On periodic phenomena of queuing systems M/G/1 with bulk arrivals and batch services

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ON PERIODIC PHENOMENA OF QUEUEING SYSTEM

M/G/1 WITH GROUP ARRIVALS AND BATCH SERVICE

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Abstract

In this paper several models of queueing system M/G/1 with group arrivals and batch service are considered, and the following fundamental questions are replied : <i> What is the structure of the phase space of the imbedded Markov chain ? <ii> What are the sufficient and necessary conditions causing the imbedded Markov chain to be reducible or irreducible and periodic or aperiodic ? <iii> What are the sufficient and necessary conditions of existence of stationary distribution ? The generating function of stationary distribution is obtained.

Résumé

Dans cet article on étudie plusieurs modèles de type M/G/1 avec services et arrivées par groupes. On analyse (en exhibant des conditions nécessaires et suffisantes) les causes de périodicité ainsi que les conditions d'ergodicité. On calcule enfin les distributions stationnaires du nombre de groupes et du nombre de clients.

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certain carelessnesses on the periodicity and the irreducibility [5. p.393]. Sahbazov had also some carelessnesses on the periodicity [6]. Teghem, Loris-Teghem and Lambotte have noted the periodicity and the irreducibility of the imbedded Markov chain, but they did not analyse in detail this problem and in addition it exists also several faults in their conclusion [7]. Zhang discussed the periodicity for M/G/1 with group arrivals and single service [11]. In the present paper, we developed the results of [11] and [7].

Since long time no sufficient attention has been given to this problem and the following fundamental questions have not been solved : What is the structure of the phase space of the imbedded Markov chain ? In what case does the imbedded chain be periodic or aperiodic ? What are the sufficient and necessary condition of existence of stationary distribution ? In this paper, we have analysed concretely several models of queueing system M/G/1 with group arrivals and batch service, and get a clear understanding of the condition of periodicity and irreducibility, the structure of phase space of imbedded Markov chain and the sufficient and necessary conditions of existence of stationary distribution, so that we replied thoroughly these questions and corrected several carelessness or faults of some articles.

2 - ANALYSIS OF SYSTEM

In the queueing system M/G/1 discussed here, the following assumptions are made :

1) Customers arrive in groups. The arrival instants process is a homogeneous Poisson process with parameter λ. Denote the total number of customers of the nth arrival group by η_n^*. $\{\eta_n^*\}_{n\geq 1}$ are i.i.d.r.v's (the independent and identically distributed random variables) with the common g.f. (generating function)

$$\phi(x) = Ex^{\eta_n^*} = \sum_{\gamma=1}^{\infty} \phi_\gamma x^r, \tag{2.1}$$

and its expectation $E\eta_n^* = d \geq 1$.

2) The service time of the n-th batch is Y_n. $\{Y_n\}_{n\geq 1}$ are i.i.d. non-negative r.v's with the common d.f. (distribution function) B(t) and the expectation b >0.

3) Customers are served in batches. We denote the capacity for service of the n-th batch by K_n. $\{K_n\}_{N\geq 1}$ are i.i.d. r.v's with the common g.f.

$$\psi(x) = EX^{K_n} = \sum_{r=1}^{\infty} \psi_r x^r \tag{2.2}$$

If the number of customers in the system at the moment that the nth batch begins to be served is not less than K_n, then K_n customers are served. However, if this number is less than K_n, then the present customers are completely served. The customers arrriving later will be served in the postrior batches. The families of the above r.v'.s are independent. We shall denote all the above r.v's by F.

For the queueing system described above, the three cases are distinguished in detail, viz.

a) If the customers arrive at an empty system, then they will be served after an additive random waiting time Z_n. $\{Z_n\}_{n\geq 1}$ are i.i.d. whose d.f. is C(t) with the expectation $c\geqq 0$. This discipline can be explained practically by "servers idles, system closed". The model of individual service for this queueing system is considered in [6]. Again if in addition $Z_n \equiv 0$, then the model transforms into the model considered by Miller in [4].

b) The arriving customers at the idle moment of system have a special service time Y'_n. $\{Y'_n\}_{n\geq 1}$ are i.i.d. whose common d.f. is B'(t), and the expectation $b'>0$. $\{Z_n\}_{n\geq 1}$, $\{Y'_n\}_{n\geq 1}$ are independent of F.

For the individual service, a) is a particular case of b).

c) As for the service of fixed duration, for example, whether there are customers or no, the train shall depart on time. The system for individual arrival and single service transforms into the ordinary M/G/1 system.

Several authors considered another service discipline that is a little different from 3), viz to start the service if and only if the number of waiting customers is not less than the service capacity.

In this paper, this case is not discussed.

By means of the theory of I.M.C. (imbedded Markov chain), taking the moments of the batches departure from the system to be renewal points, and the number of customers in system, i.e. the length of queue (containing the customers being served) to be the states of system, we obtain the following I.M.C. for (a), (b) and (c) respectively :

$$\text{(a)}\quad \xi_{n+1} = \begin{cases} \max(\xi_n - K_{n+1}, 0) + V_{n+1}, & \xi_n > 0 \\ \max(\zeta_{n+1} + \eta_{n+1} - K_{n+1}, 0) + V_{n+1}, & \xi_n = 0 \end{cases} \qquad (2.3)$$

$n=0,1,2,\ldots,$

where ξ_0 is the length of queue at $t=0$,

ξ_n is the length of queue at the instant r_n+0, where r_n is the time of the nth batch departure,

v_n is the number of arriving customers during the nth batch service,

ζ_{n+1} is the number of the first group of arriving customers after $\xi_n=0$, the g.f. of ζ_{n+1} is (2.1),

η_{n+1} is the number of arriving customers before the starting of service for the first group of arriving customers after $\xi_n=0$ during the additional time interval Z_n.

$$\text{(b)}\quad \xi_{n+1} = \begin{cases} \max(\xi_n - K_{n+1}, 0) + V_{n+1}, & \xi_n > 0 \\ \max(\zeta_{n+1} - K_{n+1}, 0) + V'_{n+1}, & \xi_n = 0 \end{cases} \qquad (2.4)$$

$n=0,1,2,\ldots,$

where V'_{n+1} is the number of arriving customers during the special service time y'_n when $\xi_n=0$. The rest of the r.v's have same meanings as above.

$$\text{(c)}\quad \xi_{n+1} = \begin{cases} \max(\xi_n - K_{n+1}, 0) + V_{n+1}, & \xi_n > 0 \\ V_{n+1}, & \xi_n = 0 \end{cases} \qquad (2.5)$$

$n=0,1,2,\ldots,$

where the r.v's have the same meanings as above.

It follows from the above assumptions that all of $\{V_n\}$, $\{\eta_n\}$, $\{\zeta_n\}$ and $\{V'_n\}$ are respectively the families of i.i.d. r.v's.

It is clear from (a) that the common g.f. of η_n is

$$H(x)=Ex^{\eta_n}=\sum_{i=0}^{\infty} h_i x^i=\sum_{i=0}^{\infty} x^i\int_0^{\infty} P\,(\text{i customers arrive during}$$

$$z_n \mid Z_n=t)\,dC(t)=\int_0^{\infty}\sum_{i=0}^{\infty} V_i(t)\,x^i dC(t)=\int_0^{\infty} e^{-\lambda t(1-\phi(x))}dC(t)=\tilde{C}(\lambda-\lambda\phi(x))$$

where $\tilde{C}$ is the L.S. transform (i.e. Laplace-Stieltjes transform) of $C(t)$.

It follows from the assumption (2) that the common g.f. of V_n is

$$K(x)=Ex^{V_n}=\sum_{i=0}^{\infty} k_i x^i=\sum_{i=0}^{\infty} x^i\int_0^{\infty} P(\text{i customers arrive during } Y_n \mid Y_n=t)$$

$$dB(t)=\int_0^{\infty}\sum_{i=0}^{\infty} x^i\, V_i(t)\,dB(t)=\int_0^{\infty} e^{-\lambda t(1-\phi(x))}dB(t)=\tilde{B}(\lambda-\lambda\phi(x)),$$

where $\tilde{B}$ is the L.S. transform of $B(t)$.

Noting that $(\zeta_{n+1},\eta_{n+1},V_{n+1},K_{n+1})$ and $(\xi_0,\xi_1,\dots,\xi_n)$ are independent, it is seen that $\{\xi_n\}_{n\geq 1}$ for (a) is a M.C. It follows by the same way that $\{\xi_n\}_{n\geq 1}$ for (b) and (c) are also M.C.

When $K_n \equiv 1$, (a) can be considered as a special case of (b), because we can take Y_n+Z_n to be Y'_n and $\eta_{n+1}+V_{n+1}$ to be V'_{n+1}, so that

$$\max\{\zeta_{n+1}+\eta_{n+1}-K_{n+1},0\}+V_{n+1}=\zeta_{n+1}+\eta_{n+1}-1+V_{n+1}=\zeta_{n+1}-1+V'_{n+1}=\max\{\zeta_{n+1}-K_{n+1},0\}+v'_{n+1}.$$

It should be noted that when $k_n \not\equiv 1$, (a) is uncertainly a special case of (b). For example, if $\xi_n=0$, $\zeta_{n+1}=1$, $\eta_{n+1}=3$, $K_{n+1}=5$, $V_{n+1}=2$, $V'_{n+1}=\eta_{n+1}+V_{n+1}=5$, then $\xi_{n+1}=2$ for (a) and $\xi_{n+1}=5$ for (b). Because for (a) the arriving customers during Z_n can be served together with the first group of arriving customers after $\xi_n=0$; but for (b) they will be served in posterior batches.

Let $Z_n \equiv 0$ (i.e. $\eta_{n+1}\equiv 0$) for (a) or $Y'_n\equiv Y_n$ (i.e. $V'_{n+1}\equiv V_{n+1}$) for (b). We obtain the model considered by Meller. For the case of single arrival and individual service, (c) is the ordinary system M/G/1, and so is (a) when $Z_n \equiv 0$ or (b) when $Y'_n = Y_n$. But these deduced systems M/G/1 have a little difference.

Finally, it should be noted that when $\xi_n > 0$ the systems (a), (b) and (c) have the common formula, viz only the first row of their transition matrixes of I.M.C. are different. Thus we shall investigate mainly the system (a).

We define the set

$$P \overset{\text{def}}{=} \{i \mid P(\eta_n^* = i) > 0,\ i \in N_+ = \{1,2,\dots\}\},$$

whose elements will be briefly called the possible values of η_n^*.

Now we consider the g.f. of V_n, V'_n and η_n. It is obvious that

$$K(x) = EX^{V_n} = \int_0^\infty e^{-\lambda t(1-\phi(x))} dB(t),$$

$$\bar{K}(x) = EX^{V'_n} = \int_0^\infty e^{-\lambda t(1-\phi(x))} dB'(t),$$

$$H(x) = EX^{\eta_n} = \int_0^\infty e^{-\lambda t(1-\phi(x))} dC(t),$$

where $\phi(x)$ is the g.f. of η_n^*.

Noting

$$K(x) = \sum_{j=0}^\infty P(V_n = j)X^j = \int_0^\infty e^{-\lambda t} e^{\lambda t\phi(x)} dB(t) = \sum_{h=0}^\infty \int_0^\infty \frac{(\lambda t)^h [\phi(x)]^h}{h!} e^{-\lambda t} dB(t)$$

$$= \sum_{h=0}^\infty \frac{[\phi(x)]^h}{h!} \int_0^\infty (\lambda t)^h e^{-\lambda t} dB(t),$$

and $\int_0^\infty (\lambda t)^h e^{-\lambda t} dB(t) > 0$, then, we consider only $\{(\phi(x))^h\}_{h \geq 0}$. If $P(V_n = j) > 0$, it exists a certain $[\phi(x)]^s$ whose coefficient of the term X^j is positive. Since this coefficient is the jth term of the s-fold convolution of $\{\phi_r\}$, j is a finite sum of several possible values of η_n^*.

We note that if $j_1, j_2 \in N = \{0,1,2,\dots\}$, $P(V_n = j_1) > 0$, $P(V_n = j_2) > 0$, then $S_1, S_2 \in N$ so that the coefficient of $[\phi(s)]^{S_1}$, $\ell_1 > 0$ and the coefficient of $[\phi(x)]^{S_2}$, $\ell_2 > 0$. From the fact that the coefficient of $x^{j_1+j_2}$ in $[\phi(x)]^{S_1+S_1}$ is greater than zero it follows that $P(V_n = j_1 + j_2) > 0$. This fact shows that the set of possible values of V_n, denoting by **G**, is closed for addition, in other words, **G** is a semimodule (i.e. additive semigroup formed by integers), and **P** is the generator of **G**.

It follows similarly that the sets of possible values of η_n, V'_n and $\zeta_n+\eta_n$ are also **G**.

We denote the set of possible values of service capacity K_n by $\overline{G}$, and the g.c.d. (greatest common divisor) of all elements of $\overline{G}$ by $\overline{\sigma}$, and the g.c.d. of all integers of **G** (also **P**) by σ, and the g.c.d. of σ and $\overline{\sigma}$ by m.

Arranging all integers of **G** in increasing sequence, i.e. $n_1<n_2<\dots$, and denoting the g.c.d. of first i numbers by t_i, then it is clear that $t_1\geq t_2\geq\dots\geq\sigma\geq1$. Because t_1-1 is finite, it follows immediately that $\ell\in N_+=\{1,2,\dots\}$ so that $t_\ell=t_{\ell+1}=\dots=\sigma$, viz σ is the g.c.d. of $n_1,n_2,\dots,n_\ell$. We obtain from the elementary number theory that $n_0\in N_+$, $\forall n\geq n_0$,

$$n\sigma = \alpha_1n_1+\alpha_2n_2+\dots+\alpha_\ell n_\ell, \quad \alpha_i\in N_+,\ 1\leq i\leq\ell \tag{2.6}$$

holds. It follows that every element of **G** takes the form of $n\sigma$, $n\in N_+$, and $n_0\in N_+$ so that $\forall n\geq n_0$, $n\sigma\in$ **G**.

<u>**Theorem 1**</u> The phase spaces of M.C. definited by (2.3), (2.4) and (2.5) contain the irreducible close set

$$G^* = \{km \mid k=0,1,2,\dots\},$$

the rest of states are transient (in fact, inessential)

<u>proof</u> By the same argument as above, we obtain that for **G** and $\overline{\mathbf{G}}$, $n_0,\overline{n}_0\in N_+$ so that $\forall$ $t\geq n_0$, $\overline{t}\geq\overline{n}_0$, $\{\alpha_i\}_{1\leq i\leq\ell}$, $\{\overline{\alpha}_j\}_{1\leq j\leq s}\in N_+$,

and
$$\begin{aligned} t\sigma&=\alpha_1n_1+\alpha_2n_2+\dots+\alpha_\ell n_\ell, && n_1,\dots,n_\ell\in \mathbf{G}\\ \overline{t}\,\overline{\sigma}&=\overline{\alpha}_1\overline{n}_1+\overline{\alpha}_2\overline{n}_2+\dots+\overline{\alpha}_s\overline{n}_s, && \overline{n}_1,\dots,\overline{n}_s\in \overline{\mathbf{G}}, \end{aligned} \tag{2.7}$$

hold, where $n_1,n_2,\dots,n_\ell$ have the same meanings as above; $\overline{n}_1<\overline{n}_2<\dots<\overline{n}_s$ are the first s numbers of $\overline{G}$ arranging in increasing sequence. It follows from the elementary number theory that $\forall k\in N$, the Diophantine equation

$$t\sigma - \bar{t}\bar{\sigma} = km \tag{2.8}$$

has the sufficient large solutions of positive integers. It follows from (2.7) and (2.8) that $\forall k \in N$, sufficient large $t\sigma \in \mathbf{G}$, $\bar{t}\bar{\sigma} \in \bar{\mathbf{G}}$ so that $km = t\sigma - \bar{t}\bar{\sigma}$, and $t\sigma$ and $\bar{t}\bar{\sigma}$ take the form of (2.7).

We consider first the system (a). One can see that the system starting from the state 0 can enter the state km. Noting that since n_1 is the smallest number of G and P is the generator of G, $n_1 \in P$, i.e. n_1 is a possible value of η_n^* (and ζ_n), hence the system starting from 0 can transform to the state $\alpha_1 n_1 + \ldots + \alpha_\ell n_\ell - \bar{n}_1 > 0$ by one step with a positive probablity. In fact, put $\zeta_{n+1} = n_1$, $\eta_{n+1} = (\alpha_1 - 1)n_1 + \alpha_2 n_2 + \ldots + \alpha_\ell n_\ell$, $V_{n+1} = 0$, $K_{n+1} = \bar{n}_1$ in the latter formula of (2.3), it follows that the above conclusion is valid. Put again $K_{n+2} = \bar{n}_1$, $V_{n+2} = 0$ in the first formula of (2.3), it is easily seen that the system can enter the state $\alpha_1 n_1 + \ldots + \alpha_\ell n_\ell - 2\bar{n}_1$ with a positive probability. The successive procedure permits that the system can reach the state $km = \alpha_1 n_1 + \ldots + \alpha_\ell n_\ell - \bar{\alpha}_1 \bar{n}_1 - \ldots - \bar{\alpha}_s \bar{n}_s \geq 0$. Conversely, it follows from the first formula of (2.3) that the system starting from km can enter 0. In fact, if $\xi_n = km$, put $K_{n+1} = \bar{n}_1$, $V_{n+1} = 0$, then $\xi_{n+1} = \max\{km - \bar{n}_1, 0\}$. If $km - \bar{n}_1 \leq 0$, then $\xi_{n+1} = 0$, if $\xi_{n+1} = km - \bar{n}_1 > 0$, then put again $K_{n+2} = \bar{n}_1$, $V_{n+2} = 0$ and so on. By writing $p = \min\{t \mid km - t\bar{n}_1 \leq 0,\ t \in N_+\}$, it follows that the system can enter 0 at the p-th step with a positive probability.

It should be noted that since m is the g.c.d. of $\bar{\sigma}$ and σ, the possible values of r.v's in (2.3) take the form of km. Hence if the system is in G^*, then in every transition the states that the system can reach with a positive probability take always the form of km. From what we described above it is easily seen that G^* is a close set, and that since every state and the state 0 are communicating, G^* is irreducible.

For (c) we can also achieve our aim along the same lines as (a). Now we consider (b). Denote the intersection of P and $\{n_1, \ldots, n_\ell\}$ by $\{n_1^*, \ldots, n_k^*\}$. Since $n_1 \in P$, the intersection is not empty. We should distinguish the two cases : from the latter formula of (2.4) <i>. If n_j^*, $\bar{n}_i$ so that $n_j^* > \bar{n}_i$, then put $\zeta_{n+1} = n_j^* \overset{\text{def}}{=} n_p$, $K_{n+1} = \bar{n}_i$, $V'_{n+1} = \alpha_1 n_1 + \ldots + (\alpha_p - 1)n_p + \ldots + \alpha_\ell n_\ell$. It implies that the system starting from 0 can enter $\alpha_1 n_1 + \ldots + \alpha_\ell - \bar{n}_i > 0$ by one step.

From the first formula of (2.4) by the same means as that of (a), it follows that the system starting from 0 can enter km.

<ii> If $\forall\ n_j^*, \bar{n}_i,\ n_j^* \leq \bar{n}_i$, then in the latter formula of (2.4), put ζ_{n+1} be an arbitrary $n_j^* \overset{def}{=} n_p$, K_{n+1} be an arbitrary $\bar{n}_i$, $V'_{n+1} = \alpha_1 n_1 + \ldots + \alpha_\ell n_\ell$. It follows that the system starting from 0 can enter $\alpha_1 n_1 + \ldots + \alpha_\ell n_\ell > 0$. Imitating the demonstration of (a), it follows that the system starting from 0 can enter an arbitrary state km.

Because the first formulas of (a), (b) and (c) are the same, it implies that for (b) and (c), $\mathbf{G}^*$ is also an irreducible and close set, and for (a), (b) and (c), starting from any state $\in \mathbf{G}^*$, the system can enter 0 with a positive probability. In fact, put $\xi_n = k \in \mathbf{G}^*$, $V_{n+1} = 0$, $K_{n+1} = \bar{n}_1$, then $\xi_{n+1} = \max\{k - \bar{n}_1, 0\}$. If $k - \bar{n}_1 \leq 0$, then $\xi_{n+1} = 0$; otherwise put $V_{n+2} = 0$, $K_{n+2} = \bar{n}_1$, then $\xi_{n+2} = \max\{k - 2\bar{n}_1, 0\} \ldots$

Generally, put $p = \min\{t \mid k - t\bar{n}_1 \leq 0,\ t \in N_+\}$, it follows that by p-steps the system starting from $k \in \mathbf{G}^*$ can enter 0 with a positive probability. Since $\mathbf{G}^*$ is a close set, it proved that $\forall\ k \in \mathbf{G}^*$ is inessential. It implies that $\forall\ k \in \mathbf{G}^*$ is transient.

□

Corollary If and only if m=1 (in particular, for single arrival and individual service) M.C. is irreducible, and when $m \neq 1$, M.C. contains the infinite transient states.

Theorem 2 For (a) and (b), if $P(K_n \equiv m_0) = 1$, $m_0 \in N_+$, and σ divides by m_0, then the periode of I.M.C.$\mathbf{G}^*$ is $\frac{\sigma}{m_0}$ (if $\frac{\sigma}{m_0} = 1$, then aperiodic); otherwise, $\mathbf{G}^*$ is aperiodic.

For (c), every state is aperiodic.

Proof We consider first the system (a). The following three cases are distinguished for the proof :

<i>. If $P(K_n \equiv m_0) = 1$, i.e. $\bar{G} = \{m_0\}$, and σ divides by m_0, then the period of G^* is $\frac{\sigma}{m_0}$.

Since G^* is irreducible, it is sufficient to consider the state 0. From the latter formula, it follows that every state that the system starting from 0 with a positive probability by one step can enter takes the form of $n\sigma-m_0$.

We now demonstrate that to start from 0 and return to 0, it is necessary to pass by p-steps, $p \in \{\frac{n\sigma}{m_0}\}_{n\in N_+}$. Evidently, it is only necessary to prove that to return to 0 for the first time it pass certainly by p-steps, $p \in \{\frac{n\sigma}{m_0}\}_{n\in N_+}$. Since the case of $\frac{\sigma}{m_0} = 1$ is trivial, we suppose below always $\frac{\sigma}{m_0} > 1$. There are only the two mutually exclusive cases after starting from a certain state $n\sigma-m_0$, $n \in N_+$.

<i> Every state that the system passes belongs to $\{n\sigma-n'm_0\}$, and enter last into 0. Evidently, starting from 0 and returning to 0, the total number of steps takes the form of $\frac{n\sigma}{m_0}$, $n \in N$.

<ii> Starting from a certain $n\sigma-m_0$, $n\in N_+$, the system passes several states $n\sigma-n'm_0$ $(n'\neq 0,\ n,n' \in N)$ and enter into a certain $n\sigma$. In order to obtain this result, iff $V_{n+1}=n\sigma$, $n\sigma-m_0-(n^*-1)m_0 > 0$, $n\sigma-m_0-n^*m_0 = 0$ $(n,n^* \in N_+)$. We indicate that if $n\sigma-m_0-(n^*-1)m_0 > 0$, then $n\sigma-m_0-n^*m_0<0$. Otherwise, we have $m_0+(n^*-1)m_0<n\sigma<m_0+n^*m_0$. Denoting $\sigma=tm_0$, $t \in N_+$, we obtain a contradictory inequatliy $n^*<nt<n^*+1$. Since $\sigma>m_0$, the system starting from $n\sigma$, $n \in N_+$ can not reach 0 by one step, but only $n\sigma-m_0$. It is easily seen that for <ii> the number of steps of starting from 0 and entering into 0 belongs to $\{\frac{n\sigma}{m_0}\}_{n\in N}$.

Again note that $\forall\ t \geq n_0$, $t\sigma=\alpha_1n_1+\alpha_2n_2+\ldots+\alpha_\ell n_\ell$ (for n_0, cf. theorem 1). If $\xi_n=0$ and we put $\zeta_{n+1}=n_1$, $\eta_{n+1}=(\alpha_1-1)n_1+\alpha_2n_2+\ldots+\alpha_\ell n_\ell$, $V_{n+1}=0$, $K_{n+1}=m_0$, then we obtain $P_{0,t\sigma-m_0}>0$. In the following successive transitions, putting always $V=0$, we see that the system can return to 0 in the following way with a positive probability :

$$t\sigma-m_0 \rightarrow t\sigma-2m_0 \rightarrow \ldots \rightarrow m_0 \rightarrow 0,$$

i.e., we obtain $p_{00}^{(\frac{t\sigma}{m_0})} > 0$, $\forall\ t \geq n_0$.

It is easily seen that the period of I.M.C. G^* is $\frac{\sigma}{m_0}$.

<2>. If $P(K_n \equiv m_0)=1$, m_0 can't divide σ, then I.M.C. G^* is aperiodic.

Let $\sigma=m_0\alpha+\beta$, where $0<\beta<m_0$, $m_0>1$,. When $t\geq n_0$ (cf. theorem 1), the system starting from 0 can enter the state $t\sigma m_0-m_0$. In fact, since $m_0>1$, $(tm_0 > n_0,\ tm_0\sigma \in G)$, $\zeta_{n+1}+\eta_{n+1}$ can take $tm_0\sigma=\alpha_1n_1+\ldots+\alpha_\ell n_\ell$.

Put $\zeta_{n+1}=n_1$, $\eta_{n+1}=(\alpha_1-1)n_1+\alpha_2n_2+\ldots+\alpha_\ell n_\ell$, $V_{n+1}=0$, and $K_{n+1}=m_0$, then it is easily seen that the conclusion is sure. By the following way the system can reach 0 :

$$t\sigma m_0-m_0 \to t\sigma m_0-2m_0 \to \ldots \to 0.$$

It implies that

$$P_{00}^{(t\sigma)} > 0. \tag{2.9}$$

Since $tm_0-1 \geq n_0$, $(tm_0-1)\sigma \in G$, $\zeta_{n+1}+\eta_{n+1}$ can take $(tm_0-1)\sigma$. Put $V_{n+1}=0$, we get that the system starting from 0 can reach $(tm_0-1)\sigma-m_0$ by one step. This procedure proceeds. Finally the system can enter into 0 in the following way :

$$(m_0t-1)\sigma-m_0 \to (m_0t-1)\sigma-2m_0 \to \ldots \to 0.$$

Denoting the greatest integer which is not more than y by [Y], it follows that the system starting from 0 after the transitions by

$$\left[\frac{(m_0t-1)\sigma}{m_0}\right]+1=\left[\sigma t-\frac{\sigma}{m_0}\right]+1=\left[\sigma t-\frac{m_0\alpha+\beta}{m_0}\right]+1=\sigma t-\alpha$$

steps returns to the state 0. Hence we obtain

$$P_{00}^{(\sigma t-\alpha)} > 0. \tag{2.10}$$

Since $t \geq n_0$, $tm_0+1 > n_0$, imitating the above procedure, it is easily seen that the system starting from 0 can reach $(m_0t+1)\sigma-m_0$ by one step. The system can enter into 0 in the following way :

$$(m_0t+1)+\sigma-m_0 \to (m_0t+1)\sigma-2m_0 \to \ldots \to 0.$$

Hence it implies that the system starting from 0 returns to 0 after transitions by

$$\left[\frac{(m_0 t+1)\sigma}{m_0}\right] + 1 = \left[t\sigma + \frac{\sigma}{m_0}\right] + 1 = t\sigma+\alpha+1$$

Steps, i.e.

$$P_{00}^{(\sigma t+\alpha+1)} > 0. \tag{2.11}$$

We obtain from (2.9), (2.10) and (2.11) that the period of the state 0 divides $t\sigma$, $t\sigma-\alpha$ and $t\sigma+\alpha+1$ respectively; then it divides the difference of every two numbers, for example, $\alpha+1$ and α. Since the two neighbour integers are mutually primary, the period of the I.M.C. is 1.

<3>. If the number of the possible values of K_n is more than one, the I.M.C. $\mathbf{G}^*$ is aperiodic.

Let m_0 and m_0' $(m_0 > m_0')$ be two arbitrary possible values of K_n. If m_0 (or m_0') does not divide σ, then by means of the demonstration of the case <2> it can be proved that the I.M.C. $\mathbf{G}^*$ is aperiodic. If m_0 and m_0' divide σ, putting $t \geq n_0$, $\zeta_{n+1}+\eta_{n+1} = t\sigma \in \mathbf{G}$, $K_n=m_0$, $V_{n+1}=0$, it follows that the system starting from 0 can reach $t\sigma-m_0$ by one step. And it can enter into 0 along the following way :

$$t\sigma-m_0 \to t\sigma-2m_0 \to \dots \to m_0 \to 0.$$

It follows that

$$P_{00}^{\left(\frac{t\sigma}{m_0}\right)} > 0, \qquad P_{0\,m_0}^{\left(\frac{t\sigma}{m_0}-1\right)} > 0 \tag{2.12}$$

Putting $K_{n+1}=m_0'$, $V_{n+1}=0$, we get from the first formula of (a)

$$P_{m_0,m_0-m_0'} > 0 \tag{2.13}$$

put $K_{n+1}=m_0$, $V_{n+1}=0$, then we obtain

$$P_{m_0-m_0',0} > 0 \tag{2.14}$$

From (2.12), (2.13) and (2.14) we get

$$P_{00}^{\left(\frac{\sigma t}{m_0}+1\right)} \geq P_{0m_0}^{\left(\frac{t\sigma}{m_0}-1\right)} P_{m_0,m_0-m_0'} \cdot P_{m_0-m_0',0} > 0 \qquad (2.15)$$

It implies easily that the period of the state 0 divides $\frac{\sigma t}{m_0}+1$ and $\frac{\sigma t}{m_0}$. Hence the period is 1, i.e. the I.M.C. is aperiodic.

Now we consider the system (b) by means of an analogic way as (a).

<1>. If a certain possible value of ζ_{n+1}, t_1, so that t_1 is not more than a certain possible value t_2 of K_{n+1}, then the I.M.C. G^* is aperiodic. In fact, put $\zeta_{n+1}=t_1$, $K_{n+1}=t_2$, $V'_{n+1}=0$, it follows from the latter formula of (b) that $p_{00} > 0$. Hence the state 0 is aperiodic.

<2>. If any possible value of ζ_{n+1} is more than any possible value of K_{n+1}, then we consider the following three cases :

<2.1>. If K_n takes the unique possible value m_0 and m_0 divides σ, then the period of G^* is $\frac{\sigma}{m_0}$ (if $\frac{\sigma}{m_0}=1$, then aperiodic). For this case the demonstration is similar to the correspondent demonstration of (a). Since the possible values of V'_{n+1} and ζ_{n+1} take the form of $n\sigma$, $n \in \mathbf{N}$, the system starting from 0 can only reaches a certain state $n\sigma-m_0$ by one step. Because the first formula of (a) and (b) are same, the way starting from a non-zero state and entering into 0 can be taken in the same way as (a). It implies that the I.M.C.G^* has also the period $\frac{\sigma}{m_0}$.

<2.2>. If K_n takes the unique possible value m_0 and m_0 does not divide σ, then the I.M.C. G^* is aperiodic. Let n' be an arbitrary possible value of ζ_{n+1} and denote n' by $n^*\sigma$. For $\forall$ $t \geq n_0+n'$, put $\zeta_{n+1} = n'$, $K_{n+1} = m_0$, $V'_{n+1} = t\sigma m_0-n'$. Since $n' > m_0$, the system starting from 0 can enter into $t\sigma m_0-m_0$ by one step. The following demonstration imitates that of the case (2) of (a).

<2.3>. If the number of possible values of K_n is more than one, then the I.M.C. G^* is aperiodic.

Let m_0, m_0' ($m_0 > m_0'$) be two possible values of K_n. If m_0 (or m_0') does not divide σ, then a similar way of (2.2) implies that G^* is aperiodic.

If m_0 and m_0' divide σ, put $\zeta_{n+1}=n'>m_0$ (n' is an arbitrary possible value of ζ_{n+1}), $K_{n+1}=m_0$, $V'_{n+1}=t\sigma-n'$. It is easily seen that $P_{0,t\sigma-m_0} > 0$.

The following treatement imitates that of the case (3) of (a).

Lastly, we consider (c). Since V_{n+1} can be 0, it follows that $P_{00}>0$ and the state 0 is aperiodic. Hence the I.M.C. G^* is aperiodic.

□

Now we give several examples of (a) which may present in practice.

Example 1 $\phi(x) = x^6$, $P(K_n = 4) = 1$
$\sigma = 6$, $\bar{\sigma} = 4$, $m = 2$, $m_0 = 4$. $G = \{6,12,\ldots,6n,\ldots\}$,
$G^* = \{0,2,4,\ldots,2k,\ldots\}$. Since m_0 does not divide σ, the I.M.C. G^* is aperiodic.

Example 2 $\phi(x) = x^6$, $P(K_n = 3) = 1$.
$\sigma = 6$, $\bar{\sigma} = 3$, $m_0 = 3$, $m = 3$, $\sigma/m_0 = 2$. $G = \{3k \mid k = 0,1,2,\ldots\}$.
The period of the I.M.C. G^* is 2.

Example 3 $\phi(x) = x^6$, $P(K_n = 2) = P(K_n = 4) = \frac{1}{2}$.
$\sigma = 6$, $\bar{\sigma} = 2$, $m = 2$. $G = \{2k \mid k = 0,1,2,\ldots\}$.
The I.C.M. G^* is aperiodic.

Example 4 $\phi(x) = x^6$, $P(K_n = 1) = 1$.
$\sigma = 6$, $\bar{\sigma} = 1$, $m = m_0 = 1$, $\sigma/m_0 = 6$. $G^* = \{0,1,2,\ldots,k,\ldots\}$.
The period of the I.M.C. G^* is 6.

Remark 1 and example 5 $\phi(x) = x$.

For the model of single arrival, the I.M.C. $G^* = \{0,1,2,\ldots,k,\ldots\}$ is aperiodic, i.e. the entire I.M.C. is aperiodic and irreducible. Whether singly serve or in batches serve. But for the model of single service, yet it is uncertain whether the chain is periodic or aperiodic, although the entire I.M.C. is irreducible.

For the model (a) with single service, [6] concludes that the limit distribution of I.M.C. exists, but it is not right. For this case, in fact, it is possible that the chain is periodic (cf. example 4). And [5) studied the model (a) of $\eta \equiv 0$ (i.e. $Z \equiv 0$), [5] considers that the

I.M.C. is irreducible and aperiodic, it is also faulty. In [7], it exists also several faults on the conditions of irreducibility and periodicity. In fact, if the minimum capacity of service (i.e. quorum of [7]) $\theta_m \equiv 1$, then the I.M.C. of [7] is the same as that of the model (a) of $\eta \equiv 0$ in the present paper. For $\theta_m \equiv 1$, we consider the following examples :

<1>. If

$$c_i = \begin{cases} i, & i \equiv 0 \pmod 8 \\ 0, & i \not\equiv \pmod 8 \end{cases} \qquad a_i = \begin{cases} i, & i \equiv 0 \pmod 4 \\ 0, & i \not\equiv 0 \pmod 4); \end{cases}$$

by means of the method of [7], we may take h=2 so that $c_i = a_i = 0$, $i \not\equiv 0 \pmod 2$, according to [7], $\{2k \mid k=0,1,2,\dots\}$ is irreducible. But, in fact, $\{4k \mid k=0,1,2,\dots\}$ is only irreducible (cf. theorem 1).

<2>. If

$C_m \equiv 1$, (i.e. $K \equiv 1$ of this paper), $a_i = \begin{cases} i, & i \equiv 0 \pmod 4 \\ 0, & i \not\equiv 0 \pmod 4 \end{cases}$, according to [7], we may take g=2, so that h =2. [7] concludes that h=2 is the periode of the I.M.C. (the entire I.M.C. is irreducible). But by theorem 2, 4 is its period.

In fact, it is easily seen that the possible values of the I.M.C. of [7] are the same as that of the model (a) of the present paper. Because for the I.M.C. $\{X_m\}$ of [7], we have

$$X_{m+1} = \begin{cases} \max(X_m + Y_m - C_m, 0) + V_{m+1}, & X_m < \theta_m \\ \max(X_m - C_m, 0) + V_{m+1}, & X_m \geq \theta_m, \end{cases}$$

where Y_m is the number of customers of several groups arriving when $X_m < \theta_m$ and $Y_m + X_m \geq \theta_m$; V_{m+1} is the number of arriving customers during the (m+1)-th batch service, so that theorem 1 and 2 are also valid for the I.M.C. $\{X_m\}$ of [7].

3 - BEHAVIOURS OF SYSTEM

Generally speaking, Foster's method is no use to a periodic and reducible chain. We indicate that an appropriate "compression" of phase space is convenient for the investigation of a periodic chain. In the pratical problems, in fact, the original length of queue ξ_0 is merely the total of customers of several arriving groups. The possible values of ξ_0 are evidently in $\mathbf{G}^*$ and after any transition they remain in $\mathbf{G}^*$. Thus we may consider $\mathbf{G}^*$ as the phase space. From now on, if without necessity, we shall drop the suffix of r.v., for example, η represents any of $\{\eta_n\}$.

Theorem 3 For the system (c) and the aperiodic cases of the systems (a) and (b), if the original length of queue $\xi_0 \in \mathbf{G}^*$, the maximal possible value of K is N, a finite positive integer, and taking $\mathbf{G}^*$ to be the phase space, we have the following three propositions :

Proposition 1 For the system (c) (abr.(c)) and the aperiodic case of the system (a) (abr.(a)), they are ergodic iff

$$\rho = d\lambda b < d'.$$

For the aperiodic case of the system (b) (abr.(b)), if

$$\rho = d\lambda b < d',$$

then it is ergodic; and if it is ergodic and

$$b < b' + \frac{1}{\lambda} \tag{3.1}$$

then $\rho = d\lambda b < d'$.

Proof We shall prove first that if $d\lambda b < d'$, then all of the three systems are ergodic.

According to Foster's criterion [5], it suffices to prove that the inequalities

$$Y_i - 1 \geq \sum_{j=0}^{\infty} Y_j P_{ij}, \qquad i > N,$$

$$\sum_{j=0}^{\infty} Y_j P_{ij} < \infty, \qquad i = 0,1,\dots,N$$

has a non-negative solution.

We consider first the case of $i > N$. Since $\psi_\ell = 0$ $(\ell > N)$,

$$P_{ij} = \sum_{\ell<i} k_{j-i+\ell}\ \psi_\ell.$$

Putting

$$Y_j = j(d'-d\lambda b)^{-1}, \quad j = 0,1,2,\ldots, \tag{3.2}$$

We get

$$\sum_{j=0}^{\infty} P_{ij}Y_j = \sum_{j=0}^{\infty}\ \sum_{\ell<i} jk_{j-i+\ell}\ \psi_\ell(d'-d\lambda b)^{-1}$$

$$= \sum_{j=0}^{\infty}\ \sum_{\ell<j} jk_\ell\ \psi_{\ell+i-j}(d'-d\lambda b)^{-1} = \sum_{\ell=0}^{\infty} k_\ell \sum_{m=1}^{\infty} (m+\ell)\psi_{i-m}(d'-d\lambda b)^{-1}$$

$$= (d\lambda b - \sum_{m=1}^{i-1} m\ \psi_m + i)(d'-d\lambda b)^{-1} = (d\lambda b-d'+i)(d'-d\lambda b)^{-1} = Y_i - 1. \tag{3.3}$$

Secondly we consider the case $i \leq N$. For the case $i = 0$ of (a), we have

$$\sum_{j=0}^{\infty} jP_{oj} \leq \sum_{j=0}^{\infty} j\{k_j + P(\zeta+\eta-K+V = j\} = d\lambda b + E(\zeta+\eta-K+V) -$$

$$\sum_{j=-N}^{-1} jP(\zeta+\eta-K+V=j) \leq 2d\lambda b+d+d\lambda c-d+N < \infty \tag{3.4}$$

If $N \geq i > 0$, then we get

$$\sum_{j=0}^{\infty} jP_{ij} = d\lambda b+i-d' + \sum_{m\geq i} (m-i)\psi_m \leq d\lambda b+i \leq d\lambda b+N < \infty \tag{3.5}$$

For (b) and (c), it suffices to consider the case $i=0$. For (c), we have

$$P_{oj} = k_j, \quad \sum_{j=0}^{\infty} jP_{oj} = d\lambda b < \infty. \tag{3.6}$$

For (b), we have

$$p_{oj} \le k'_j + P(\zeta - K + V' = j),$$

where $\{k'_j\}$ is the distribution of V', and

$$\sum_{j=0}^{\infty} jP_{oj} \le d\lambda b' + \sum_{j=-N}^{\infty} jP(\xi - K + V' = j) - \sum_{j=-N}^{-1} jP(\zeta - K + V' = j)$$

$$\le d\lambda b' + E(\zeta - K + V') + N < \infty \tag{3.7}$$

Bellow we shall demonstrate the necessity. Assuming that the system is ergodic, and $(\Pi_0, \Pi_1, \dots)$ is the ergodic distribution, we get

$$\Pi_j = \sum_{i=0}^{\infty} \Pi_i P_{ij}, \qquad \forall\ j \in \{0,1,2,\dots\}.$$

Note that iff $i \in \mathbf{G}^*$, $M_i > 0$ and that $0 \in \mathbf{G}^*$ for (a), (b) and (c), i.e., $\Pi_0 > 0$. For (c), we get

$$\Pi(z) = \sum_{j=0}^{\infty} \Pi_j Z^j = \Pi_0 K(z) + \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \Pi_i P_{ij} z^j.$$

Note that

$$\Pi(z)K(z)\psi\left(\frac{1}{z}\right) = \sum_{j=-N}^{\infty} \sum_{i=0}^{\infty} \sum_{1 \le \ell \le N} \Pi_i \psi_\ell k_{j-i+\ell} z^j,$$

$$K(z)\psi\left(\frac{1}{z}\right) = \sum_{j=-N}^{\infty} \sum_{1 \le \ell \le N} \psi_\ell k_{j+\ell} z^j,$$

we get

$$\sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \Pi_i P_{ij} z^j = \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \left\{ \sum_{i \le \ell \le N} \psi_\ell k_j + \sum_{1 \le \ell \le i-1} \psi_\ell k_{j-i+\ell} \right\} \Pi_i z^j$$

$$= K(z) \sum_{i=1}^{\infty} \sum_{i \le \ell \le N} \Pi_i \psi_\ell + \Pi(z)K(z)\psi\left(\frac{1}{z}\right) - \Pi_0 K(z)\psi\left(\frac{1}{z}\right)$$

$$- \sum_{-N \le j \le -1} \sum_{i=1}^{\infty} \sum_{1 \le \ell \le N} \Pi_i \psi_\ell k_{j-i+\ell} z^j - \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \sum_{i \le \ell \le N} \Pi_i \psi_\ell k_{j-i+\ell} z^j$$

Noting

$$\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}\sum_{i\le \ell\le N}\Pi_i\psi_\ell k_{j-i+\ell}z^j = K(z)\sum_{i=1}^{\infty}\sum_{i\le \ell\le N}\Pi_i\psi_\ell z^{i-\ell} -$$

$$-\sum_{i=1}^{\infty}\sum_{i\le \ell\le N}\sum_{i-\ell\le j\le -1}\Pi_i\psi_\ell k_{j-i+\ell}z^j,$$

$$\sum_{-N\le j\le -1}\sum_{i=1}^{\infty}\sum_{1\le \ell\le N}\Pi_i\psi_\ell k_{j-i+\ell}z^j = \sum_{i=1}^{\infty}\sum_{i\le \ell\le N}\sum_{i-\ell\le j\le -1}\Pi_i\psi_\ell k_{j-i+\ell}z^j,$$

We have

$$\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}\Pi_i P_{ij}z^j = K(z)\Big[\sum_{i=1}^{N}\sum_{i\le \ell\le N}\Pi_i\psi_\ell(1-z^{i-\ell})-\Pi_0\psi(\frac{1}{z})\Big]+\Pi(z)K(z)\psi(\frac{1}{z}),$$

$$\Pi(z) = \Pi_0 K(z)+K(z)\Big[\sum_{i=1}^{N}\sum_{i\le \ell\le N}\Pi_i\psi_\ell(1-z^{i-\ell})-\Pi_0\psi(\frac{1}{z})\Big]+\Pi(z)K(z)\psi(\frac{1}{z}).$$

It follows from above that

$$\Pi(z) = \frac{K(z)}{1-K(z)\psi(\frac{1}{z})}\Big\{\Pi_0\Big[1-\psi(\frac{1}{z})\Big] + \sum_{i=1}^{N-1}\sum_{i+1\le \ell\le N}\Pi_i\psi_\ell(1-z^{i-\ell})\Big\}. \quad (3.8)$$

Letting $z\to 1-0$ in the both sides of (3.8), we have

$$1 = \frac{1}{d'-d\lambda b}\Big\{\Pi_0 d' + \sum_{i=1}^{N-1}\sum_{i+1\le \ell\le N}(\ell-i)\Pi_i\psi_\ell\Big\}. \quad (3.9)$$

thus we have $d\lambda b < d'$.

For (a), denoting the distribution of $\zeta+\eta-K$ by $\{q_i\}_{-N\le i<\infty}$ whose g.f. is $\phi(z)H(z)\psi(\frac{1}{z})$, since

$$P_{0j} = \sum_{i\le 0} q_i k_j + \sum_{i>0} q_i\, k_{j-i}, \quad (3.10)$$

after a somewhat lengthy computation (noting

$$\phi(z)H(z)\psi(\frac{1}{z})K(z) = \sum_{j=-N}^{\infty}\sum_{i=-N}^{\infty} q_i k_{j-i} z^j,)$$

we have

$$\sum_{j=0}^{\infty} P_{oj} z^j = \phi(z)H(z)K(z)\psi(\frac{1}{z})+K(z)\{\sum_{-N\le i\le -1} q_i(1-z^i)\}, \tag{3.11}$$

such that

$$\Pi(z) = \frac{K(z)}{1-K(z)\psi(\frac{1}{z})}\{\Pi_0[\sum_{-N\le i\le -1} q_i(1-z^i)+\phi(z)H(z)\psi(\frac{1}{z})-\psi(\frac{1}{z})]$$

$$+\sum_{i=1}^{N-1}\sum_{i+1\le \ell\le N} \Pi_i\psi_\ell(1-z^{i-\ell})\} \tag{3.12}$$

Letting $z \to 1-0$ in the both sides of (3.12), we get

$$1 = \frac{1}{d'-d\lambda b}\{\Pi_0[q_{-1}+2q_{-2}+\ldots+Nq_{-N}+d+d\lambda c]+\sum_{i=1}^{N-1}\sum_{i+1\le\ell\le N}(\ell-i)\Pi_i\psi_\ell\}. \tag{3.13}$$

Such that $d\lambda b < d'$.

Last we consider (b). Denoting the distribution of V' by $\{k'_i\}_{i\ge 0}$ whose g.f. is $\overline{K(z)}$ and the distribution of ζ-K by $\{q'_i\}_{-N\le i<\infty}$ whose g.f. is $\phi(z)\psi(\frac{1}{z})$.

It is obvious that

$$P_{oj} = \sum_{-N\le i\le 0} q'_i k'_j + \sum_{i>0} q'_i k'_{j-i},$$

and such that

$$\Pi(z) = \frac{K(z)}{1-K(z)\psi(\frac{1}{z})} \sum_{i=1}^{N-1} \sum_{i+1\leq \ell \leq N} \Pi_i \psi_\ell (1-z^{i-\ell})$$

$$+ \frac{\Pi_0}{1-K(z)\psi(\frac{1}{z})} \{\overline{K(z)}[\sum_{-N\leq i\leq -1} q'_i(1-z^i)]+\psi(\frac{1}{z})[\overline{K(z)}\phi(z)-K(z)]\}. \tag{3.14}$$

Letting $z\to 1-0$ in the both sides of (3.14), we have

$$1 = \frac{1}{d'-d\lambda b} \{\Pi_0[q'_{-1}+2q'_{-2}+\ldots+Nq'_{-N}+d+d\lambda b'-d\lambda b] + \sum_{i=1}^{N-1} \sum_{i+1\leq \ell \leq N} (\ell-i)\Pi_i\psi_\ell\}. \tag{3.15}$$

If $-\sum_{-N\leq i\leq -1} iq'_i+d+d\lambda b'-d\lambda b > 0$, then $d' > d\lambda b$. Particularly if the hypothesis for (b), i.e. $b < b' + \frac{1}{\lambda}$, holds, then $d' > d\lambda b$.

<u>Remark 2</u> The hypothesis $b < b' + \frac{1}{\lambda}$ seems too strong. But for the model "singly serve", i.e. $K \equiv 1$, the above condition seems best possible. Since, for this case, (3.15) has the following form :

$$1 = \frac{\Pi_0}{d'-d\lambda b}(d+d\lambda b'-d\lambda b).$$

Below we shall consider the recurrence and the transience. Since (a), (b) and (c) in Pedro vit's sense are equivalent for recurrence and transience [8], it suffices to consider the system (c).

<u>Proposition 2</u> (a) and (c) are null-recurrent iff $\rho = d\lambda b = d'$; if $b<b'+\frac{1}{\lambda}$, then (b) is null-recurrent iff $\rho = d\lambda b = d'$.

<u>Proof</u> According to the proposition 1, it is sufficient to prove if $d\lambda b=d'$, then (c) is recurrent.

If $N = 1$, then $\psi_1 = 1$, $d' = d\lambda b = 1$, and

$$P_{ij} = \begin{cases} k_j & , i = 0,1 \\ k_{j-i+1} & , i \geq 2,\ j-i+1 \geq 0, \end{cases} \tag{3.16}$$

Putting $Y_j = j$ (j=0,1,2,...,), it is easily seen that for $i > 0$,

$$\sum_{j=0}^{\infty} P_{ij}Y_j = i = y_i,\ Y_j \to \infty \text{ as } j \to \infty. \tag{3.17}$$

According to Foster's criterion [cf.(5)], (c) is recurrent

If $N > 1$, we put again $Y_j = j$, $j = 0,1,2,\ldots$, such that

$$\sum_{j=0}^{\infty} Y_j P_{ij} = \sum_{j=0}^{\infty} jP_{ij} = \sum_{j=0}^{\infty} \sum_{1\leq \ell \leq N} \psi_\ell k_j j + \sum_{j=0}^{\infty} \sum_{1\leq \ell \leq i-1} \psi_\ell k_{j-i+\ell} j$$

$$= i + \sum_{1\leq \ell \leq N} (\ell - i)\psi_\ell.$$

For $i \geq N$, we clearly have

$$\sum_{j=0}^{\infty} Y_j P_{ij} = i = Y_i,\ Y_j \to \infty, \text{ as } j \to \infty. \tag{3.18}$$

It follows from Pakes's recurrence criterion (cf.[9]) that (c) is recurrent.

Proposition 3 (a), (b) and (c) are transient iff $\rho = d\lambda b > d'$.

Proof According to Pedro vit's result [8], it suffices to prove that if $\rho = d\lambda b > d'$, then (c) is transient. We can also prove this proposition without Pedro's vit's result. Since the following demonstration is not related to $\{P_{oj}\}$, the demonstration is valid for all of (a), (b) and (c). We consider the function

$$F_i(z) = \sum_{j=0}^{\infty} P_{ij} z^j - z^i , \qquad 1 \le i \le N. \tag{3.19}$$

Noting that

$$F_i(1) = 0, \; F_i'(1) = d\lambda b - d' + \sum_{1 \le \ell \le N} (\ell - i)\psi_\ell > 0, \tag{3.20}$$

it is easily seen that it exists $z_i \in (0,1)$ so that $\forall\, x \in (z_i, 1)$, $F_i(x) < 0$, i.e.

$$\sum_{j=0}^{\infty} P_{ij} x^j < x^i, \quad 0 < Z_i < x < 1, \quad 1 \le i \le N. \tag{3.21}$$

Letting $\alpha = \max(z_i, \ldots, Z_N)$ and putting

$$Y_j = \alpha^j, \qquad 0 \le j < \infty, \tag{3.22}$$

we have

$$\sum_{j=0}^{\infty} P_{ij} Y_j \le Y_i, \quad 0 < i \le N. \tag{3.23}$$

Below we shall consider the case $i > N$. We have

$$Ez^{V-K} = \sum_{j=-N}^{\infty} P(V-K=j) z^j = \sum_{j=-N}^{\infty} \sum_{\ell=1}^{N} \psi_\ell k_{j+\ell} z^j , \tag{3.24}$$

where Ez^{V-K} is the g.f. of V-K.

Putting $Y_j = \beta^j$ $(0 \le j < \infty)$ where β is a constant, we get, for $i > N$,

$$\sum_{j=0}^{\infty} P_{ij} Y_j = \sum_{j=0}^{\infty} \sum_{1 \le \ell \le i-1} \psi_\ell k_{j-i+\ell}\, \beta^j = \beta^i \sum_{m=-i}^{\infty} \sum_{1 \le \ell \le i-1} \psi_\ell k_{m+\ell} \beta^m$$

$$= \beta^i \sum_{m=-N}^{\infty} \sum_{\ell=1}^{N} \psi_\ell k_{m+\ell}\, \beta^m = \beta^i E\beta^{V-K} .$$

Choosing $\beta = \beta^*$ such that

$$1 > \beta^* > \alpha = \max(z_1, \dots z_N),$$

and putting $Y_j = \beta^{*j}$, $0 \le j < \infty$,

since Ez^{V-K} is differentiable for z in a left neighbourhood of 1 and

$$\frac{d}{dz}(Ez^{V-K})\Big|_{z=1} = E(V-K) = d\lambda b - d' > 0, \tag{3.25}$$

we have

$$E\beta^{*V-K} \le Ez^{V-K}\Big|_{z=1} = 1,$$

so

$$\sum_{j=0}^{\infty} P_{ij} Y_j \le Y_i \quad (i > N).$$

For $1 \le i \le N$, we have also obviously

$$\sum_{j=0}^{\infty} P_{ij} Y_j \le Y_i,$$

and

$$Y_i < Y_0 = 1 \quad (1 \le i < \infty).$$

According to the Foster's criterion [5], (a), (b) and (c) are transient.

Now we shall study the periodic cases. It is seen from theorem 2 that (c) is aperiodic. Hence we consider only (a) and (b). For (a) and (b), only in the case 1 of theorem 2, i.e., $K \equiv m_0$, the system can be periodic. By means of the method of compression of phase space, we assume that the original length of queue ξ_0 is the total of customers of several arriving groups. Because m_0 divides σ, the possible values of $\{\xi_n\}_{n\ge0}$ are several times of m_0, so that $G^* = \{km_0 \mid k=0,1,2,\dots\}$. We construct an I.M.C. G^{**} in G^* by considering the instant of departure of the last customer of an arriving customers group as a renewal point, and the numbers of groups of arriving customers groups in the system as the states.

For the two cases, we have the following relations respectively :

$$(a^\circ)\quad \xi^0_{n+1} = \begin{cases} \xi^0_n - 1 + V^0_{n+1}, & \xi^0_n > 0 \\ \eta^0_{n+1} + V^0_{n+1}, & \xi^0_n = 0, \end{cases}$$

$$(b^\circ)\quad \xi^0_{n+1} = \begin{cases} \xi^0_n - 1 + V^0_{n+1}, & \xi^0_n > 0 \\ V'^0_{n+1}, & \xi^0_n = 0, \end{cases}$$

where

ξ^0_n = the number of groups of arriving customers in the system at r^0_n+0; r^0_n is the departure time of the n-th arriving customers group,

V^0_n = the number of groups of arriving customers during the service time of the n-th arriving customers group,

η^0_n = the number of groups arriving customers before the starting of service for the first group of arriving customers after $\xi^0_n = 0$ during the additional time interval z_n,

V'^0_{n+1} = the number of groups of arriving customers during the special service time Y'_n when $\xi^0_n = 0$.

The correspondents of K_n and ζ_n are K^0_n and ζ^0_{n+1}, and evidently $K^0_n = \zeta^0_{n+1} = 1$.

We consider every m_0 customers as a "super-customer". Let $\phi^0(x)$ be the common g.f. of the number of "super-customer" in an arriving customers group, then we seen that $\phi^0(x) = \phi(x^{\frac{1}{m_0}})$ which expectation is d/m_0.

Writing $\phi^0(x) = \sum_{r=1}^{\infty} \phi^0_r x^r$, it is seen that the common d.f. of the service time of an arriving customers group is

$$B^0(t) = \sum_{r=1}^{\infty} \phi_r^0 B^{*r}(t),$$

and $K^0(x) = Ex^{V_n^0} = \int_0^{\infty} e^{-\lambda t(1-x)} dB^0(t) = \tilde{B}^0(\lambda-\lambda x)$, $EV_n^0 = \dfrac{d\lambda b}{m_0}$.

Similarly, we have

$$B'^0(t) = \sum_{r=1}^{\infty} \phi_r^0 B'^{*r}(t),$$

$$\overline{K}^0(x) = EX^{V_n'^0} = \tilde{B}'0\ (\lambda-\lambda x),$$

$$EV_n'^0 = \frac{d\lambda b'}{m_0}.$$

Considering an arriving customers group as a "hyper-customer", the problem transforms into the queueing model M/G/1 with single arrival and individual service. The arriving instants process is a homogeneous Poisson process with parameter λ. The d.f. of service time is $B^0(t)$ (or $B'^0(t)$, for the special service time).

From the remark 1 we seen that I.M.C. G^{**} is obviously irreducible and aperiodic. The phase space $G^{**} = \{0,1,2,...\}$. Noting that (a°) and (b°) are respectively the special cases of the models (a) and (b) when $\zeta_{n+1}=K_{n+1}=1$, $\phi(x) = x$ (cf. (2.3), (2.4)), we seen that the theorem 3 holds for (a°) and (b°), i.e. :

Theorem 4 For (a°), the I.M.C. G^{**} is ergodic iff $\rho = \dfrac{d\lambda b}{m_0} < 1$,

For (b°) if $\rho = \dfrac{d\lambda b}{m_0} < 1$,

then it is ergodic; and if it is ergodic and

$$\frac{d\lambda b}{m_0} < \frac{d\lambda b'}{m_0} + \lambda \ (\text{or } db < db' + m_0)$$

then

$$\rho = \frac{d\lambda b}{m_0} < 1.$$

4 - CALCULATION OF STATIONARY DISTRIBUTION

In order to calculate the stationary(i.e. limiting) distribution, we establish first the following lemma.

Lemma If $d\lambda b < d'$, $K(z)$ is analytic in $|z| < 1 + \delta$ ($\delta > 0$, sufficient arbitrary small) and $k_0 \neq 0$, then the equation $K(z)\psi(z^{-1}) = 1$ have N mutually different roots in unit circle $|z| \leq 1$, and 1 is its simple root, 0 is not its root.

Proof Because

$$K(z)\psi(z^{-1})z^N \Big|_{z=0} = K(z)(\psi_1 z^{N-1} + \psi_2 z^{N-2} + \dots + \psi_N) \Big|_{z=0} = k_0 \psi_N \neq 0,$$

and

$$K(z)\psi(z^{-1}) \Big|_{z=0} \overset{\text{def}}{=} \lim_{\substack{z\to 0 \\ z\neq 0}} K(z)\psi(z^{-1}) = \lim_{\substack{z\to 0 \\ z\neq 0}} \frac{K(z)\psi(z^{-1})z^N}{z^N} = \infty,$$

it follows that 0 is not a root of the equation $K(z)\psi(z^{-1}) = 1$. We consider the following equation which is equivalent to $K(z)\psi(z^{-1}) = 1$ in $z \neq 0$:

$$K(z)\psi(z^{-1})\, z^N = z^N \tag{4.1}$$

Since $\psi(z^{-1})z^N$ is a polynomial, the two sides of (4.1) are analytic in $|z| < 1+\delta$.

$$\text{Noting } \frac{d}{dz}\psi(z^{-1})K(z)z^N \Big|_{z=1} = d\lambda b + N - d' < N = \frac{d}{dz} z^N \Big|_{z=1}, \tag{4.2}$$

we get $\psi((1+\delta)^{-1})K(1+\delta)(1+\delta)^N < (1+\delta)^N$. Utilizing Rouche's theorem in $|z| < 1+\delta$ and noting the arbitrariness of δ, it follows that in unit circle $|z| \leq 1$ the equation (4.1) and $z^N = 0$ have the same number of roots. It is obvious that 1 is a root of (4.1) and that 0 is not a root of (4.1). And 1 is its simple root; otherwise, the two sides of (4.2) have the equal value which contradicts the assumption $d\lambda b < d'$. □

Denote the roots of the equation $K(z)\psi(z^{-1}) = 1$ (i.e. the roots of (4.1)) in $|z| \leq 1$ by $1, \delta_1, \delta_2, \ldots, \delta_{N-1}$. From (3.8), (3.12) and (3.14) it is seen that for the determination of $\Pi(z)$, it is sufficient to determine $\Pi_0, \Pi_1, \Pi_2, \ldots \Pi_N$. Noting that $\Pi(z)$ is analytic in $|z| < 1$ and that the equation $K(z) = 0$ and the equation $K(z)\psi(z^{-1}) = 1$ have not any common root, hence if δ_i is a root of $K(z)\psi(z^{-1})$ in $|z| < 1$, then δ_i is also a root of the i-th one of the following equations :

$$\sum_{j=1}^{N-1} \Pi_j \sum_{k=j+1}^{N} \psi_k (1-\delta_i^{j-k}) = \Pi_0 \mu_i, \quad i=1,2,\ldots,N-1. \tag{4.3}$$

where μ_i is the coefficient of Π_0 substituting z by δ_i in the numerable of $\Pi(z)$ for the systems (a) and (c); but for (b), this coefficient divides by $K(\delta_i)$.

It we resolve the equations (4.3), then all Π_i, $i \geq 1$, are represented by Π_0 which is determined by the formula

$$\lim_{z\to 1} \Pi(z) = \sum_{i=0}^{\infty} \Pi_i = 1.$$

In order to resolve (4.3), we consider the following two cases :

Case 1 The all δ_i are mutually different. For this case, the coefficient matrix of (4.3) is

$$\left\| \begin{matrix} \sum\limits_{k=2}^{N} \psi_k (1-\delta_1^{1-k}) & \sum\limits_{k=2}^{N} \psi_k (1-\delta_2^{1-k}) & \ldots & \sum\limits_{k=2}^{N} \psi_k (1-\delta_{N-1}^{1-k}) \\ \sum\limits_{k=3}^{N} \psi_k (1-\delta_1^{2-k}) & \sum\limits_{k=3}^{N} \psi_k (1-\delta_2^{2-k}) & \ldots & \sum\limits_{k=3}^{N} \psi_k (1-\delta_{N-1}^{2-k}) \\ \sum\limits_{k=N}^{N} \psi_k (1-\delta_1^{N-k-1}) & \sum\limits_{k=N}^{N} \psi_k (1-\delta_2^{N-k-1}) & \ldots & \sum\limits_{k=N}^{N} \psi_k (1-\delta_{N-1}^{N-k-1}) \end{matrix} \right\|$$

$$= \left\| \begin{matrix} \psi_N\psi_{N-1}\psi_{N-2} & \cdots & \psi_3\psi_2 \\ \psi_N & \psi_{N-1} \cdots & \psi_4\psi_3 \\ 0 & & \\ & & \psi_N \end{matrix} \right\| \cdot \left\| \begin{matrix} 1-\delta_1^{1-N} & 1-\delta_2^{1-N} & \cdots\cdots & 1-\delta_{N-1}^{1-N} \\ 1-\delta_1^{2-N} & 1-\delta_2^{2-N} & \cdots\cdots & 1-\delta_{N-1}^{2-N} \\ & & & \\ 1-\delta_1^{-1} & 1-\delta_2^{-1} & \cdots\cdots & 1-\delta_{N-1}^{-1} \end{matrix} \right\| , \quad (4.4)$$

Denoting the three matrixes by A,B and C respectively, we have A=BC. Below we shall demonstrate A is invertible. Since $|A| = |B| \cdot |C|$, and $|B| = \psi_N^{N-1} \neq 0$, it is sufficient to demonstrate $|C| \neq 0$. Utilizing Vandermonde determinant, we obtain

$$|C| = (\delta_1\cdots\delta_{N-1})^{-(N-1)} \begin{vmatrix} \delta_1^{N-1} - 1 & \delta_2^{N-1} - 1 & \cdots & \delta_{N-1}^{N-1} - 1 \\ \delta_1^{N-1} - \delta_1 & \delta_2^{N-1} - \delta_2 & \cdots & \delta_{N-1}^{N-1} - \delta_{N-1} \\ \delta_1^{N-1} - \delta_1^2 & \delta_2^{N-1} - \delta_2^2 & \cdots & \delta_{N-1}^{N-1} - \delta_{N-1}^2 \\ & & & \\ \delta_1^{N-1} - \delta_1^{N-2} & \delta_2^{N-1} - \delta_2^{N-2} & \cdots & \delta_{N-1}^{N-1} - \delta_{N-1}^{N-2} \end{vmatrix}$$

$$= (\delta_1\cdots\delta_{N-1})^{1-N}(\delta_1-1)\cdots(\delta_{N-1}-1).$$

$$\begin{vmatrix} \delta_1^{N-2}+\delta_1^{N-3}+\cdots+\delta_1+1 & \delta_2^{N-2}+\cdots+1 & \delta_{N-1}^{N-2}+\cdots+1 \\ \delta_1^{N-2}+\delta_1^{N-3}+\cdots+\delta_1 & \delta_2^{N-2}+\cdots+\delta_2 & \delta_{N-1}^{N-2}+\cdots+\delta_{N-1} \\ \delta_1^{N-2}+\delta_1^{N-3}+\cdots+\delta_1^2 & \delta_2^{N-2}+\cdots\delta_2^2 & \delta_{N-1}^{N-2}+\cdots+\delta_{N-1}^2 \\ & & \\ \delta_1^{N-2} & \delta_2^{N-2} \cdots\cdots & \delta_{N-1}^{N-2} \end{vmatrix}$$

$$= (\delta_1 \ldots \delta_{N-1})^{1-N}(\delta_1 - 1)\ldots(\delta_{N-1}-1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \delta_1 & \delta_2 & \cdots & \delta_{N-1} \\ \delta_1^2 & \delta_2^2 & \cdots & \delta_{N-1}^2 \\ \\ \delta_1^{N-2} & \delta_2^{N-2} & \cdots & \delta_{N-1}^{N-2} \end{vmatrix} \qquad (4.5)$$

$$= \left(\prod_{i=1}^{N-1} \delta_i\right)^{1-N} \prod_{i=1}^{N-1} (\delta_i - 1) \prod_{1\le i<j\le N-1} (\delta_j - \delta_i). \qquad (4.6)$$

Since $\delta_i \neq 0,1$ and δ_i are mutually different, $|C| \neq 0$ so that $|A| \neq 0$.

Case 2 If δ_i is n_i multiple roots, $i=1,2,\ldots,s$; $n_1+n_2+\ldots+n_s = N-1$, then

$$\frac{d^\ell}{dx^\ell} \sum_{j=1}^{N-1} \Pi_j \left[\sum_{k=j+1}^{N} (1-x^{j-k})\right]\Bigg|_{x=\delta_i} = 0$$

$$\ell = 1,2,\ldots,n_i-1. \qquad (4.7)$$

For this case, the correspondent coefficient matrix is changed to the matrix D :

$$D = \left\| \begin{matrix} \sum_{k=2}^{N} \psi_k (1-\delta_1^{1-k}) & \frac{d}{dx_2} \sum_{k=2}^{N} \psi_k (1-x_2^{1-k}) \Big|_{x_2=\delta_1} & \cdots \\ \\ \sum_{k=3}^{N} \psi_k (1-\delta_1^{2-k}) & \frac{d}{dx_2} \sum_{k=3}^{N} \psi_k (1-x_2^{2-k}) \Big|_{x_2=\delta_1} & \cdots \\ \\ \sum_{k=N}^{N} \psi_k (1-\delta_1^{N-k-1}) & \frac{d}{dx_2} \sum_{k=N}^{N} \psi_k (1-x_2^{N-k-1}) \Big|_{x_2=\delta_1} & \cdots \end{matrix} \right\|$$

$$= B.\left\| \begin{matrix} 1-X_1^{1-N}, & \frac{d}{dX_2}(1-X_2^{1-N}),\dots & \frac{d^{(n_1-1)}}{dX_{n_1}^{(n_1-1)}}(1-X_{n_1}^{1-N}), & 1-X_{n_1+1}^{1-N},\dots \\ 1-X_1^{2-N}, & \frac{d}{dX_2}(1-X_2^{2-N}),\dots & \frac{d^{(n_1-1)}}{dX_{n_1}^{(n_1-1)}}(1-X_{n_1}^{2-N}), & 1-X_{n_1+1}^{2-N},\dots \\ 1-X_1^{-1}, & \frac{d}{dX_2}(1-X_2^{-1}),\dots & \frac{d^{(n_1-1)}}{dX_{n_1}^{(n_1-1)}}(1-X_{n_1}^{-1}), & 1-X_{n_1+1}^{-1},\dots \end{matrix} \right\|$$

$$X_1 = X_2 = \dots = X_{n_1} = \delta_1$$

$$X_{n_1+1} = \dots\dots = X_{n_1+n_2} = \delta_2$$

$$X_{n_1+n_2+\dots+n_{s-1}+1} = \dots = X_{N-1} = \delta_s$$

$$\stackrel{\text{def}}{=} B.E$$

Below we demonstrate $|E| \neq 0$. First we point

$$|E| = \frac{\partial^{\sum_{i=1}^{s} \frac{n_i(n_i-1)}{2}}}{\partial X_2 \partial X_3^2 \ldots \partial X_{n_1}^{n_1-1} \; \partial X_{n_1+2} \, \partial X_{n_1+3}^2 \ldots \partial_{n_1+n_2}^{n_2-1} \ldots \partial X_{N-1}^{n_s-1}}$$

$$\begin{vmatrix} 1-X_1^{1-N} & 1-X_2^{1-N} & \ldots & 1-X_{n_1}^{1-N} & 1-X_{n_1+1}^{1-N} & 1-X_{n_1+2}^{1-N} & \ldots \\ 1-X_1^{2-N} & 1-X_2^{2-N} & \ldots & 1-X_{n_1}^{2-N} & 1-X_{n_1+1}^{2-N} & 1-X_{n_1+2}^{2-N} & \ldots \\ & & & & & & \\ 1-X_1^{-1} & 1-X_2^{-1} & \ldots & 1-X_{n_1}^{-1} & 1-X_{n_1+1}^{-1} & 1-X_{n_1+2}^{-1} & \ldots \end{vmatrix}$$

$$\begin{aligned} &X_1 = X_2 = \ldots = X_{n_1} = \delta_1 \\ &X_{n+1} = \ldots = X_{n_1+n_2} = \delta_2 \\ &\ldots\ldots\ldots \\ &X_{n_1+\ldots+n_{s-1}+1} = \ldots = X_{N-1} = \delta_s. \end{aligned} \tag{4.8}$$

Noting that in the right determinant, every element of the i-th column is only a function of X_i, and that in the expansion of a determinant every term contains one and only one element of every column, it is easily seen that (4.8) is valid.

Denote the differential operator $\dfrac{\partial^{\sum_{i=1}^{s} n_i(n_i-1)/2}}{\partial X_2 \partial X_3^2 \ldots \partial X_{n_1}^{n_1-1} \; \partial X_{n_1+2} \ldots \partial X_{N-1}^{n_s-1}}$ by T.

From

$$|E| = T(X_1 \ldots X_{N-1})^{1-N}(X_1-1)\ldots(X_{N-1}-1) \begin{vmatrix} 1 & 1 & \ldots & 1 \\ X_1 & X_2 & & X_{N-1} \\ X_1^2 & X_2^2 & & X_{N-1}^2 \\ & & & \\ X_1^{N-2} & X_2^{N-2} & & X_{N-1}^{N-2} \end{vmatrix}$$

$$\begin{aligned} & X_1 = X_2 = \ldots = X_{n_1} = \delta_1 \\ & \ldots\ldots\ldots \\ & X_{n_1} + \ldots + n_{s-1} + 1 = \ldots = X_{N-1} = \delta_s. \end{aligned}$$

it is easily proved that

$$|E| = (X_1 \ldots X_{N-1})^{1-N}(X_1-1)\ldots(X_{N-1}-1)T. \begin{vmatrix} 1 & 1 & \ldots & 1 \\ X_1 & X_2 & & X_{N-1} \\ X_1^2 & X_2^2 & & X_{N-1}^2 \\ & & & \\ X_1^{N-2} & X_2^{N-2} & & X_{N-1}^{N-2} \end{vmatrix} \qquad (4.9)$$

$$\begin{aligned} & X_1 = \ldots = X_{n_1} = \delta_1 \\ & \ldots\ldots\ldots \\ & X_{n_1+\ldots+n_{s-1}+1} = \ldots = X_{N-1} = \delta_s. \end{aligned}$$

In fact, according to the differentiation of product, we have

$$|E| = \sum_{ij} T_i(X_1 \ldots X_{N-1})^{1-N}(X_1-1)\ldots(X_{N-1}-1)T_j.F \left| \begin{aligned} & X_1 = \ldots = X_{n_1} = \delta_1 \\ & \ldots\ldots \\ & X_{n_1+\ldots+n_{s-1}+1} = \ldots = X_{N-1} = \delta_s, \end{aligned} \right. \qquad (4.10)$$

where

$$F = \begin{vmatrix} 1 & 1 & \dots & 1 \\ X_1 & X_2 & & X_{N-1} \\ X_1^2 & X_2^2 & & X_{N-1}^2 \\ & & & \\ X_1^{N-2} & X_2^{N-2} & & X_{N-1}^{N-2} \end{vmatrix}$$

T_i, T_j are certain differential operators.

In the right side of (4.10), only the term

$$(X_1 \dots X_{N-1})^{1-N}(X_1-1)\dots(X_{N-1}-1)T.F \Bigg|_{\substack{X_1=\dots=X_{n_1}=\delta_1 \\ \dots\dots \\ X_{n_1+\dots+n_{s-1}+1}=\dots=X_{N-1}=\delta_s}}$$

does not equal zero, for all the rest, since in their correspondent determinants it exists not less than two same columns, they equal zero, so that (4.9) being proved.

By use of the result of [10], we have

$$|E| = (\prod_{i=1}^{s} \delta_i^{n_i})^{1-N} . \prod_{i=1}^{s} (\delta_i-1)^{n_i} . (\prod_{i=1}^{s} \prod_{j=1}^{n_i-1} j!) .$$

$$\{[(\delta_2-\delta_1)^{n_2}(\delta_3-\delta_1)^{n_3}\dots(\delta_s-\delta_1)^{n_s}]^{n_1} . [(\delta_3-\delta_2)^{n_3}\dots(\delta_s-\delta_2)^{n_s}]^{n_2} .$$

$$. [(\delta_s-\delta_{s-1})^{n_s}]^{n_{s-1}}\} \neq 0. \qquad (4.11)$$

If δ is the unique N-1 multiple roots, by a directing calculation we obtain

$$|E| = \prod_{i=1}^{N-2} i! \ \delta^{-(N-1)^2} (\delta-1)^{N-1} \neq 0. \qquad (4.12)$$

For this case, the result of [10] can not be used.

Denoting the solution $\{\Pi_j\}_{1\leq j\leq N}$ of (4.7) ((4.3) is its special case of $n_i \equiv 1$) by $\{\Pi_0 n_j\}_{1\leq j\leq N}$, it is evidently that

$$n_j = \frac{\rho_j}{\psi_N^{N-1}} \cdot \{ \prod_{i=1}^{s} (\delta_i^{n_i(1-N)} (\delta_i - 1)^{n_i} \prod_{j=1}^{n_i - 1} j!) . \prod_{1\leq i<j\leq s} (\delta_j - \delta_i)^{n_j n_i} \}, \qquad (4.13)$$

where ρ_j is the value of determinant obtained by replacing the j-row,

i.e. $\left(\sum_{k=j+1}^{N} \psi_k (1-\delta_1^{j-k}), \frac{d}{dX_2} \sum_{k=j+1}^{N} \psi_k (1-X_2^{j-k}) \,|\, X_2=\delta_1, \ldots \right)$, with

$(\mu_1, \mu_2, \ldots, \mu_N)$ in the determinant $|D|$.

For the system (c), it follows from (3.9) that

$$\Pi_0 = (d' - d\lambda b)/\{d' + \sum_{i=1}^{N} \sum_{i+1\leq \ell \leq N} (\ell - i) n_i \psi_\ell\}.$$

For the system (b), it follows from (3.15) that

$$\Pi_0 = (d' - d\lambda b)/\{ \sum_{1\leq i\leq N} i q'_{-i} + d(1+\lambda b' - \lambda b) + \sum_{i=1}^{N} \sum_{i+1\leq \ell\leq N} (\ell - i) n_i \psi_\ell \}.$$

For the system (a), it follows from (3.13) that

$$\Pi_0 = (d' - d\lambda b)/\{ \sum_{1\leq i\leq N} i q_{-i} + d(1+\lambda c) + \sum_{i=1}^{N} \sum_{i+1\leq \ell\leq N} (\ell - i) n_i \psi_\ell \}.$$

Therefore, the g.f. of stationary distribution, $\Pi(z)$, are obtained for the three systems.

The calculation above is valid for both aperiodic case and periodic case. But we must note that the I.M.C. are different for the two cases (cf. theorem 4).

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