# On a class of Boltzmann-type schemes for hyperbolic conservation laws

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PAPIER RÉCUPÉRÉ ET RECYCLÉ

## ABSTRACT

In this paper we consider a class of numerical schemes of Boltzmann type for hyperbolic conservation laws. We give comparaison results of this class with the Roe-Glimm scheme on one hand and the Roe-Godunov scheme on the order hand.

## KEY-WORDS

Hyperbolic conservation laws - Boltzmann scheme - Roe-Glimm scheme - Roe-Godunov scheme.

## RESUME

Dans ce papier on considère une classe de schémas numériques de type Boltzmann pour les lois de conservation hyperboliques.

On donne des résultats de comparaison de cette classe avecle schéma de Roe-Glimm d'une part et le schéma de Roe-Godunov d'autre part.

## MOTS-CLEFS

Lois de conservation hyperboliques - schéma de type Boltzmann - Schéma de Roe-Glimm - Schéma de Roe-Godunov.

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## INTRODUCTION

We consider numerical solutions of the initial-value problem for hyperbolic systems of conservation laws

$$(*)\qquad \begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial B(u)}{\partial x} = 0 & \text{in} \quad (0,\infty) \times \mathbb{R} \\ u(0,x) = u_0(x) & \text{in} \quad \mathbb{R} \end{cases}$$

where $B$, $u_0$ are given data; $u = (u_1,\dots,u_M)^*$.

Writen in matrix form the system $(*)$ is given by

$$(1)\qquad \frac{\partial u}{\partial t} + B'(u)\,\frac{\partial u}{\partial x} = 0$$

where $B'$ is the Jacobian matrix.

We assume that the eigenvalues $\lambda_1(u),\dots,\lambda_M(u)$ are real distinct and arranged in an increasing order. Thus we have a complete set of eigenvectors, and according to the spectral theory of matrices, we see that

$$(2)\qquad B'(u) = \sum_{k=1}^{M} \lambda_k(u) P_k(u) \quad , \quad \sum_{k=1}^{M} P_k(u) = I \ ,$$

where $P_k(u)$ are the corresponding projector of $\lambda_k(u)$.

In this paper we give a class of upstream schemes obtained by approximating (*) by collisionless Boltzmann equations :

(3) $$\frac{\partial z_k}{\partial t} + \xi \frac{\partial z_k}{\partial x} = 0 \quad , \qquad 1 \le k \le M \ , \ \xi \in \mathbb{R} \ .$$

Let us consider a vector distribution $F(u,\xi)$ such that :

(4) $$u = \int_{\mathbb{R}} F(u,\xi)d\xi$$
$$B(u) = \int_{\mathbb{R}} \xi\ F(u,\xi)d\xi \ .$$

In the linear case, an explicit form of $F$ is found in [1]. Here we give $F$ for the general case :

(5) $$F(u,\xi) = \sum_k \int_0^u \delta(\xi-\lambda_k(w))P_k(w)dw$$

where the integration in (5) is carried out on a path in state-space connecting $u$ and $0$.

Clearly this path is not unique (except in the scalar case when $M = 1$ for which

(6) $$F(u,\xi) = \int_0^u \delta(\xi-B'(w))dw \ .$$

Let us solve the problem

(7) $$\frac{\partial z}{\partial t} + \xi \frac{\partial z}{\partial x} = 0$$
$$z(0,x,\xi) = F(u_0(x),\xi) \ .$$

Indeed, we have

$$z(t;x;\xi) = F(u_0(x-\xi t),\xi) \tag{8}$$

and if we set

$$\left\{\begin{aligned} z(t,x) &= \int_{\mathbb{R}} z(t,x,\xi)d\xi \\ \omega(t,x) &= \int \xi\, z(t,x,\xi)d\xi \end{aligned}\right. \tag{9}$$

then $z(t,x)$ is an approximation to the solution $u(t,x)$ of (*) at least for small time. Moreover $z(t,x)$ can be interpreted as the total density and $\omega(t,x)$ as the total flux.

## I. GODUNOV TYPE SCHEME WITH LARGE TIME STEP

Let us introduce a uniform spectral grid

$$x_i = ih \quad , \quad i \in \mathbb{Z} \quad , \quad h > 0$$

be the mesh width and let $\theta > 0$ be the time step.

Let $u_n(x)$ be a piecewise constant function which approximates the solution $u(t,x)$ at time $t_n = n\theta$ :

$$u_n(x) = u^n_{i+1/2} \quad , \quad x_i < x < x_{i+1} \,. \tag{10}$$

Next, to define an approximation $u_{n+1}(x)$ at time $t_{n+1}$ that is piecewise constant we take

$$u^{n+1}_{i+1/2} = \frac{1}{h} \int_{I_i} \int_{\mathbb{R}} F(u_n(x-\theta\xi),\xi)d\xi dx \tag{11}$$

where $I_i = (x_i, x_{i+1})$.

Thus $u^{n+1}_{i+1/2}$ is the average of $z(x,\theta)$ over $I_i$.

We do not impose any stability condition on the time step. On the other hand, integrating (3) over $(0,\theta) \times I_i$ and then with respect to $\xi$ yields the conservation form

(12) $$h(u^{n+1}_{i+1/2} - u^{n}_{i+1/2}) + \theta(B^n_{i+1} - B^n_i) = 0$$

where

(13) $$B^n_i = \frac{1}{\theta} \int_0^\theta \int_{\mathbb{R}} \xi \, F(u_n(x_i - t\xi),\xi) d\xi dt$$

$B^n_i$ is called the numerical flux associated with the scheme (11).

To see that this numerical flux is consistent, we have only to remark that there is an integer $m$, which depends on $\theta$, such that

(14) $$B^n_i = B(u^n_{i-m+1/2}, \ldots, u^n_{i+m+1/2})$$

since we have (13) and the fact that $u_n$ is piecewise constant. Now for $u^n_{i+1/2+\ell} = \bar{u}$, $\ell = -m,\ldots,m$; we have

(15) $$u_n(x_i - t\xi) = \bar{u} \quad \forall\ \xi$$

and from (4) we get that

$$B^n_i = B(\bar{u})$$

which proves the consistency.

Remark 1 : As the support of $F$ extends from $\min_k \min_w \lambda_k(w)$ to $\max_k \max_w \lambda_k(w)$, we have the following decomposition of $F$ :

(16) $$F = F_+ + F_-$$

where

(17) $$F_+ = \int_0^u \sum_{k:\lambda_k(w)\geq 0} \delta(\xi-\lambda_k(w))P_k(w)dw$$
$$F_- = \int_0^u \sum_{k:\lambda_k(w)<0} \delta(\xi-\lambda_k(w))P_k(w)dw \ .$$

Remark 2 : We have clearly that this class of schemes is linearly stable.

## II. GLIMM TYPE SCHEME WITH LARGE TIME STEP

Another way to obtain an approximation $u_{n+1}$ to $u$ at time $t_{n+1}$, that is piecewise constant, is to define $u_{i+1/2}^{n+1}$ with Glimm's random choice method,

(18) $$u_{i+1/2}^{n+1} = \int F(u_n(x_i-\theta\xi+\theta_n h),\xi)d\xi$$

where $(\theta_n)$ is a sequence of random numbers equidistributed in $(0,1)$.

Thus,

(19) $$u_{i+1/2}^{n+1} = \sum_k \int \int_0^{u_n(x_i-\theta\xi+\theta_n h)} \delta(\xi-\lambda_k(w))P_k(w)dwd\xi \ .$$

In practice we can give a linear congruential method for pseudo-random number generation with adequate choice of parameters in this method which guarantee statistical almost-independence of successive pseudo-random numbers.

Let $m_0 \geq 3$ and $r$ be integers, let $\alpha_0$ be an integer such that

$$0 \leq \alpha_0 < m_0$$

and let $\beta$ be an integer relatively prime to $m_0$

$$2 \leq \beta < m_0$$

and

$$(\beta-1)\alpha_0 + r \not\equiv 0 \pmod{m_0}.$$

Then we take

$$\alpha_{n+1} = \beta\alpha_n + R \pmod{m_0} \quad , \quad n \geq 0$$

and

$$a_n = \alpha_n/m_0 \; . \tag{20}$$

## III. A PARAMETRIC TIME DISCRETIZATION OPERATOR

For $(t,y) \in [0,\infty) \times \mathbb{R}$ we associate an operator $T_{t,y}$ which transforms any initial value $u(x)$ onto some approximation of the solution of $(*)$ at time $t$,

$$T_{t,y}\, u(x) = \sum_k \int \int_{u(y)}^{u(x-\xi t)} \delta(\xi-\lambda_k(w))P_k(w)dwd\xi \tag{21}$$

where the first integration in (21) is carried out on a path in state-space connecting $u(y)$ to $u(x-\xi t)$.

It is easy to verify that (21) defines a consistent time discretization of $(*)$.

Because the propagation speed is finite we can solve exactly $(*)$ with the initial value

$$u_n(x) = u^n_{i+1/2} \qquad x_i < x < x_{i+1} \quad , \quad n \in \mathbb{N} \quad , \quad i \in \mathbb{Z}$$

up to time $t_{n+1} = (n+1)\theta$, at least for small $\theta$ .

Let us consider for each mesh point $x_i$ and each step time $t_n$ the solution of the Riemann problem $(*)$ with the initial data

$$(22) \qquad u_{n,i}(x) = \begin{cases} u^n_{i-1/2} & x \le x_i \\ u^n_{i+1/2} & x > x_i \end{cases} .$$

Then

$$(23) \qquad T_{t,x_i} u_{n,i}(x) = \sum_k \int \int_{u_{n,i}(x_i)}^{u_{n,i}(x-\xi t)} \delta(\xi-\lambda_k(w)) P_k(w) dw d\xi$$

$$= \sum_k \iint_{u^n_{i-1/2}}^{u_{n,i}(x-\xi t)} \delta(\xi-\lambda_k(w)) P_k(w) dw d\xi$$

and we define,

$$(24) \qquad \hat{u}_{n+1}(x) = \sum_i \sum_k \iint_{u^n_{i-1/2}}^{u_{n,i}(x-\xi\theta)} \delta(\xi-\lambda_k(w)) P_k(w) dw d\xi .$$

But, we have

$$u_{n,i}(x-\xi\theta) = u^n_{i-1/2} + H(x-x_i-\xi\theta)(u^n_{i+1/2}-u^n_{i-1/2})$$

thus,

$$(25) \qquad \hat{u}_{n+1}(x) = \sum_i \sum_k \int \int_{u^n_{i-1/2}}^{u^n_{i+1/2}} \delta(\xi-\lambda_k(w))H(x-x_i-\xi\theta)P_k(w)dwd\xi$$

and for a test function $f$ we have

$$(26) \qquad \int \hat{u}_{n+1}(x)\ f(x)dx = \sum_i \sum_k \iiint_{u^n_{i-1/2}}^{u^n_{i+1/2}} H(x-x_i-\xi\theta)f(x)\delta(\xi-\lambda_k(w))P_k(w)dwdxd\xi$$

$$= \sum_i \sum_k \iiint_{u^n_{i-1/2}}^{u^n_{i+1/2}} H(x-x_i)f(x+\xi\theta)\delta(\xi-\lambda_k(w))P_k(w)dwdxd\xi$$

$$(23) = \quad \sum_i \sum_k \iint_{u^n_{i-1/2}}^{u^n_{i+1/2}} f(x+\lambda_k(w)\theta)H(x-x_i)P_k(w)dwdx$$

$$= \quad \sum_i \sum_k \iint_{u^n_{i-1/2}}^{u^n_{i+1/2}} f(x)H(x-x_i-\lambda_k(w)\theta)P_k(w)dwdx.$$

Hence, we also have

$$(27) \qquad \hat{u}_{n+1}(x) = \sum_i \sum_k \int_{u^n_{i-1/2}}^{u^n_{i+1/2}} H(x-x_i-\lambda_k(w)\theta)P_k(w)dw\ .$$

## IV - A NUMERICAL SCHEME

As $\hat{u}_{n+1}(x)$ is no longer of the form (10), that is piecewise constant on the grid $(x_i,\ i \in \mathbb{Z})$, we use the following random process to get $u_{n+1}(x)$ at time $t_{n+1}$.

Let

$$(28) \qquad \hat{\hat{u}}_{n+1}(x) = \sum_k \sum_i \int_0^1 H(x-x_i-\theta\lambda_k(u^n_{i-1/2}+sw^n_i))P_k(u^n_{i-1/2}+sw^n_i)dsw^n_i$$

where $w^n_i = u^n_{i+1/2} - u^n_{i-1/2}$.

Then we take

$$u_{n+1}(x) = \sum_k \sum_i H(x_j - x_i + a_n h - \theta\lambda_k(u^n_{i-1/2} + b_n w^n_i))P_k(u^n_{i-1/2} + b_n w^n_i)w^n_i \tag{29}$$

$$\text{for} \quad x_j < x < x_{j+1} \quad , \quad j \in \mathbb{Z} ,$$

where $(a_n)$, $(b_n)$ are independent sequences of random numbers equidistributed in $(0,1)$.

In the sequel we apply our scheme to an illustrative example and we compare the obtained result with the one given by the Roe-Glimm scheme on one hand and the Roe-Godunov scheme on the other hand.

Let us consider the Riemann problem :

$$\frac{\partial u_1}{\partial t} - \frac{\partial u_2}{\partial x} = 0$$

$$\frac{\partial u_2}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{u_1}\right) = 0$$

$$(30) \qquad u_1(0,x) = \begin{cases} u^1_L & x < 0 \\ u^1_R & x > 0 \end{cases}$$

$$u_2(0,x) = \begin{cases} v^2_L & x < 0 \\ v^2_R & x > 0 \end{cases}$$

thus initial data represent a single shock.

Problem (30) arises in fluid mechanics and it is equivalent to the second-order nonlinear wave equation

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(1/\frac{\partial w}{\partial x}\right) = 0 \tag{31}$$

with $u_1 = \frac{\partial w}{\partial x}$ , $u_2 = \frac{\partial w}{\partial t}$ .

Here,

$$B'(u) = \begin{bmatrix} 0 & -1 \\ 1/u_1^2 & 0 \end{bmatrix}$$

with eigenvalues

(32) $$\lambda_1(u) = -1/u_1 \quad , \quad \lambda_2(u) = 1/u_1$$

and corresponding projectors

(33) $$P_1(u) = \begin{bmatrix} 1/2 & u_1/2 \\ 1/2u_1 & 1/2 \end{bmatrix}$$

(34) $$P_2(u) = \begin{bmatrix} 1/2 & -u_1/2 \\ -1/2u_1 & 1/2 \end{bmatrix} .$$

On the other hand to obtain the Roe-scheme we have to find a real matrix $B' = B'(u_L,u_R)$ such that :

(35) $$B'(u_L,u_R)(u_R-u_L) = B(u_R) - B(u_L) .$$

In the general case the existence of such a matrix is discussed in [1].

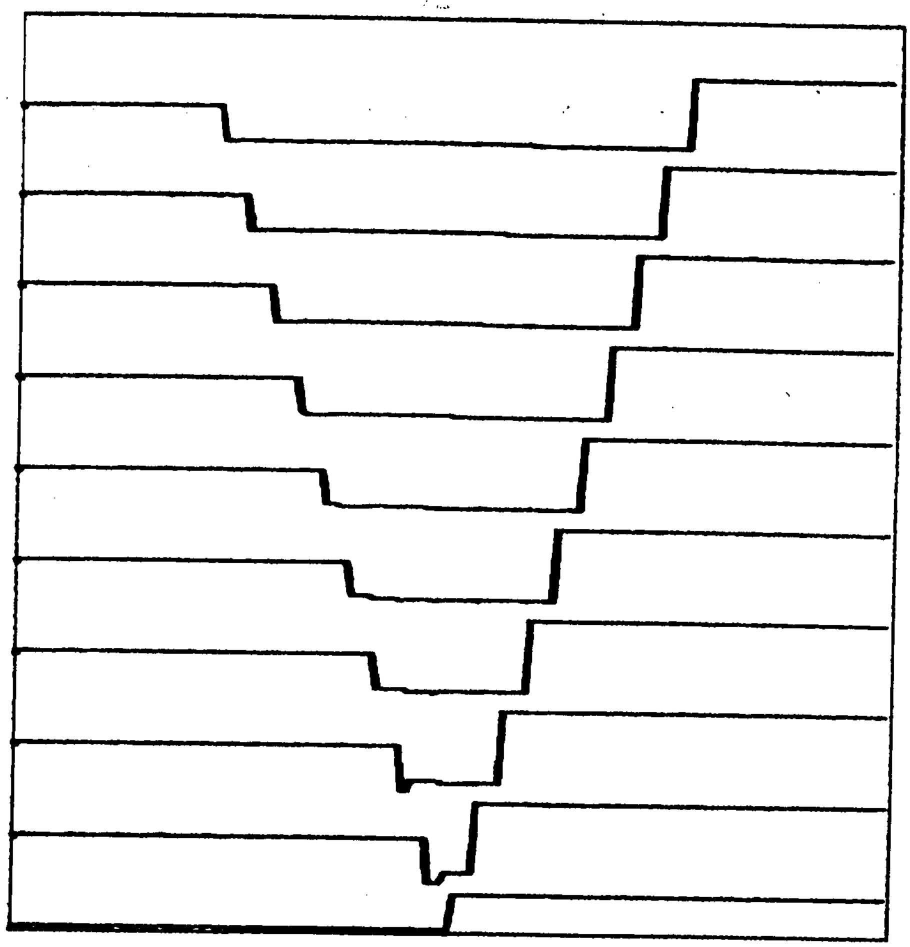
Here we have

(36) $$B'(u_L,u_R) = \begin{bmatrix} 0 & -1 \\ 1/u_L^1 u_R^1 & 0 \end{bmatrix} .$$

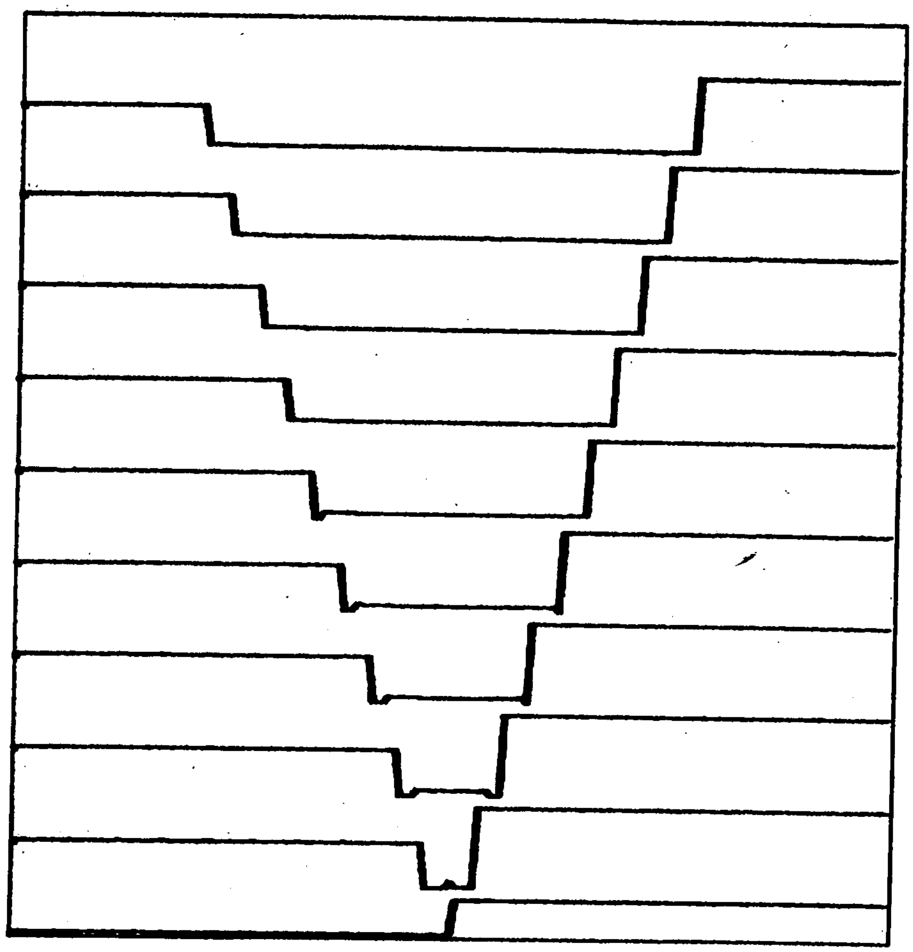
We assume in the sequel that

$$u_L^1 > 0 \quad , \quad u_R^1 > 0 \; .$$

In our numerical calculations we take $\theta = 0.1$, $h = 0.7$, thus using our numerical scheme and then the Roe-Glimm scheme we obtain the following results :



- OUR SCHEME -



ROE-GLIMM SCHEME

The program is available from the author. Hence, we expect that our scheme give good numerical results even in the case of problems not admitting Roe-matrix.

## REFERENCE

[1] A. HARTEN, P.D. LAX, B.V. LEER, "On upstream differencing and Godunov-type schemes for hyperbolic conservation laws", SSAM Review, Vol. 25, n° 1 (1983).

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