta t A)^{-1} \tilde{H}_n(z) dz, \quad (33)$$

where  $\Gamma_\theta$  denotes the oriented boundary of  $S_\theta$ .

From Lemma 2 and (30), we deduce that

$$\forall z \in \Gamma_\theta, \quad \|R(z)^n\| \leq C n^k,$$

and

$$\forall z \in \Gamma_\theta \text{ with } |z| \leq \eta, \quad \|R(z)^n\| \leq C n^k e^{-\mu n |z|};$$

therefore

$$\forall z \in \Gamma_\theta \text{ with } z \neq 0, \quad \|\tilde{H}_n(z)\| \leq C n^k |z|^{-p}$$

and, since from (9),  $\tilde{H}_n(z) = \sum_{j=0}^{n-1} e^{jz} R(z)^{n-j-1} \tilde{H}_1(z) = O(z^{p+1})$ .

$$\forall z \in \Gamma_\theta \text{ with } |z| \leq \eta, \quad \|\tilde{H}_n(z)\| \leq C n^{k+1} e^{-\mu n |z|} |z| \quad (34)$$

Then, using (33) and (28),

$$\begin{aligned} \|\tilde{H}_n(\Delta tA)\| &\leq \frac{1}{\pi} \int_0^\infty \frac{C}{r} \tilde{H}_n(re^{i\theta}) dr, \\ &\leq C \int_0^\eta n^{k+1} e^{-\mu nr} dr + C n^k \int_\eta^\infty r^{-p-1} dr, \\ &\leq C n^k. \end{aligned}$$

We cannot use directly the Dunford-Taylor spectral representation for  $\tilde{G}_n(\Delta tA)$  when  $\tilde{G}_n(0) \neq 0$ , but we can write

$$\tilde{G}_n(\Delta tA) = e^{n\Delta tA} \tilde{G}_n(0) + \frac{1}{2\pi i} \int_{\Gamma_\theta} (zI - \Delta tA)^{-1} (\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)) dz;$$

therefore

$$\|\tilde{G}_n(\Delta tA)\| \leq C n^k + \frac{1}{\pi} \int_0^\infty \frac{C}{r} \|\tilde{G}_n(re^{i\theta}) - \exp(nre^{i\theta}) \tilde{G}_n(0)\| dr,$$

and we obtain easily

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)\| \leq C n^k |z|^{-p},$$

and

$$\forall z \in \Gamma_\theta \text{ with } |z| \leq \eta, \quad \|\tilde{G}_n(z) - e^{nz} \tilde{G}_n(0)\| \leq C n^{k+1} e^{-\mu n|z|} |z|.$$

Thus,  $\|\tilde{G}_n(\Delta tA)\| \leq C n^k$  and Theorem 7 follows from (32).

**Proof of Theorem 8.** From the representation formula (8), we have

$$\|u(t_n) - u_n\|_X \leq C (\|H_n(\Delta tA)\| + \|G_n(\Delta tA)\|) \|u_0\|_X.$$

Now we write

$$H_n(\Delta tA) = (I - e^{\Delta tA}) H_n(\infty) + \frac{1}{2\pi i} \int_{\Gamma_\theta} (zI - \Delta tA)^{-1} (H_n(z) - (1 - e^z) H_n(\infty)) dz.$$

From Lemma 2, there exists  $\beta$ ,  $0 < \beta < 1$ , such that

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|R(z)^n\| \leq C \beta^n;$$

using also  $R(z) - R(\infty) = O(z^{-1})$ , (when  $z \rightarrow \infty$ ), we obtain

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|H_n(z) - (1 - e^z) H_n(\infty)\| \leq C n^{k+1} \beta^n |z|^{-1},$$

and from (34)

$\forall z \in \Gamma_\theta$  with  $|z| \leq \eta$ ,  $\|H_n(z) - (1 - e^z)H_n(\infty)\| \leq C(n^{k+1}e^{-\mu n|z|}|z|^{p+1} + \beta^n |z|)$ .

Therefore

$$\begin{aligned} \|H_n(\Delta t A)\| &\leq C(n^{k+1}\beta^n + \int_0^\eta n^{k+1}e^{-\mu n r} r^p dr), \\ &\leq C(n^{k+1}\beta^n + n^{k-p} \int_0^\infty e^{-\mu t} t^p dt), \\ &\leq C n^{k-p} = C(\Delta t/t_n)^{p-k}. \end{aligned}$$

Similarly we get

$$\forall z \in \Gamma_\theta \text{ with } |z| \geq \eta, \quad \|G_n(z) - (1 - e^z)G_n(\infty)\| \leq C n^{k+1} \beta^n |z|^{-1},$$

and

$\forall z \in \Gamma_\theta$  with  $|z| \leq \eta$ ,  $\|G_n(z) - (1 - e^z)G_n(\infty)\| \leq C(n^k e^{-\mu n|z|}|z|^p + \beta^n |z|)$ , which yields

$$\|G_n(\Delta t A)\| \leq C(\Delta t/t_n)^{p-k}$$

and completes the proof of Theorem 8.

#### 4. Remarks.

Theorem 1 is valid in particular when  $X$  is a Hilbert space but in this case it may be improved by taking Condition (29) in the place of Condition (5). Furthermore an easier proof can be given; indeed, changing possibly the norm in  $X$ , we can assume that  $e^{tA}$  is a semigroup of contraction on  $X$ , i.e.  $\|e^{tA}\| \leq 1$ , ( $\forall t > 0$ ); then, from a theorem of von Neumann, we have

$$\|\tilde{H}_n(\Delta t A)\| \leq C \max_{\operatorname{Re} z = 0} \|\tilde{H}_n(z)\| \leq C n^{k+1},$$

similarly,  $\|\tilde{G}_n(\Delta t A)\| \leq C n^k$ , which completes the proof.

Now, if we assume further that  $A$  is a (semidefinite negative) selfadjoint operator, Theorem 7 and Theorem 8 are valid with  $\theta = 0$ . In this case, the proof becomes very simple since, from the spectral theory,

$$\|H_n(\Delta t A)\| \leq \sup_{x \leq 0} \|H_n(x)\|, \quad \|G_n(\Delta t A)\| \leq \sup_{x \leq 0} \|G_n(x)\|.$$

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