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► **To cite this version:**

O. Bennouna. On global smooth solution of Cauchy problem for a class of quasilinear parabolic systems in several spaces variables. RR-0487, INRIA. 1986. inria-00076067

HAL Id: inria-00076067

<https://hal.inria.fr/inria-00076067>

Submitted on 24 May 2006

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Rapports de Recherche

N° 487

**ON GLOBAL SMOOTH SOLUTION
OF CAUCHY PROBLEM
FOR A CLASS OF QUASILINEAR
PARABOLIC SYSTEMS
IN SEVERAL SPACES VARIABLES**

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Février 1986

ON GLOBAL SMOOTH SOLUTION OF CAUCHY
PROBLEM FOR A CLASS OF QUASILINEAR
PARABOLIC SYSTEMS IN SEVERAL SPACES
VARIABLES

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Abstract- In this paper we consider a class of quasilinear parabolic systems in several spaces variables :

$$(*) \left\{ \begin{aligned} \frac{\partial u}{\partial t} + \sum_{i=1}^N B_i(t, x, u) \frac{\partial u}{\partial x_i} + C(t, x, u) &= \mu \Delta u \text{ in } \pi_T \\ u(x, 0) &= u_0(x) \text{ in } \mathbb{R}^N \end{aligned} \right.$$

Where B_i , C and u_0 are given data, $\mu > 0$ fixed and $\pi_T =]0, T[\times \mathbb{R}^N$, $u = (u^1, u^2, \dots, u^M)$.

Using probabilistic methods we introduce a notion of generalized solution of (*) and we prove existence and uniqueness results.

Key words : quasilinear - generalized solution

Résumé- Dans ce papier on considère une classe de systèmes quasilinéaires paraboliques avec plusieurs variables d'espace :

$$(*) \left\{ \begin{aligned} \frac{\partial u}{\partial t} + \sum_{i=1}^N B_i(t, x, u) \frac{\partial u}{\partial x_i} + C(t, x, u) &= \mu \Delta u \text{ dans } \pi_T \\ u(x, 0) &= u_0(x) \text{ dans } \mathbb{R}^N \end{aligned} \right.$$

où B_i , C et u_0 sont des données, $\mu > 0$ fixé et $\pi_T =]0, T[\times \mathbb{R}^N$, $u = (u^1, u^2, \dots, u^M)$.

On introduit une notion de solution généralisée pour (*) pour laquelle on a un résultat d'existence et d'unicité ceci à l'aide de techniques probabilistes.

Mots clefs : quasilinéaire - solution généralisée

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INTRODUCTION

We consider in this paper the problem of existence and uniqueness of global smooth solutions for the Cauchy problem

$$(*) \quad \frac{\partial u}{\partial t} + \sum_{i=1}^N B_i(t, x, u) \frac{\partial u}{\partial x_i} + C(t, x, u) = \mu \Delta u \quad \text{in } \Pi_T$$
$$u(x, 0) = u_0(x) \quad \text{in } \mathbb{R}^N$$

where B_i , C and u_0 are given data, $\mu > 0$ fixed, and $\Pi_T =]0, T[\times \mathbb{R}^N$,

$$u = (u^1, u^2, \dots, u^M).$$

Using probabilistic methods we introduce a notion of generalized solution of (*). When this generalized solution is smooth enough then it is a classical solution. In the proof of the existence result we use a successive approximation process.

The paper is organized as follows : the part I contains definitions and some usefull lemmas, and the part II is concerned with the existence and uniqueness results.

This paper generalizes the result of D.W. STROOCK [2] on the probabilistic representation of the solution of Cauchy problem for a class of linear parabolic systems.

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I. DEFINITIONS AND PRELIMINARY RESULTS

I.1. Notations and assumptions

In the following we assume that :

- (1) $B_i^{k\ell}(t,x,v)$, $C^{k\ell}(t,x,v)$ are bounded and have bounded derivatives with respect to v
- (2) $u_0^k(x)$ is a bounded measurable function.

Let us denote

$$(3) \quad G(t,x,v) = \int_0^1 \frac{\partial C}{\partial u}(t,x,\theta v) d\theta \quad , \quad \forall (t,x,v) \in \Pi_T \times \mathbb{R}^M$$

then

$$C(t,x,v) = G(t,x,v)v + C(t,x,0) .$$

In the sequel, for convenience, we take

$$C(t,x,0) \equiv 0 .$$

Let $\Omega = C([0,\infty), \mathbb{R}^N)$. Given $t \geq 0$ and $\omega \in \Omega$ denote by $y(t) \equiv y(t,\omega)$ the value of ω at t . For $0 \leq s \leq t$, define

$$F_t^s = \mathcal{B}[y(\lambda) : s \leq \lambda \leq t]$$

$$F^s = \mathcal{B}[y(\lambda) : \lambda \geq s] .$$

Then it was shown in [1] that for each $s \geq 0$ and $x \in \mathbb{R}^N$ there is a unique probability measure $P_{s,x}$ on (Ω, F^s) such that

$$(4) \quad y(t) = x + \sqrt{2\mu} [w(t) - w(s)] \quad \text{a.s. } P_{s,x}$$

where $w(t)$ is the standard Wiener process.

Let $u(t,x)$ be a bounded measurable vector function and denote by $X_u^s(t)$, $t \geq s$ the solution to the stochastic integral equation

$$(5) \quad X_u^s(t) = I + \frac{1}{\sqrt{2\mu}} \int_s^t X_u^s(\lambda) B_i(\lambda, y(\lambda), u(\lambda, y(\lambda))) dw_i(\lambda) + \int_s^t X_u^s(\lambda) G(\lambda, y(\lambda), u(\lambda, y(\lambda))) d\lambda \quad \text{a.s. } P_{s,x}.$$

For convenience we consider problem (*) with final condition.

I.2. Definitions and lemmas

We now give the definition of a generalized solution.

Definition

A bounded measurable vector function $u(t,x)$ is a generalized solution of (*) if $u(t,x)$ satisfies the integral equation :

$$(6) \quad u(t,x) = E_{t,x} X_u^t(T) u_0(y(T)) \quad , \quad \forall (t,x) \in \Pi_T.$$

Remark 1

From the weak continuity of $P_{s,x}$ we have that $u \in C(\Pi_T)$.

Remark 2

When $u \in C_b^{2,1}(\Pi_T)$, using the Itô formula we see that u is a classical solution.

Let us give some usefull results :

Lemma 1 : Under conditions (1), (2), we have the estimate

$$(7) \quad \|u(t)\|_\infty^2 \leq M \|u_0\|_\infty^2 \exp\left[\frac{(T-t)}{2\mu} (\|B\|_\infty^2 + 4\mu \|G\|_\infty)\right]$$

$$\forall t \in [0, T].$$

Proof : We have

$$(8) \quad |u(t,x)| \leq \|u_0\|_{\infty} (E_{t,x} \|X_u^t(T)\|^2)^{1/2} .$$

On the other hand we see that

$$(9) \quad \begin{aligned} \|X_u^t(T)\|^2 &= M + \frac{1}{\sqrt{2\mu}} \sum_i \int_t^T \text{Tr}[X_u^t(s) B_i(s, y(s), u(s, y(s))) X_u^t(s)^*] dw_i(s) + \\ &+ \frac{1}{2\mu} \sum_i \int_t^T \text{Tr}[X_u^t(s) B_i(s, y(s), u(s, y(s))) \\ &\quad B_i^*(s, y(s), u(s, y(s))) X_u^t(s)^*] ds + \\ &+ 2 \int_t^T \text{Tr}[X_u^t(s) G(s, y(s), u(s, y(s))) X_u^t(s)^*] ds . \end{aligned}$$

Then, using condition (3) we get that

$$(10) \quad E \|X_u^t(T)\|^2 \leq M + \left[\frac{1}{2\mu} \|B\|_{\infty}^2 + 4\mu \|G\|_{\infty} \right] \int_t^T E \|X_u^t(s)\|^2 ds$$

which implies

$$E \|X_u^t(T)\|^2 \leq M \exp \left[\frac{(T-t)}{2\mu} (\|B\|_{\infty}^2 + 4\mu \|G\|_{\infty}) \right]$$

and then (7).

Remark 3

If we assume that : $\text{Tr}[AG(t,x,v)A^*] \leq -\alpha \|A\|^2$ for any $(t,x,v) \in \Pi_T \times \mathbb{R}^M$ and any $M \times M$ real valued matrix A we obtain

$$(11) \quad E \|X_u^t(s)\|^2 \leq M \exp \left[\frac{(s-t)}{2\mu} (\|B\|_{\infty}^2 - 4\alpha\mu) \right] \quad \forall s, 0 \leq t \leq s \leq T .$$

We assume in the following that

$$(12) \quad ||G(t,x,v_1) - G(t,x,v_2)|| \leq K ||v_1 - v_2|| \quad \forall (t,x) \in \Pi_T.$$

Lemma 2 : Under conditions (1), (2) and (12) we have

$$(13) \quad E ||X_u^t(s) - X_v^t(s)||^2 \leq M_0(s-t) \int_t^s ||u(\theta) - v(\theta)||_\infty^2 d\theta$$

where

$$(14) \quad M_0(s-t) = 4M \cdot \exp\left[\frac{4}{u} (s-t) ||B||_\infty^2\right] \exp\left[4(s-t) ||G||_\infty (1+(s-t) ||G||_\infty)\right] \\ \cdot \left[||\frac{\partial G}{\partial u}||_\infty^2 (s-t) + \frac{1}{2u} ||\frac{\partial B}{\partial u}||_\infty^2 \right] (*).$$

Proof :

$$(15) \quad X_u^t(s) - X_v^t(s) = \frac{1}{\sqrt{2u}} \int_t^s X_u^t(\theta) [B_i(\theta, y(\theta), u(\theta, y(\theta))) - \\ - B_i(\theta, y(\theta), v(\theta, y(\theta)))] dw_i(\theta) + \\ + \frac{1}{\sqrt{2u}} \int_t^s [X_u^t(\theta) - X_v^t(\theta)] B_i(\theta, y(\theta), v(\theta, y(\theta))) dw_i(\theta) + \\ + \int_t^s X_u^t(\theta) [G(\theta, y(\theta), u(\theta, y(\theta))) - G(\theta, y(\theta), v(\theta, y(\theta)))] d\theta \\ + \int_t^s [X_u^t(\theta) - X_v^t(\theta)] G(\theta, y(\theta), v(\theta, y(\theta))) d\theta \\ = I + II + III + IV.$$

We have

(*) $||\frac{\partial f}{\partial u}||_\infty$ is the smallest Lipschitz constant.

$$(16) \quad E(|I|^2) \leq \frac{1}{2\mu} E \int_t^s \|X_u^t(\theta)[B_i(u(\theta, y(\theta))) - B_i(v(\theta, y(\theta)))]\|^2 d\theta$$

$$\leq \frac{1}{2\mu} \|\frac{\partial B}{\partial u}\|_\infty^2 \sup_{t \leq \theta \leq s} (E \|X_u^t(\theta)\|^2) \int_t^s \|u(\theta) - v(\theta)\|^2 d\theta$$

$$(17) \quad E(|II|^2) \leq \frac{1}{2\mu} E \int_t^s \|[X_u^t(\theta) - X_v^t(\theta)]B_i(v(\theta, y(\theta)))\|^2 d\theta$$

$$\leq \frac{1}{2\mu} \|B\|_\infty^2 \int_t^s E \|X_u^t(\theta) - X_v^t(\theta)\|^2 d\theta$$

$$(18) \quad E(|III|^2) \leq \|\frac{\partial G}{\partial u}\|_\infty^2 \sup_{t \leq \theta \leq s} (E \|X_u^t(\theta)\|^2) \int_t^s \|u(\theta) - v(\theta)\|^2 d\theta (s-t)$$

$$(19) \quad E(|IV|^2) \leq \|G\|_\infty^2 \int_t^s E \|X_u^t(\theta) - X_v^t(\theta)\|^2 d\theta (s-t)$$

Hence, we get that

$$(20) \quad E \|X_u^t(s) - X_v^t(s)\|^2 \leq 4 \left(\frac{1}{2\mu} \|B\|_\infty^2 + \|G\|_\infty^2 (s-t) \right) \int_t^s E \|X_u^t(\theta) - X_v^t(\theta)\|^2 d\theta +$$

$$+ 4 \sup_{t \leq \theta \leq s} E \|X_u^t(\theta)\|^2 \left[\|\frac{\partial G}{\partial u}\|_\infty^2 (s-t) + \frac{1}{2\mu} \|\frac{\partial B}{\partial u}\|_\infty^2 \int_t^s \|u(\theta) - v(\theta)\|^2 d\theta \right]$$

and then, using (11) and the Gronwall lemma,

$$(21) \quad E \|X_u^t(s) - X_v^t(s)\|^2 \leq 4M(\exp[2(s-t)] \left(\frac{1}{\mu} \|B\|_\infty^2 + 2\|G\|_\infty \right))$$

$$\left[\|\frac{\partial G}{\partial u}\|_\infty^2 (s-t) + \frac{1}{2\mu} \|\frac{\partial B}{\partial u}\|_\infty^2 \int_t^s \|u(\theta) - v(\theta)\|^2 d\theta \right]$$

$$\left(\exp[4(s-t)] \left(\frac{1}{2\mu} \|B\|_\infty^2 + \|G\|_\infty^2 (s-t) \right) \right)$$

$$\int_t^s \|u(\theta) - v(\theta)\|^2 d\theta$$

which completes the proof.

II. EXISTENCE AND UNIQUENESS RESULTS

Let us consider the following process of successive approximations :

$$(22) \quad \begin{aligned} u^0(t, x) &= u_0(x) \\ u^n(t, x) &= E_{t, x} X_{u^{n-1}}^t(T) u_0(y(T)) \quad , \quad \forall n, n \geq 1 \end{aligned}$$

where $X_{u^{n-1}}^t(s)$ is the solution of

$$(23) \quad \begin{aligned} X_{u^{n-1}}^t(s) &= I + \frac{1}{\sqrt{2\mu}} \int_t^s X_{u^{n-1}}^t(\theta) B_i(\theta, y(\theta), u^{n-1}(\theta, y(\theta))) dw_i(\theta) \\ &\quad + \int_t^s X_{u^{n-1}}^t(\theta) G(\theta, y(\theta), u^{n-1}(\theta, y(\theta))) d\theta \end{aligned}$$

then, we have

Theorem 1 : Under conditions (1), (2) and (12) we have

$$(24) \quad \sup_{t \in [0, T]} \| |u^{n+1}(t) - u^n(t)| \|_\infty^2 \leq \frac{(M_1(T) \cdot T)^n}{n!} \sup_{t \in [0, T]} \| |u^1(t) - u^0(t)| \|_\infty^2$$

where

$$(25) \quad M_1(T) = \| |u_0| \|_\infty^2 M_0(T) .$$

Proof : Indeed, we have for $n \geq 1$

$$(26) \quad u^{n+1}(t, x) - u^n(t, x) = E_{t, x} X_{u^n}^t(T) u_0(y(T)) - E_{t, x} X_{u^{n-1}}^t(T) u_0(y(T)) .$$

Thus

$$\begin{aligned} \| |u^{n+1}(t, x) - u^n(t, x)| \| &\leq E_{t, x} \| |u_0(y(T))| \| \| |X_{u^n}^t(T) - X_{u^{n-1}}^t(T)| \| \\ &\leq \| |u_0| \|_\infty E_{t, x} \| |X_{u^n}^t(T) - X_{u^{n-1}}^t(T)| \| . \end{aligned}$$

From (13), we get that

$$(27) \quad \|u^{n+1}(t) - u^n(t)\|_\infty^2 \leq \|u_0\|_\infty^2 M_0(T-t) \int_t^T \|u^n(\theta) - u^{n-1}(\theta)\|_\infty^2 d\theta .$$

Hence

$$(28) \quad \sup_{t \in [0, T]} \|u^{n+1}(t) - u^n(t)\|_\infty^2 \leq \left(\|u_0\|_\infty^2 \right)^n \frac{(M_0(T) \cdot T)^n}{n!} \sup_{t \in [0, T]} \|u^1(t) - u^0(t)\|_\infty^2$$

which completes the proof.

Now from (24) we have the uniform convergence of the sequence $u^n(t, x)$ on $\Pi_{T, R}$ (*) to a vector valued function $u(t, x)$.

On the other hand, there is a matrix-valued function $X_u^s(t)$ such that

$$(29) \quad X_u^s(t) = I + \frac{1}{\sqrt{2\mu}} \int_s^t X_u^s(\theta) B_i(\theta, y(\theta), u(\theta, y(\theta))) d\omega_i(\theta) + \\ + \int_s^t X_u^s(\theta) G(\theta, y(\theta), u(\theta, y(\theta))) d\theta \quad , \quad \text{a.s. } P_{s, x}$$

and $u(t, x)$ satisfies

$$(30) \quad u(t, x) = E_{t, x} X_u^t(T) u_0(y(T)) .$$

Thus $u(t, x)$ is a generalized solution of (*).

To obtain the uniqueness result, let $(X_v^s(t), v(t, x))$ be another generalized solution of (*).

We have

$$(31) \quad u(t, x) - v(t, x) = E_{t, x} X_u^t(T) u_0(y(T)) - E_{t, x} X_v^t(T) u_0(y(T))$$

(*) $\Pi_{T, R} =]0, T[\times B(0, R) .$

hence,

$$\|u(t) - v(t)\|_{\infty}^2 \leq \|u_0\|_{\infty}^2 E_{t,x} \|X_u^t(T) - X_v^t(T)\|^2.$$

From this and (13) we obtain that

$$(32) \quad \|u(t) - v(t)\|_{\infty}^2 \leq \|u_0\|_{\infty}^2 M_0(T) \cdot \int_t^T \|u(\theta) - v(\theta)\|_{\infty}^2 d\theta.$$

Thus, by Gronwall's inequality we have that

$$(33) \quad \|u(t) - v(t)\|_{\infty}^2 \equiv 0, \quad \forall t \in [0, T]$$

i.e. $u(t, x) = v(t, x)$ in $\bar{\Pi}_T$.

Remark 4

When $B_i^{k\ell}(t, x, v)$, $G(t, x, v)$ and $u_0(x)$ have two continuous derivatives in x and v then we can obtain a priori estimate of derivatives of the generalized solution.

Remark 5

The results of this paper remain valid for the more general parabolic systems with Cauchy condition :

$$\frac{\partial u^k}{\partial t} + \sum_{i=1}^N \sum_{\ell=1}^M B_i^{k\ell}(t, x, u) \frac{\partial u^{\ell}}{\partial x_i} + C^k(t, x, v) = \sum_{i,j=1}^N a_{ij}(t, x) \frac{\partial^2 u^k}{\partial x_i \partial x_j}$$

in Π_T

$$u^k(0, x) = u_0^k(x) \text{ in } \mathbb{R}^N, \quad 1 \leq k \leq M.$$

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Imprimé en France

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